Theta and Selberg Zeta Function

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Abstract

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1 The Theta function

Let M be a closed Riemannian manifold. We consider the Laplace-Beltrami operator Δ_M.

We use the scalar products on the spaces of smooth differential forms $\Omega^i(M)$ induced by the Riemannian volume density vol_M and the induced metric on T^*M in order to define the formal adjoint d^* of the de Rham differential $d:\Omega^0(M)\to\Omega^1(M)$. Then the Laplace-Beltrami operator is given by

$$\Delta_M := d^*d : C^\infty(M)\to C^\infty(M)\ .$$

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It is an elliptic second order differential operator. We consider it as an unbounded operator with domain $C^\infty(M) = \Omega^0(M)$ on the Hilbert space $L^2(M)$ obtained by completing $C^\infty(M)$ with respect to the L^2-scalar product. By definition, Δ_M is symmetric and positive. It is in fact essentially selfadjoint.

The Laplace-Beltrami operator has a discrete spectrum $\mathtt{spec}(\Delta_M) \subset \mathbb{R}$ consisting of positive eigenvalues λ of finite multiplicity $m_\lambda \in \mathbb{N}$.

Example 1.1. *For $M = \mathbb{R}/\mathbb{Z}$ we have $\Delta_M = -\frac{d^2}{dt^2}$ and the eigenvalues and multiplicities are given by*

$$\begin{array}{ccc} 0 & m_0 = 1 & \\ 4\pi^2 n^2 & m_{4\pi^2n^2} = 2 & n \in \mathbb{N} \end{array}$$

By Weyl's asymptotic we know that

$$\sum_{\lambda \in \mathtt{spec}(\Delta_M) \cap [0,R]} m_\lambda \stackrel{R\to\infty}{\sim} R^{n/2} ,$$

where $n := \dim(M)$.

Definition 1.2. *The theta function of M is the holomorphic function defined on $\{\mathtt{Re}(t) > 0\} \subset \mathbb{C}$ by*

$$\theta_M(t) := \sum_{\lambda \in \mathtt{spec}(\Delta_M)} e^{-t\sqrt{\lambda}} .$$

Weyl's asymptotic implies that the sum defining the theta function converges. It moreover shows that the operator $e^{-t\sqrt{\Delta_M}}$ defined by functional calculus is of trace class. We can write the theta function in the form

$$\theta_M(t) = \mathtt{Tr} e^{-t\sqrt{\Delta_M}} .$$

Example 1.3. *For the circle $\mathbb{R}/\mathbb{Z}$ we can calculate the theta function explicitly. We have*

$$\theta_{S^1}(t) = 1 + 2\sum_{n=1}^{\infty} e^{-2\pi n t} = 1 + \frac{2}{e^{2\pi t} - 1} .$$

Observe that this function extends meromorphically to all of $\mathbb{C}$ with first order poles in the set $i\mathbb{Z}$. Moreover, we have a functional equation

$$\theta_{S^1}(t) + \theta_{S^1}(-t) = 0 .$$

Note that the closed geodesics on $\mathbb{R}/\mathbb{Z}$ have length in $\mathbb{N}$. Further note that

$$\mathtt{lim}_{\epsilon\to 0} \frac{1}{it + \epsilon}$$

exists in the sense of a distribution on $\mathbb{R}$. We see that

$$\mathtt{lim}_{\epsilon\to 0} \theta_{S^1}(it + \epsilon)$$

has a limit in the sense of distributions which is smooth outside $i\mathbb{Z}$.

One can now ask in general, how the theta function $\theta_M(t)$ of a closed Riemannian manifold behaves near the imaginary axis.

The Riemannian metric induces a geodesic flow Φ_t, $t \in \mathbb{R}$ on the unit sphere bundle $\pi : S(TM) \to M$.

We will describe its generator, a vector field on $S(TM)$. Let ∇^{LC} be the Levi-Civita connection. The associated parallel transport induces a splitting

$$T(S(TM)) \cong T^h\pi \oplus T^v\pi$$

into a horizontal and a vertical subspace. For $\xi \in S(TM)$ let $\xi^H \in T_\xi(S(TM))$ denote the unique horizontal vector such that $d\pi(\xi^h) = \xi$. Then the generator of the geodesic flow is the vector field $\xi \mapsto \xi^h$ on $S(TM)$.

The orbit $t \to \Phi_t(\xi)$ of a point $\xi \in S(TM)$ is called a geodesic. A closed geodesic γ of length $l_\gamma > 0$ is represented by a point $\xi \in S(TM)$ such that $\Phi_{l_\gamma}(\xi) = \xi$. By convention, then all the points $\Phi_s(\xi)$, $s \in \mathbb{R}$, represent the same closed geodesic. The multiplicity $n_\gamma \in \mathbb{N} \setminus \{0\}$ of the closed geodesic is the minimal integer such that $\Phi_{l_\gamma/n_\gamma}(\xi) = \xi$. We let $L(M)$ denote the set of closed geodesics. We define the length spectrum of the Riemannian manifold as the set of length of closed geodesics

$$|L(M)| := \{l_\gamma \mid \gamma \in L(M)\} \subseteq \mathbb{R} \ .$$

Example 1.4. *We have* $|L(\mathbb{R}/\mathbb{Z})| = \mathbb{N}$.

Theorem 1.5 (Duistermaat-Guillemin)**.** *If M is a closed Riemannian manifold, then*

$$\mathtt{lim}_{\epsilon\to 0}\theta_M(it+\epsilon)$$

exists as a tempered distribution. This distribution is smooth on the subset $\mathbb{R} \setminus (\{0\} \cup |L(M)|)$.

Proof. Let f be a Schwarz function. Then we have

$$\mathtt{lim}_{\epsilon\to 0}\int_{\mathbb{R}} f(t)\theta_M(it+\epsilon)dt = \sum_{\lambda\in\mathtt{spec}(\Delta_M)} m_\lambda \mathtt{lim}_{\epsilon\to 0} e^{-\epsilon\lambda}\int_{\mathbb{R}} f(t)e^{-it\lambda}dt = \sum_{n\in\mathbb{N}} m_\lambda \hat{f}(\lambda) \ . \quad (1)$$

In view of Weyl's asymptotics this gives the limit in the sense of tempered distributions.

We now use facts on the wave equation in order to analyse the singularities of the theta function. For $\mathtt{Re}(t) > 0$ the operator $e^{-t\sqrt{\Delta}}$ has a smooth integral kernel, called the heat kernel, $W(t,x,y)$ so that

$$(\exp(-t\sqrt{\Delta})\phi)(x) = \int_M W(t,x,y)\,\phi(y)\,d\mathrm{vol}_M(y) \ .$$

We can express the theta function in terms of the heat kernel as

$$\theta_M(t) = \int_M W(t,x,x)\, d\mathrm{vol}_M(x) \ .$$

For real t the heat kernel solves the heat equation

$$\partial_t W(t,x,y) = \sqrt{\Delta_{M,x}} W(t,x,y) \ , \quad \mathtt{lim}_{t\to\infty} W(t,x,y) = \delta_y(x)$$

with distributional initial condition. In order to study the kernel on the imaginary axis we consider the equation

$$-i\partial_t W(it,x,y) = \sqrt{\Delta_{M,x}} W(it,x,y) \ , \quad \mathtt{lim}_{t\to\infty} W(it,x,y) = \delta_y(x) \ .$$

If we differentiate again we get rid of the square root of the Laplacian and the more common wave equation

$$-\partial_t^2 W(t,x,y) = \Delta_{M,x} W(t,x,y) \ .$$

Depending on the initial conditions the solution represents the following operators:

$$\cos(t\sqrt{\Delta_M}) \ : \ \mathtt{lim}_{t\to 0} W^{ev}(t,x,y) = \delta_y(x) \ , \quad \mathtt{lim}_{t\to 0}\partial_t W^{ev}(t,x,y) = 0$$

$$\frac{\sin(t\sqrt{\Delta_M})}{\sqrt{\Delta_M}} \ : \ \mathtt{lim}_{t\to 0} W^{odd}(t,x,y) = 0 \ , \quad \mathtt{lim}_{t\to 0}\partial_t W^{odd}(t,x,y) = \delta_y(x)$$

We can then use the relation

$$W(it,x,y) = W^{ev}(t,x,y) + \sqrt{\Delta_{M_x}} W^{odd}(i,x,y)$$

in order to analyse the the kernel.

Example 1.6. *If $M = \mathbb{R}$, then*

$$W^{ev}_{\mathbb{R}}(t,x,y) = \frac{1}{2}(\delta(x-y+t)+\delta(x-y-t)) \ , \quad W^{odd}_{\mathbb{R}}(t,x,y) = \frac{1}{2}(\Theta(x-y+t)-\Theta(x-y-t)) \ ,$$

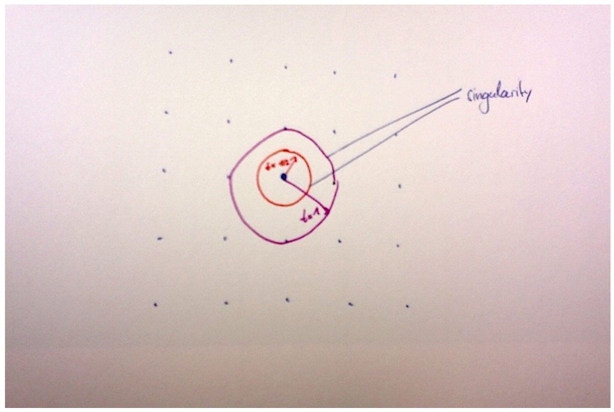
where Θ is the Heaviside distribution. For $M = \mathbb{R}/\mathbb{Z}$ we get by averaging

$$W^{ev}_{S^1}(t,x,y) = \sum_{n\in\mathbb{Z}} W^{ev}_{\mathbb{R}}(t,x,y+n) \ , \quad W^{odd}_{S^1}(t,x,y) = \sum_{n\in\mathbb{Z}} W^{odd}_{\mathbb{R}}(t,x,y+n) \ .$$

In the general case, by finite propagation speed and Egorov's theorem we know that

1. $W^{ev/odd}(it,x,y) = 0$ if $\mathrm{dist}(x,y) > t$
2. $W^{ev/odd}(it,x,y)$ is smooth if there is no geodesic of length t between x and y.

Since $\sqrt{\Delta_M}$ is pseudodifferential, it does not increase singular supports. We conclude that $W(it, x, y)$ is smooth if there is no geodesic of length t between x and y. □



Remark 1.7. *Note that the Theorem 1.5 gives a very rough picture. One can actually describe the singularity of $\theta_M(t)$ at $t = 0$ very precisely using Hadamard's asymptotics of the wave kernel. Moreover, one can describe the nature of the singularities on the real axis in a much more precise manner. The mathematical framework here is the theory of Fourier integral operators.*

2 Hyperbolic space

We now assume that M is an orientable closed n-dimensional Riemannian manifold with constant sectional curvature K. In this case the universal covering of M is a symmetric space. We distinguish three cases.

1. If $K > 0$, then the universal covering of M is the round sphere S^n_K. We have a presentation
$$M \cong \Gamma\backslash S^n \cong \Gamma\backslash SO(n+1)/SO(n)$$
for a finite group $\Gamma \cong \pi_1(M)$

2. If $K = 0$, then M is covered by a torus and its universal covering is $\mathbb{R}^n$. We have a presentation
$$M \cong \Gamma^{fin}\backslash T^n \cong \Gamma\backslash SO(n) \ltimes \mathbb{R}^n \ ,$$
where Γ is an extension of a finite group Γ^{fin} by $\mathbb{Z}^n$. We will not consider the flat case further.

3. If $K < 0$, then the universal covering of M is a hyperbolic space H^n_K and
$$M \cong \Gamma\backslash H^n_K \cong \Gamma\backslash SO(n,1)/SO(n)$$
for some discrete torsion-free subgroup $\Gamma \cong \pi_1(M)$.

While the Duistermaat-Guillemin theorem 1.5 is robust against perturbations the analytic properties of the theta function beyond the imaginary axis are very delicate. It turns out to be useful to consider theta function of the shifted operator

$$A_M := \Delta_M + \frac{(n-1)^2}{4} K \ .$$

This operator may have finitely many negative eigenvalues.

Remark 2.1. *The effect of the shift is that the essential spectrum of A_{H^n} is $[0, \infty)$. The resolvent kernel is the distributional kernel of $(A_{H^n} - \lambda)^{-1}$. It is apriori defined for $\lambda \notin [0, \infty)$. It extends as a meromorphic function to the Riemann surface of $\sqrt{\lambda}$. Without the shift we would have to consider the Riemann surface of the function $\sqrt{\frac{(n-1)^2}{4} K + \lambda}$.*

Note that Riemannian manifolds of constant sectional curvature are real analytic. If we scale the metric by $g \rightsquigarrow \lambda g$, then $K \rightsquigarrow \lambda^{-2} K$ and $A_M \rightsquigarrow \lambda^{-2} A_M$. So formally we can relate the case $K = -1$ with the case $K = 1$ by the scaling $g \rightsquigarrow ig$. This is well reflected by the heat/wave kernels. There is a strong relation between $W_{H^n}(t, x, y)$ and $W_{S^n}(it, x, y)$.

The wave kernels $W_{H^n}^{ev/odd}(t, x, y)$ can be calculate explicitly.

Example 2.2. *Here is the formula for even $n = 2m$ and $K = -1$:*

$$W_{H^n}^{odd}(t, x, y) = \frac{1}{2^{m+1/2} \pi^m} \left(\frac{1}{\sinh(t)} \frac{\partial}{\partial t} \right)^{m-1} \mathtt{Re} \left(\frac{1}{\sqrt{\cosh(t) - \cosh(\mathrm{dist}(x, y))}} \right)$$

The wave kernel for M is given as in Example 1.6 by averaging

$$W(it, x, y) = \sum_{\gamma \in \Gamma} W_{H^n}(it, x, \gamma y) \ . \tag{2}$$

This formula can be employed to describe the singularities of $\theta_M(t)$ explicitly in terms of Γ. Here we use the bijection

$$Conj(\Gamma) \setminus \{[1]\} \cong L(M)$$

between the set of conjugacy classes of non-identity elements in Γ and closed geodesics on M which holds true in general for manifolds with negative sectional curvature.

The geodesic flow Φ_t on a Riemannian manifold of negative sectional curvature is hyperbolic. By definition of hyperbolicity this means that there is a Φ-invariant decomposition

$$T(S(TM)) \cong T^+ \oplus T^- \oplus T_0$$

of the tangent bundle of $S(TM)$ into a uniformally exponentially expanding and contracting directions, and the flow direction. For a hyperbolic flow on a compact manifold we know that the length spectrum $|L(M)| \subseteq \mathbb{R}$ is discrete. If $\gamma \in L(M)$ is a closed geodesic of length l_γ and $\xi \in \gamma$, then we define the Poincaré section

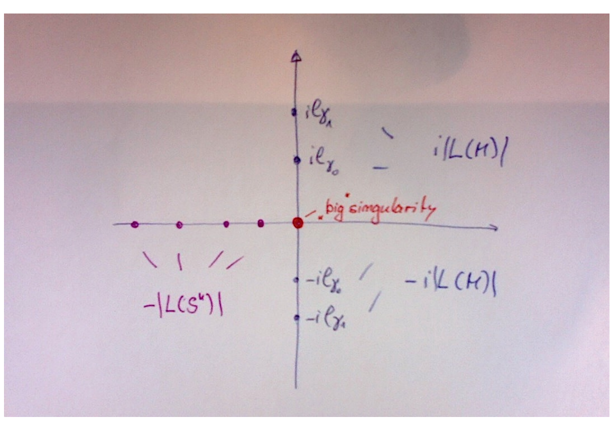
$$P_\gamma^\pm := (d\Phi_{l_\gamma})_{|T_\xi^\pm} \in \mathtt{End}(T_\xi^\pm) \ .$$

A detailed analysis leads to:

Theorem 2.3 (Cartier-Voros (n=2), Juhl, B.-Olbrich)**.** *Let M be a closed even-dimensional manifold of constant sectional curvature $K = -1$.*

1. *The θ-function $\theta_M(t)$ has a meromorphic continuation to all of $\mathbb{C}$.*
2. *It satisfies the functional equation*
$$\theta_M(t) + \theta_M(-t) = \chi(M)\theta_{S^n}(it) \ . \tag{3}$$
3. *The singularities of θ_M are contained in the set*
$$i|L(M)| \cup -i|L(M)| \cup -|L(S^n)| \ .$$
4. *For $\gamma \in L(M)$ the singularity at $\pm il_\gamma$ is a first order pole with residue*
$$\mathtt{res}_{t=il_\gamma}\theta_M(t) = \frac{(-1)^{n-1} l_\gamma e^{\frac{n-1}{2}l_\gamma}}{2\pi n_\gamma \det(1 - P_\gamma^+)}$$
(the contributions of different geodesics of the same length add up).
5. *The function $\theta_{S^n}(t)$ is explicitly known:*
$$\theta_{S^n}(t) = Q(\frac{d}{dt})\frac{\cosh(t)}{\sinh^2(t/2)} \ .$$
for an explicit even polynomial Q of degree $n-2$.

Proof. One can prove this theorem by an explicit calculation of the theta function using the explicit formula (2) for the wave kernel. We refer to [BO95b], [BO94] for details. □



Remark 2.4. Note that singularities of $\theta_{S^n}(t)$ are in

$$i\pi\mathbb{Z} = i|L(S^n)| \cup -i|L(S^n)|$$

as expected. The Euler characteristic $\chi(M)$ comes in through the proportionality principle as a topological expression of the volume of M via

$$\mathrm{vol}(M) = -\frac{\chi(M)}{2}\mathrm{vol}(S^n) \ .$$

Remark 2.5. *There are the following generalizations:*

1. *all rank compact one locally symmetric spaces,*
2. *Laplace operators on bundles*
3. *surface of finite volume.*

3 The Selberg Trace Formula

We have the identity of distributions

$$\mathtt{lim}_{\epsilon\to 0}\left[\frac{1}{ix+\epsilon} - \frac{1}{ix-\epsilon}\right] = 2\pi\delta_0(x)$$

(which is also an ingredient of the proof of Theorem 2.3). If we apply $\theta_M(it)$ to a symmetric, smooth and compactly supported function $f(t)$ and use the functional equation (3) of θ_M and the symmetry of θ_{S^n} in order to replace

$$\int_{-\infty}^{0} f(t)\theta(it+\epsilon)dt \quad \text{by} \quad -\int_0^\infty \theta_M(it-i\epsilon)f(t) + \int_0^\infty \chi(M)\theta_{S^n}(t-i\epsilon)f(t)dt \ ,$$

then we get

$$\begin{aligned}\mathtt{lim}_{\epsilon\to 0}\int_{\mathbb{R}} \theta_M(it+\epsilon)f(t)dt &= \chi(M)\int_0^\infty \theta_{S^n}(t)f(t)dt \\ &\quad + \sum_{\gamma\in L(M)} \frac{(-1)^{n-1} l_\gamma e^{\frac{n-1}{2}l_\gamma}}{n_\gamma \det(1-P_\gamma^+)} f(l_\gamma) \ .\end{aligned}$$

We get the following version of Selberg's Trace formula:

Theorem 3.1 (Selberg Trace Formula)**.** *For a symmetric $f \in C_c^\infty(\mathbb{R})$ we have*

$$\begin{aligned}\sum_{\lambda\in\mathtt{spec}(A_M)} m_\lambda \hat{f}(\sqrt{\lambda}) &= \chi(M)\int_0^\infty \theta_{S^n}(t)f(t)dt \\ &\quad + \sum_{\gamma\in L(M)} \frac{(-1)^{n-1} l_\gamma e^{\frac{n-1}{2}l_\gamma}}{n_\gamma \det(1-P_\gamma^+)} f(l_\gamma) \ .\end{aligned}$$

Remark 3.2. *The left-hand side needs the analytic continuation of $\hat{f}$ in order to evaluate at the square roots of the negative eigenvalues of A_M. The extreme one is $i\frac{n-1}{2}$. Hence we can extend this side of the trace formula to functions which exponentially decay like $e^{-(\frac{n-1}{2}+\epsilon)t}$, but not to Schwarz functions. The same applies to the terms on the right-hand side.*
Since $\theta_{S^n}(t)$ is singular at $t = 0$ the integral has to be interpreted appropriately. This requires smoothness of f.

The left-hand side of the trace formula is called the spectral side of the trace formula. The right-hand side is the geometric side. It is the sum of the identity contribution and the hyperbolic contribution.

Remark 3.3. *The Selberg Trace Formula generalizes the Poisson summation formula for a symmetric Schwarz function on $\mathbb{R}$. For the present discussion we write it in the form*

$$\hat{f}(0) + \sum_{n\in\mathbb{N}} 2\hat{f}(\sqrt{2\pi n}) = f(0) + \sum_{n\in\mathbb{Z}} f(n) \ .$$

The left-hand side is the sum over the spectrum of Δ_{S^1}, while the right-hand side is the sum over the set of closed geodesics and the identity contribution.

Remark 3.4. *The formula given in 3.1 this is a very geometric formulation of the Selberg Trace Formula. There are much more general versions, usually formulated in terms of representation theory. For a Lie group G the right-regular representation R of G on $L^2(\Gamma\backslash G)$ decomposes as a direct integral of irreducible unitary representations. We let $L^2_d(\Gamma\backslash G) \subseteq L^2(\Gamma\backslash G)$ be the subspace which decomposes as a direct sum. The trace formula in general is an identity of the form*

$$\mathtt{Tr}(R(f)_{|L^2_d(\Gamma\backslash G)}) = \sum_{\gamma\in\Gamma} \int_{G/G_\gamma} f(hgh^{-1})dh$$

for $f \in C_c^\infty(G)$ and suitable normalizations of the Haar measure on G/G_γ. The summands on the right-hand side are called orbital integrals and denoted by $\mathcal{O}_\gamma(f)$. The left-hand side can be written as a sum over the contributions of the unitary representations

$$\mathtt{Tr}(R(f)_{|L^2_d(\Gamma\backslash G)}) = \sum_{\pi\in\hat{G}} N_\gamma(\pi)\theta_\pi(f)$$

where $\hat{G}$ is the unitary dual of G, θ_π is the character of π, and $N_\Gamma(\pi)$ is the multiplicity of π in $L^2_d(\Gamma\backslash G)$. Using the Fourier transform $\hat{f}(\pi) := \theta_\pi(f)$ we get the trace formula in the form

$$\sum_{\pi\in\hat{G}} \hat{f}(\pi) = \sum_{\gamma\in\Gamma} \mathcal{O}_\gamma(f)$$

4 The Selberg Zeta Function

Note the equality

$$\frac{1}{s^2+\lambda^2} = \int_{-\infty}^{\infty} \frac{e^{-s|t|}}{2s} e^{it\lambda} dt \ .$$

We will apply the Selberg Trace Formula 3.1 to the family of functions

$$f_s(t) := \frac{e^{-s|t|}}{2s}$$

for $\mathtt{Re}(s) > 0$. The spectral side is then formally (since the resolvent is not of trace class)

$$\mathtt{Tr}\frac{1}{s^2+A_M} \ .$$

The hyperbolic contribution can be expressed in terms of the function

$$L_M(s) := \sum_{\gamma \in L(M)} \frac{(-1)^{n-1} l_\gamma e^{\frac{n-1}{2} l_\gamma}}{n_\gamma \det(1-P_\gamma^+)} e^{-s l_\gamma} \ .$$

The sum converges for $\mathtt{Re}(s) > \frac{n-1}{2}$. We get the identity

$$\sum_{\lambda \in \mathtt{spec}(A_M)} \frac{m_\lambda}{s^2+\lambda} - \chi(M) \int_0^\infty \theta_{S^n}(t) \frac{e^{-s|t|}}{2s} dt = \frac{L_M(s)}{2s} \ . \tag{4}$$

Remark 4.1. *Since we insert a function which is not smooth at $t=0$ the spectral and the identity contributions require a regularization. One idea is to consider a sufficiently high derivative with respect to s of all terms. Furthermore, one must take* $\mathtt{Re}(s) > \frac{n-1}{2}$.

We study the identity contribution

$$\chi(M) \int_0^\infty \theta_{S^n}(t) \frac{e^{-s|t|}}{2s} dt$$

more closely. It is useful consider the decomposition

$$\chi(M) \int_0^\infty \theta_{S^n}(t) \frac{e^{-s|t|}}{2s} dt = \frac{\chi(M)}{2} \sum_{\lambda \in \mathtt{spec}(A_{S^n})} \frac{m_\lambda}{s^2-\lambda^2} + I_{odd}(s)$$

into an even and odd part. The odd part can be calculated explicitly and one can see from the explicit formula that it extends meromorphically to all of $\mathbb{C}$. It follows from the explicit formulas for $I^{odd}(s)$ that the poles of the even and odd part cancel for $\mathtt{Re}(s) > 0$ and add up for $\mathtt{Re}(s) < 0$.

Example 4.2. *For $n=1$ we get*

$$I^{odd}(s) = -\frac{\pi\chi(M)}{2} \tan(\pi s) \ .$$

Corollary 4.3. *The function $L_M(s)$ has a meromorphic continuation to all of $\mathbb{C}$.*

From (4) we read off following information about the singularities of L_M.

Corollary 4.4. 1. *An eigenvalue $\lambda \in \mathrm{Spec}(A_M)\setminus\{0\}$ contributes a pole in $\pm i\sqrt{\lambda}$ with residue m_λ*

2. *If $0 \in \mathrm{Spec}(A_M)$, then it contributes a first order pole with residue $2m_0$.*

3. *An eigenvalue $\lambda \in \mathrm{Spec}(A_{S^n})$ contributes a pole at $-\sqrt{\lambda}$ with residue $\chi(M)m_\lambda$.*

Since the residues of $L_M(s)$ are all in $\mathbb{Z}$ we can make the following definition:

Definition 4.5. *We define the Selberg Zeta function $Z_M(s)$ to be the meromorphic function on $\mathbb{C}$ characterized uniquely by*

$$\frac{Z'_M(s)}{Z_M(s)} = L_M(s)\ , \quad \mathtt{lim}_{s\to\infty} Z_M(s) = 1\ .$$

Remark 4.6. *Note that the normalization condition is justified since $\mathtt{lim}_{s\to\infty} L_M(s) = 0$.*

5 A Panorama of results on Z_S

In this section we give a list of results on the Selberg zeta function. The whole theory can be developed in a similar manner for all compact locally symmetric manifolds of negative section curvature and form auxiliary bundles. Much of the theory extends to the case of finite volume and convex cocompact manifolds. For simplicity of notation we state most results in the special case of compact even-dimensional hyperbolic manifolds without bundles.

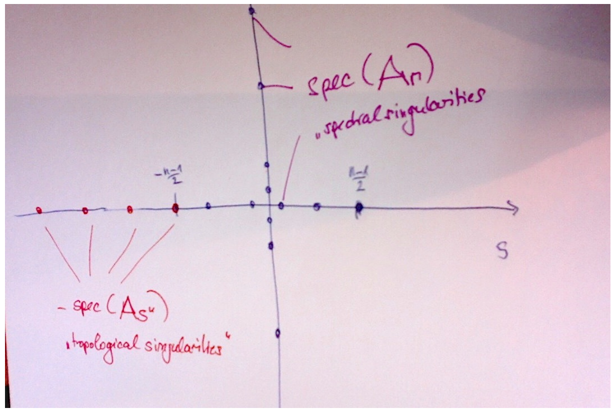
Many people contributed to this field. I want just mention some who influenced me personally: A. Juhl, M. Olbrich, W. Hoffmann, W. Müller, S. Patterson, A. Deitmar, S. Lang, H. Moscovici, R. Stanton, L. Guillope, M. Zworski, M. Wakayama, D. Fried. Further contributers (this is definitly not a complete list) were U. Bröker, J. Joergenson, R. Gangolli, D'Hoker, D.H Phong, P. Cartier, A. Voros, S. Koyama, J. Millson, P. Perry, N. Kurokawa M. Wakayama.

5.1 Product formula

For $\mathtt{Re}(s) > \frac{n-1}{2}$ the Selberg Zeta function can be represented as an infinite product (see [BO95b, Ch. 3])

$$Z_S(M) = \prod_{\gamma\ \text{primitve}} \prod_{k=0}^{\infty} \det(1 - S^k(P_\gamma^-) e^{-(s+\frac{n-1}{2})l_\gamma})$$

5.2 Singularities



In particular, the analog of Riemann hypothesis holds true. It is a consequence of the fact that A_M and A_{S^n} are selfadjoint.

5.3 Determinant formula

The Selberg zeta function can be expressed in terms of zeta regularized determinants of differential operators [BO95b, Thm. 3.19]:

$$Z_M(s) = \det(s^2 + \Delta_M) \det(s + \sqrt{\Delta_{S^n}})^{-\chi(M)} \exp(P(s))$$

where $P(s)$ is an explicitly known polynomial of degree n.

5.4 Order of singularities - Patteron's conjecture

The divisor of the Selberg Zeta function can be described in a very uniform manner using group cohomology of the group $\Gamma \cong \pi_1(M)$ in induced representations of the isometry group of $\tilde{M}$.

For every complex number $s \in \mathbb{C}$ we consider the bunde $\Lambda^s \to S^{n-1}$ of s-densities on S^{n-1}. This bundle is equivariant with respect to the action of the diffeomorphism group of S^n. The isometry group of $\tilde{M}$ acts isometrically on the universal covering H^n of M, and therefore and on its compactification at infinity $S^{n-1} \cong \partial_\infty H$. This action is in fact given by real analytic diffeomorphisms. It therefore acts in the space of hyperfunction sections

$$I^{-\omega}(s) := C^{-\omega}(S^{n-1}, \Lambda^{s-\frac{n-1}{2}})$$

of the $s - \frac{n-1}{2}$-density bundle. This is the afroamentioned induced representation. We can restrict this representation along the inclusion $\Gamma \hookrightarrow G$. We define the Poincaré polynomial

$$\chi_x(\Gamma, I^{-\omega}(s)) := \sum_{\in \mathbb{N}} (-1)^i x^i \dim H^i(\Gamma, C^{-\omega}(S^{n-1}, \Lambda^s)) \ .$$

Note that $I^{-\omega}(s)$ is huge so that this definition requires a justification.

Theorem 5.1 (Patterson's conjecture, B.-Olbrich)**.**

1. $\chi_x(\Gamma, I^{-\omega}(s))$ *is well-defined.*
2. $\chi_1(\Gamma, I^{-\omega}(s)) = 0$
3. $\mathtt{ord}_{t=s} Z_M(t) = -\frac{d}{dx}_{|x=1} \chi_x(\Gamma, I^{-\omega}(s))$ *for all* $s \in \mathbb{C}$.

We refer to [BO95a].

5.5 Bundles and spectral invariants

If $V \to S(TM)$ is a locally homogeneous bundle, then the geometric flow lifts. We let $P_\gamma^{\pm,V} \in \mathtt{End}(T_\xi^\pm \otimes V_\xi)$ the corresponding Poincaré sections. Then we can define a Selberg Zeta function of V by

$$Z_M(s, V) := \prod_{\gamma \text{ primitve}} \prod_{k=0}^{\infty} \det(1 - S^k(P_\gamma^{-,V}) e^{-(s+\frac{n-1}{2})l_\gamma}) \ .$$

For example, in the odd-dimensional case, if $V = (\pi^* S(M))^\pm$ is the canonical decomposition of the spinor bundle into the $\pm i$- eigespaces of the Clifford multiplication by the base point, then we get (see [BO95b, Kor. 5.4])

Theorem 5.2.

$$\mathtt{lim}_{s \to 0} \frac{Z_M(s, S(M)^+)}{Z_M(s, S(M)^-)} = e^{\pi i \eta(\not{D}(M))} \ .$$

Applying similar techniques to the de Rham complex one can get a presentation of the analytic torsion. This leads to the Fried conjecture.

References

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