Étale cohomology

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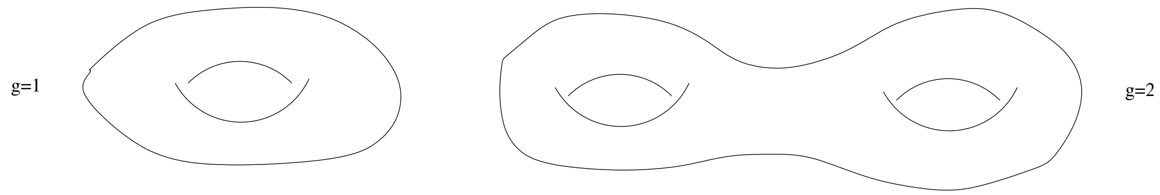
1 Introduction

In mathematics, one often looks for invariants which characterize or classify the regarded objects. Often such invariants are given by cohomology groups.

This is a long standing approach in topology, where one considers singular cohomology groups $H^i(X, \mathbb{Q})$ of a topological space X, which are defined by explicit 'cycles' and 'boundaries'. These suffice to determine the genus g of a (compact) Riemann surface: If X looks topologically like a sphere with g handles:



then $\dim_{\mathbb{Q}} H^1(X, \mathbb{Q}) = 2g$. These cohomology groups can also be obtained as sheaf cohomology (of a constant sheaf).

Riemann surfaces can also be regarded as complex algebraic curves, i.e., as algebraic curves over a field $\mathbb{C}$ of complex numbers. For any algebraic varieties X over any field k (or any scheme) one can consider sheaf cohomology with respect to the Zariski topology. This is useful for coherent sheaves, for example for the Grothendieck-Serre duality and the Riemann-Roch theorem.

However, the Zariski cohomology of an algebraic variety X over $\mathbb{C}$ does not give the singular cohomology of a topological space $X(\mathbb{C})$; this is due to the fact that this topology is much finer than the Zariski topology. Furthermore one wants to obtain an analogous topology for varieties over any field k. For fields with positive characteristic, Serre showed that there exists no cohomology theory $H^*(-, \mathbb{Q})$, such that $H^1(X, \mathbb{Q})$ has the dimension $2g$ for a smooth projective curve of genus g. But Weil had postulated such a theory to show the Weil conjectures for varieties over finite fields by a fixpoint formula, as it is known in topology.

The solution was found by Grothendieck, together with M. Artin, by creating the étale cohomology. For any prime $\ell \neq char(k)$ this provides cohomology groups $H^i(X, \mathbb{Q}_\ell)$ that have the properties postulated by Weil. With these, Deligne eventually proved the Weil conjectures.

2 Grothendieck topologies/Sites

Grothendieck's approach for the étale cohomology (and since then for many other theories) was to leave the setting of topological spaces. He noticed that one only needs the notion of 'coverings' with certain properties, to define sheaves and their cohomologies, by replacing at the same time 'open set' by 'object in a category'.

Definition 2.1 Let $\mathcal{X}$ be a category and $\mathcal{C}$ another category. A **presheaf on $\mathcal{X}$ with values in $\mathcal{C}$** is a *contravariant* functor

$$P : \mathcal{X} \to \mathcal{C}\,.$$

Morphisms of presheaves are morphisms of functors.

(Here we ignore –actually non-trivial – set theory problems by assuming that the category $\mathcal{X}$ is small). If $\mathcal{C}$ is the category $\underline{Ab}$ of abelian groups (resp. the category $\underline{Rg}$ of rings, resp. ...), then one speaks of presheaves of abelian groups [for short: abelian presheaves] (resp. of rings, resp. ...).

Example 2.2 Let X be a topological space. Then one can assign to X the following category $\underline{X}$: Objects are open sets $U \subseteq X$. Morphisms are the inclusions $V \subseteq U$.

Then one can see that a presheaf in Grothendieck's sense is just a classical presheaf: Because of the contravariant functoriality, one has an arrow $P(U) \to P(V)$ for every inclusion $V \subseteq U$. The properties of a functor provide the properties of presheaves for these 'restrictions' $res_{U,V}$.

Definition 2.3 Let $\mathcal{X}$ be a category.

(a): A **Grothendieck topology** on $\mathcal{X}$ consists of a set $\mathcal{T}$ of families $(U_i \xrightarrow{\varphi_i} U)_{i \in I}$ of morphisms in $\mathcal{X}$, called **coverings** of $\mathcal{T}$, such that the following properties hold:

(T1) If $(U_i \to U)_{i\in I}$ in $\mathcal{T}$ and $V \to U$ is a morphism in $\mathcal{X}$, then all fibre products $U_i \times_U V$ exist, and $(U_i \times_U V \to V)_{i\in I}$ is in $\mathcal{T}$.

(T2) If $(U_i \to U)_{i\in I}$ is in $\mathcal{T}$ and $(V_{ij} \to U_i)_{j\in J_i}$ is in $\mathcal{T}$ for all $i \in I$, then the family

$$(V_{ij} \to U)_{i,j}$$

obtained by the compositions $V_{ij} \to U_i \to U$ is in $\mathcal{T}$.

(T3) If $\varphi : U' \to U$ is an isomorphism, then $(U' \xrightarrow{\varphi} U)$ is in $\mathcal{T}$.

(b) A **site** is a pair $\mathcal{S} = (\mathcal{X}, \mathcal{T})$ with a category $\mathcal{X}$ and a Grothendieck topology $\mathcal{T}$ on $\mathcal{X}$. One denotes the underlying category $\mathcal{X}$ also by $Cat(\mathcal{S})$ and the topology also by $Cov(\mathcal{S})$, thus $\mathcal{S} = (Cat(\mathcal{S}), Cov(\mathcal{S}))$. Sometimes $(\mathcal{X}, \mathcal{T})$ is called a Grothendieck topology as well.

Example 2.4 If one takes the usual coverings $(U_i)_{i\in I}$ of open sets $U \subseteq X$ in example 2.2, then the corresponding families $(U_i \hookrightarrow U)_{i\in I}$ form a Grothendieck topology on $\underline{X}$. Note: The fibre product of open sets $U \subseteq X, V \subseteq X$ is the intersection $U \cap V$.

Definition 2.5 Let $\mathcal{S} = (\mathcal{X}, \mathcal{T})$ be a site, and let $\mathcal{C}$ be a category with products (e.g., the category of sets or of abelian groups). A presheaf

$$F : \mathcal{X} \to \mathcal{C}$$

is called a **sheaf** (with respect to $\mathcal{T}$), if for every covering $(U_i \to U)_{i\in I}$ in $\mathcal{T}$ the diagram

$$F(U) \xrightarrow{\alpha} \prod_i F(U_i) \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

is exact, where the arrow α_1 on the right side is induced by the first projections $U_i \times_U U_j \to U_i$ and the arrow α_2 is induced by the second projection $U_i \times_U U_j \to U_j$ (This means that α is the difference kernel of α_1 and α_2, see appendix 4.A below). Morphisms of sheaves are morphisms of the underlying presheaves.

Remark 2.6 Let $\mathcal{C}$ be the category of sets. If, for $s \in F(U)$, we denote the component of $\alpha(s)$ in $F(U_i)$ by $s|_{U_i}$ and for $(s_i) \in \prod\limits_i F(U_i)$, we denote the images of s_i and s_j in $F(U_i \times_U U_j)$ by $s_i|_{U_i\times_U U_j}$ and $s_j|_{U_i\times_U U_j}$ respectively, then we literally obtain the same conditions as for the usual sheaves on topological spaces, except that we replace $U_i \cap U_j$ with $U_i \times_U U_j$: The conditions are:
(i) If $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
(ii) If $(s_i)_{i\in I} \in \prod\limits_i \mathcal{F}(U_i)$ with $s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j}$ for all $i, j \in I$, then there is an $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all $i \in I$.

Definition 2.7 (a) A morphism $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ of sites is a (covariant) functor $f^0 : \mathcal{X} \to \mathcal{X}'$ (!) which has the following properties:
(S1) If $(U_i \xrightarrow{\varphi_i} U)$ is in $\mathcal{T}$, then $(f^0(U_i) \xrightarrow{f^0(\varphi_i)} f^0(U))$ is in $\mathcal{T}'$.
(S2) If $(U_i \to U)$ is in $\mathcal{T}$ and $V \to U$ is a morphism in $\mathcal{T}$, then the canonical morphism

$$f^0(U_i \times_U V) \to f^0(U_i) \times_{f^0(U)} f^0(V)$$

is an isomorphism for all i.

Example 2.8 If $f : X' \to X$ is a continuos map between of topological spaces, we obtain a morphism $f : \mathcal{S}(X') \to \mathcal{S}(X)$ of the associated sites (Example 2.4) by

$$\begin{array}{rccc} f^{-1}: & \underline{X} & \to & \underline{X'} \\ & U & \mapsto & f^{-1}(U)\,. \end{array}$$

3 Constructions for presheaves and sheaves

For a category $\mathcal{X}$ let $Pr(\mathcal{X})$ be the category of abelian presheaves on $\mathcal{X}$.

Definition 3.1 (Push-forward) Let $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ be a morphism of sites and let $P' : \mathcal{X}' \to \underline{Ab}$ be an abelian presheaf. Then the direct image (or push-forward) $f_P P'$ of P' is defined as the presheaf

$$f_P P' = P' f^0 : \mathcal{X} \xrightarrow{f^0} \mathcal{X}' \xrightarrow{P'} \underline{Ab}\,.$$

Explicitly we have $(f_P P')(U) = P'(f^0(U))$ for U in $\mathcal{X}$ and $f_P(\varphi) = P'(f^0(\varphi)) : P'(f^0(U_2)) \to P'(f^0(U_1))$ for $\varphi : U_1 \to U_2$ in $\mathcal{X}$). For a morphism $\psi : P_1' \to P_2'$ of abelian presheaves on $\mathcal{X}$ one obtains a morphism

(3.1.1) $$f_P \psi : f_P P_1' \to f_P P_2'$$

as follows: For U in $\mathcal{X}$ define

$$\begin{array}{ccccc} (f_P\psi)_U & : & (f_P P_1')(U) & \to & (f_P P_2')(U) \\ \shortparallel & & \shortparallel & & \shortparallel \\ \psi_{f^0(U)} & : & P_1'(f^0(U)) & \to & P_2'(f^0(U))\,. \end{array}$$

One can see easily that this produces a morphism of presheaves (3.1.1) and that one obtains a functor

$$\begin{array}{rrcl} f_P : & Pr(\mathcal{X}') & \to & Pr(\mathcal{X}) \\ & P' & \mapsto & f_P P' \\ & \psi & \mapsto & f_P \psi\,. \end{array}$$

Proposition 3.2 The functor

$$f_P : Pr(\mathcal{X}') \to Pr(\mathcal{X})$$

has a left adjoint

$$f^P : Pr(\mathcal{X}) \to Pr(\mathcal{X}')\,.$$

For presheaves $P \in Pr(\mathcal{X})$ and $P' \in Pr(\mathcal{X}')$ we thus have isomorphisms

(3.2.1) $$Hom_{\mathcal{X}'}(f^P P, P') \cong Hom_{\mathcal{X}}(P, f_P P')\,,$$

functorially in P and P'. For a presheaf P on $\mathcal{X}$, $f^P P$ is called the inverse image (or pull-back) of P.

Proof of 3.2: For U' in $\mathcal{X}'$ consider the following category $I_{U'}$: Objects are pairs (U, ψ), where U is an object in $\mathcal{X}$ and

$$\psi : U' \to f^0(U)$$

is a morphism in $\mathcal{X}'$. A morphism $(U_1, \psi_1) \to (U_2, \psi_2)$ is a morphism $\varphi : U_1 \to U_2$ in $\mathcal{X}$ for which the diagram

(3.2.2)
$$\begin{tikzcd} & f^0(U_1) \arrow[dd, "f^0(\varphi)"] \\ U' \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\ & f^0(U_2) \end{tikzcd}$$

is commutative. Then we have a functor

$$(3.2.3)\qquad \begin{array}{rrcl} P: & I_{U'}^{op} & \to & \underline{Ab} \\ & (U,\psi) & \mapsto & P(U) \\ & \varphi & \mapsto & P(\varphi) \end{array}$$

(where $I_{U'}^{op}$ denotes the dual category of $I_{U'}$) and define

$$(f^P P)(U') = \varinjlim_{(U,\psi)\in I_{U'}^{op}} P(U)$$

as the inductive limit over $I_{U'}^{op}$ (The idea is that $(f^P P)(U')$ is the inductive limit of all sets $P(U)$, where "U' is contained in $f^0(U)$", see Example 3.4 below).

If $\varphi' : U' \to V'$ is a morphism in $\mathcal{X}'$, we obtain a functor

$$I_{V'} \to I_{U'}\,,$$

by mapping an object $(V, V' \to f(V))$ in $I_{V'}$ to the object $(V, U' \xrightarrow{\varphi'} V' \to f(V))$, and mapping a morphism $\varphi : V_1 \to V_2$ to the same morphism.

This gives a morphism

$$(f^P P)(V') = \varinjlim_{I_{V'}^{op}} P(U) \to \varinjlim_{I_{U'}^{op}} P(U) = (f^P P)(U')\,.$$

With this $f^P P$ becomes a contravariant functor

$$f^P P : \mathcal{X}' \to \underline{Ab}$$

i.e., an abelian presheaf on $\mathcal{X}'$.

Now we prove the adjointness. Let P' be an abelian presheaf on $\mathcal{X}'$ and let

$$(3.2.4)\qquad v : f^P P \to P'$$

be a morphism of abelian presheaves. For all U in $\mathcal{X}$ one obtains the homomorphism

$$(3.2.5)\qquad v_{f^0(U)} : (f^P P)(f^0(U)) \to P'(f^0(U)) = (f_P P')(U)\,.$$

Furthermore the pair $(U, id_{f^0(U)})$ is an object of $I_{f^0(U)}$, and we obtain a canonical homomorphism

$$(3.2.6)\qquad P(U) \to \varinjlim_{(V,\psi)\in I_{f^0(U)}^{op}} P(V) = (f^P P)(f^0(U))\,,$$

and by composition of (3.2.6) and (3.2.5) a homomorphism

$$(3.2.7)\qquad P(U) \to (f_P P')(U)\,,$$

which is obviously functorial in U, so that we get a morphism of abelian presheaves on $\mathcal{X}$

$$(3.2.8)\qquad w : P \to f_P P'\,.$$

Conversely, consider a morphism w as in (3.2.8), and let $U' \in ob(\mathcal{X}')$. Then for every object $(U, \psi : U' \to f^0(U))$ in $I_{U'}$ one has the homomorphism

$$P(U) \xrightarrow{w_U} (f_P P')(U) = P'(f^0(U)) \xrightarrow{P'(\psi)} P'(U') .$$

This homomorphism is functorial in (U, ψ) and gives a homomorphism (universal properties of the direct limit)

$$(f^P P)(U') = \varinjlim_{(U,\psi)\in I_{U'}^{op}} P(U) \to P'(U') ,$$

which itself is functorial in U' and therefore gives a morphism

$$v : f^P P \to P'$$

of abelian presheaves on $\mathcal{X}'$.

Finally, one easily shows that the mappings $v \mapsto w$ and $w \mapsto v$ are inverse to each other.

Remark 3.3 The same holds for presheaves with values in a category $\mathcal{C}$, if all direct limits exist in $\mathcal{C}$, e.g., $\mathcal{C} = \underline{Set}, \underline{Rg}, ...$

Example 3.4 Let $f : X' \to X$ be a continuous map of topological spaces and

$$f : \mathcal{S}(X') \to \mathcal{S}(X), \quad U \mapsto f^{-1}(U) ,$$

the corresponding morphism of sites. Then

$$f_P : Pr(X') \to Pr(X), \quad f^P : Pr(X) \to Pr(X')$$

are the usual functors. This is obvious for f_P: One has $(f_P P')(U) = P'(f^{-1}(U))$. For f^P one obtains the usual construction: for $U' \subseteq X'$, $I_{U'}$ is the ordered set (!) of the open sets $U \subseteq X$ with $f(U') \subseteq U$, thus $U' \subseteq f^{-1}(U)$, and $f^P P(U') = \varinjlim_{f(U')\subseteq U} P(U)$.

For a site $(\mathcal{X}, \mathcal{T})$ let $Sh(\mathcal{X}, \mathcal{T})$ be the category of abelian sheaves (with respect to $\mathcal{T}$) on $\mathcal{X}$. We obtain a fully faithful embedding

$$i = i_{\mathcal{T}} : Sh(\mathcal{X}, \mathcal{T}) \hookrightarrow Pr(\mathcal{X}) .$$

Theorem 3.5 The embedding i has a left adjoint

$$a = a_{\mathcal{T}} : Pr(\mathcal{X}) \to Sh(\mathcal{X}, \mathcal{T}) .$$

Thus for all presheaves P and all sheaves F one has isomorphisms, functorial in P and F,

$$Hom_{Pr}(P, iF) \xrightarrow{\sim} Hom_{Sh}(aP, F) .$$

For a presheaf P, aP is called the associated sheaf (with respect to $\mathcal{T}$).

For the proof we need some preparations.

Definition 3.6 A refinement

$$(V_j \to U)_{j\in J} \to (U_i \to U)_{i\in I}$$

of coverings of U is a map $\varepsilon : J \to I$ of the index sets and a family $(f_j)_{j\in J}$ of U-morphisms $f_j : V_j \to U_{\varepsilon(j)}$.

With the refinements as morphisms and the obvious compositions, we obtain the category $\mathcal{T}(U)$ of the coverings of U (with respect to the topology $\mathcal{T}$).

Definition 3.7 Let U in $\mathcal{X}$ and P be an abelian presheaf on $\mathcal{X}$.

(a) For every covering $\mathfrak{U} = (U_i \to U)$ in $\mathcal{T}$

$$\check{H}^0(\mathfrak{U}, P) = \ker(\prod_i P(U_i) \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} \prod_{i,j} P(U_i \times_U U_j))$$

is called the **zeroth Čech cohomology** of P with respect to $\mathfrak{U}$. Here let α_1 and α_2 be defined as in Definition 2.5.

(b) Call

$$\check{H}^0(U, P) = \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, P)$$

the **zeroth Čech cohomology** of P for U, where the direct limit runs over the category $\mathcal{T}(U)^{op}$.

Remark 3.8 A presheaf P on $\mathcal{X}$ is a sheaf for $\mathcal{T}$ if and only if for all U in $\mathcal{X}$ and all $\mathfrak{U} = (U_i \to U)$ in $\mathcal{T}(U)$ the canonical homomorphism

$$P(U) \to \check{H}^0(\mathfrak{U}, P)$$

is an isomorphism. In this case $P(U) \to \check{H}^0(U, P)$ is an isomorphism as well.

Proof of Theorem 3.5 Let P be an abelian presheaf on $\mathcal{X}$. For U in $\mathcal{X}$ define

$$\tilde{P}(U) := \check{H}^0(U, P)\,.$$

This produces a presheaf, since for $\varphi : V \to U$ in $\mathcal{X}$ we have a canonical homomorphism

$$\varphi^* : \check{H}^0(U, P) \to \check{H}^0(V, P)\,, \tag{3.5.1}$$

because for every covering $\mathfrak{U} = (U_i \to U)$ of U we obtain the covering $\mathfrak{U}_V := (U_i \times_U V \to V)$ of V, thus an induced homomorphism

$$\check{H}^0(\mathfrak{U}, P) \to \check{H}^0(\mathfrak{U}_V, P)\,, \tag{3.5.2}$$

and by passing to the limit over the coverings in $\mathfrak{U}$ we obtain (3.5.1).

A morphism of abelian presheaves

$$\psi : P_1 \to P_2$$

induces a canonical morphism of presheaves

$$\tilde{\psi} : \tilde{P}_1 \to \tilde{P}_2 \tag{3.5.3}$$

as follows: For every covering $\mathfrak{U} = (U_i \to U)$, ψ induces a homomorphism

$$\check{H}^0(\mathfrak{U}, P_1) \to \check{H}^0(\mathfrak{U}, P_2)\,. \tag{3.5.4}$$

This is compatible with refinements and by passing to the limit over $\mathcal{T}(U)^{op}$ gives a map

$$\tilde{\psi}_U : \check{H}^0(U, P_1) \to \check{H}^0(U, P_2)\,. \tag{3.5.5}$$

For every morphism $\varphi : V \to U$, the diagram

$$\begin{tikzcd} \tilde{\psi}_U : & \check{H}^0(U, P_1) \arrow[r] \arrow[d, "\varphi^*"'] & \check{H}^0(U, P_2) \arrow[d, "\varphi^*"] \\ \tilde{\psi}_V : & \check{H}^0(V, P_1) \arrow[r] & \check{H}^0(V, P_2) \end{tikzcd}$$

is commutative. This gives (3.5.3). One can easily see that this defines a functor

$$\begin{array}{rcl} Pr(\mathcal{X}) & \to & Pr(\mathcal{X}) \\ P & \mapsto & \tilde{P} \\ \psi & \mapsto & \tilde{\psi}\,. \end{array}$$

Definition 3.9 A presheaf P is called *separated* with respect to $\mathcal{T}$, if for every covering $(U_i \to U)$ in $\mathcal{T}$ the homomorphism

$$P(U) \to \prod_i P(U_i)$$

is injective. (Equivalently, $P(U) \to \check{H}^0((U_i \to U), P)$ is injective).

Lemma 3.10 (a) If P is an abelian presheaf, then $\tilde{P}$ is separated.

(b) There exists a canonical morphism $P \to \tilde{P}$.

(c) If P is a separated abelian presheaf, then $P \to \tilde{P}$ is a monomorphism and $\tilde{P}$ is a sheaf.

(d) If F is a sheaf, then $F \to \tilde{F}$ is an isomorphism.

Preliminary Remark for the Proof: We will see later (see 3.11 and 3.12):

1) For every element $\overline{s} \in \check{H}^0(U, P)$ there exists a covering $\mathfrak{U} = (U_i \to U)$ in $\mathcal{T}(U)$ and an element $s \in \check{H}^0(\mathfrak{U}, P)$ which is mapped to $\overline{s}$ under

$$\check{H}^0(\mathfrak{U}, P) \to \check{H}^0(U, P)$$

(In this case we say that $\overline{s}$ is represented by s).

2) If $\overline{s}$ is represented by $s_1 \in \check{H}^0(\mathfrak{U}_1, P)$ and $s_2 \in \check{H}^0(\mathfrak{U}_2, P)$ (with $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathcal{T}(U)$), then there are refinements $\mathfrak{U}_3 \to \mathfrak{U}_1, \mathfrak{U}_3 \to \mathfrak{U}_2$, such that s_1 and s_2 have the same image in $\check{H}^0(\mathfrak{U}_3, P)$.

Proof of 3.10 (a): Let $(U_i \to U)_i$ be a covering in $\mathcal{T}$ and let $\overline{s} \in \ker(\tilde{P}(U) \to \prod_i \tilde{P}(U_i))$. We have to show $\overline{s} = 0$. There exists a covering $(V_j \to U)_j$ and an element $s \in \check{H}^0((V_j \to U)_j, P)$ which represents $\overline{s}$.

Let s_i be the image of s under

$$\check{H}^0((V_j \to U)_j, P) \to \check{H}^0((V_j \times_U U_i \to U_i)_j, P)\,.$$

This represents $\overline{s}|_{U_i} = 0 \in \check{H}^0(U_i, P)$. By the preliminary remark there exists a refinement for every $i \in I$

$$f_i : (W_{ik} \to U_i)_k \to (V_j \times_U U_i \to U_i)_j$$

such that f_i^* maps s_i to 0 in $\check{H}^0((W_{ik} \to U_i)_k, P)$.

By composition of the coverings $(W_{ik} \to U_i)_k$ and $(U_i \to U)_i$ (axiom (T2)) we obtain a covering $(W_{ik} \to U)_k$ and via the f_i a refinement

$$f : (W_{ik} \to U)_k \to (V_j \to U)_j\,.$$

Then, under

$$f^* : \check{H}^0((V_j \to U)_j, P) \to \check{H}^0((W_{ik} \to U)_k, P)\,,$$

s is mapped to 0 by construction. Thus $\overline{s} = 0$.

(b): This is given by the canonical homomorphisms

$$P(U) \xrightarrow{\sim} \check{H}^0((U \xrightarrow{id} U), P) \to \check{H}^0(U, P) = \tilde{P}(U)\,.$$

(c) Let P be a separated abelian presheaf.

Claim 3.10.1: For every covering $\mathfrak{U} = (U_i \to U)$ in $\mathcal{T}$, $\check{H}^0(\mathfrak{U}, P) \to \check{H}^0(U, P)$ is injective.

Proof By the preliminary remark it suffices to show the injectivity of

$$f^* : \check{H}^0((U_i \to U), P) \to \check{H}^0((V_j \to U), P)$$

for every refinement $f : (V_j \to U) \to (U_i \to U)$. For this consider the covering

$$(V_j \times_U U_i \to U)$$

which is the composition of the coverings $(V_j \times_U U_i \to V_i)$ and $(U_i \to U)$. It has the two refinements

$$(V_j \times_U U_i \to U) \xrightarrow{pr_2} (U_i \to U)$$

$$(V_j \times_U U_i \to U) \xrightarrow{pr_1} (V_j \to U) \xrightarrow{f} (U_i \to U)\,.$$

By the following Lemma 3.11 the two induced homomorphisms

$$\check{H}^0((U_i \to U), P) \underset{pr_1^* f^*}{\overset{pr_2^*}{\rightrightarrows}} \check{H}^0((V_j \times_U U_i \to U), P)$$

are equal. It thus suffices to prove the injectivity of pr_2^*; then $pr_1^* f^*$ and hence also f^* is injective. But pr_2^* is the restriction of

$$\prod_i P(U_i) \xrightarrow{pr_2^*} \prod_i \prod_j P(V_j \times_U U_i)$$

to $\check{H}^0((U_i \to U), P)$, and for every i, $P(U_i) \to \prod_j P(V_j \times_U U_i)$ is injective, because P is separated.

If we apply Claim 3.10.1 to the covering $(U \to U)$, then the injectivity of

$$P(U) = \check{H}^0((U \to U), P) \to \check{H}^0(U, P) = \tilde{P}(U)$$

follows and therefore the first claim of (c).

Now we prove that $\tilde{P}$ is a sheaf. Let $(U_i \to U)$ be a covering. We have to show that

$$\tilde{P}(U) \to \prod_i \tilde{P}(U_i) \rightrightarrows \prod_{i,j} \tilde{P}(U_i \times_U U_j) \tag{3.10.2}$$

is exact. By (a) $\tilde{P}$ is separated, thus the first map is injective. Now let

$$(\overline{s}_i) \in \ker(\prod_i \tilde{P}(U_i) \rightrightarrows \prod_{i,j} \tilde{P}(U_i \times_U U_j)\,.$$

For each i choose a covering $(V_{ik} \to U_i)$ and an element $s_i \in \check{H}^0((V_{ik} \to U_i), P)$, which represents $\overline{s}_i \in \tilde{P}(U_i)$. Let s^1_{ij} be the image of s_i under

$$\check{H}^0((V_{ik} \to U_i), P) \to \check{H}^0((V_{ik} \times_U U_j \to U_i \times_U U), P)\,,$$

and let s^2_{ij} be the image of s_j under

$$\check{H}^0((V_{ik} \to U_j), P) \to \check{H}^0((U_i \times_U V_{ik} \to U_i \times_U U_j), P)\,.$$

The elements represented by s^1_{ij} and s^2_{ij} in $\check{H}^0(U_i \times_U U_j, P)$ are equal to the images of $\overline{s}_i$ and $\overline{s}_j$, respectively, and thus are equal. It follows from Claim 3.10.1 that s^1_{ij} and s^2_{ij} have the same image in

$$\check{H}^0((V_{ik} \times_U V_{j\ell} \to U_i \times_U U_j), P) \subseteq \prod_{k,\ell} P(V_{ik} \times_U V_{j\ell})$$

This implies that

$$s' = (s_i) \in \ker(\prod_{i,k} P(V_{ik}) \rightrightarrows \prod_{i,k,j,\ell} P(V_{jk} \times_U V_{j\ell})) = \check{H}^0((V_{ik} \to U), P)\,.$$

The element $\overline{s}' \in \tilde{P}(U)$ represented by s' is then mapped to $(\overline{s}_i)$ under $\tilde{P}(U) \to \prod_i \tilde{P}(U_i)$. This proves the second claim of (c).

(d) follows immediately from Remark 3.8. This finishes the proof of 3.10.

Lemma 3.10 implies Theorem 3.5: If P is an abelian presheaf, then we define

$$aP = \tilde{\tilde{P}}\,.$$

By 3.10 (a) $\tilde{P}$ is separated, by 3.10 (c) $\tilde{\tilde{P}}$ is a sheaf. Furthermore by 3.10 (b) we obtain a canonical morphism of abelian presheaves

$$can : P \to \tilde{P} \to \tilde{\tilde{P}} = aP\,.$$

If now F is an abelian sheaf and

$$\psi : P \to F(= iF)$$

is a morphism of abelian presheaves, then, by functoriality of the used constructions (the assignment $P \mapsto \tilde{P}$, the morphism $P \to \tilde{P}$), we obtain a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d] & F \arrow[d, "\rho_1"', "\wr"] \arrow[dd, bend left, "\rho"] \\
\tilde{P} \arrow[r, "\tilde{\psi}"] \arrow[d] & \tilde{F} \arrow[d, "\rho_2"', "\wr"] \\
aP = \tilde{\tilde{P}} \arrow[r, "\tilde{\tilde{\psi}}"] & \tilde{\tilde{F}} = aF
\end{tikzcd}$$

where we have isomorphisms on the right hand side by 3.10 (d). Now, if we define

$$a\psi = \tilde{\tilde{\psi}}\,,$$

then we obtain a commutative diagram

$$\begin{tikzcd}
P \arrow[rr, "can"] \arrow[dr, "\psi"'] & & aP \arrow[dl, "\rho^{-1}a\psi =: \psi'"] \\
& F\,, &
\end{tikzcd}$$

in which ψ' is unique: For this it suffices to show that in a commutative diagram

$$\begin{tikzcd}
P \arrow[rr, "can"] \arrow[dr, "\psi"'] & & \tilde{P} \arrow[dl, "\mu"] \\
& F &
\end{tikzcd}$$

the morphism μ is unique (By applying it twice it follows that ψ' is unique). Because of the additivity it suffices to show this for $\psi = 0$. But if $\psi = 0$ and $(U_i \to U)$ is a covering in $\mathcal{T}$, then the commutative diagram

$$\begin{tikzcd}
P(U) \arrow[r] \arrow[d, "\psi_U"'] & \check{H}^0((U_i \to U), P) \arrow[r, hook] \arrow[d] \arrow[dl, "\mu"'] & \prod_i P(U_i) \arrow[r, "pr_i"] \arrow[d, "\psi"] & P(U_i) \arrow[d, "\psi_{U_i}=0"] \\
F(U) \arrow[r, "\sim"] & \check{H}^0((U_i \to U), F) \arrow[r, hook] & \prod_i F(U_i) \arrow[r, "pr_i"] & F(U_i)
\end{tikzcd}$$

implies that $\mu = 0$.

Lemma 3.11 Let

$$f, g : (U'_j \to U) \to (U_i \to U)$$

be two refinements of coverings in the Grothendieck topology $\mathcal{T}$. Then for every abelian presheaf P the induced maps

$$f^*, g^* : \check{H}^0((U_i \to U), P) \to \check{H}^0((U'_j \to U), P)$$

are equal.

Proof Let $f = (\varepsilon, (f_j))$ and $g = (\delta, (g_j))$. We have a diagram

$$\begin{tikzcd}
\prod_i P(U_i) \arrow[r, "d^0=\alpha_1-\alpha_2"] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & \prod_{i_1,i_2} P(U_{i_1} \times_U U_{i_2}) \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] \arrow[dl, "\Delta^1"'] \\
\prod_j P(U'_j) \arrow[r, "d^0=\alpha_1-\alpha_2"] & \prod_{j_1,j_2} P(U_{j_1} \times_U U_{j_2}) ,
\end{tikzcd}$$

where Δ^1 is defined by
$$(\Delta^1 s)_j = P((f_j, g_j)_U)(s_{\varepsilon(j),\delta(j)}) ,$$
with the canonical morphism

$$(f_j, g_j)_U : U'_j \to U_{\varepsilon(j)} \times_U U_{\delta(j)} .$$

One checks that
$$\Delta^1 \circ d^0 = g^* - f^* .$$
Thus f^* and g^* agree on $\ker d^0 = \check{H}^0((U_i \to U), P)$, which proves the claim.

With this result we are able to understand the limit

$$\check{H}^0(U, P) = \varinjlim_{\mathcal{T}(U)^0} \check{H}^0(\mathfrak{U}, P) .$$

better: For two coverings $\mathfrak{U}, \mathfrak{U}'$ in $\mathcal{T}(U)$ call $\mathfrak{U}'$ finer than $\mathfrak{U}$ (Notation $\mathfrak{U}' \geq \mathfrak{U}$), if there is a refinement $f : \mathfrak{U}' \to \mathfrak{U}$. Define the equivalence relation $\sim$ on the set $ob(\mathcal{T}(U))$ of the coverings of U by
$$\mathfrak{U} \sim \mathfrak{U}' \;\Leftrightarrow\; \mathfrak{U} \leq \mathfrak{U}' \text{ and } \mathfrak{U}' \leq \mathfrak{U} .$$
Then the set of the equivalent classes

$$\mathcal{T}(U)_0 = ob(\mathcal{T}(U))/\sim$$

becomes an ordered set, with the ordering induced by $\leq$. This ordering is inductive: For two coverings $\mathfrak{U} = (U_i \to U)_i$ and $\mathfrak{V} = (V_j \to U)_j$ there is a common refinement $\mathfrak{W} = (U_i \times_U U_j \to U)_{i,j}$ with the obvious refinements

$$\mathfrak{U} \leftarrow \mathfrak{W} \to \mathfrak{V} ,$$

given by the maps $i \leftarrow\!\shortmid (i, j) \mapsto j$ and the projections $U_i \leftarrow U_i \times_U U_j \to U_j$; hence we have $\mathfrak{U}, \mathfrak{V} \leq \mathfrak{W}$.

By Lemma 3.11, for $\mathfrak{U}' \geq \mathfrak{U}$ we further obtain, by choice of a refinement $f : \mathfrak{U}' \to \mathfrak{U}$, a uniquely determined homomorphism

$$\check{H}^0(\mathfrak{U}, P) \to \check{H}^0(\mathfrak{U}', P) . \tag{3.11.1}$$

Corollary 3.12 The zeroeth Čech Cohomology

$$\check{H}^0(U,P) = \varinjlim_{\mathcal{T}(U)_0} \check{H}^0(\mathfrak{U},P)\,,$$

is the inductive limit over the inductively ordered set $\mathcal{T}(U)_0$.

This implies the claims in the preliminary remark for the proof of Lemma 3.10.

Now we define the push-forward maps and pull-back maps for sheaves. Let

$$f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$$

be a morphism of sites.

Lemma 3.13 If F' is an abelian sheaf on $(\mathcal{X}', \mathcal{T}')$, then $f_P F'$ is again a sheaf.

Proof: Left to the readers!

Lemma/Definition 3.14 (a) $f_* F' := f_P F'$ is called the direct image (or push-forward) of F' (with respect to f).

(b) For an abelian sheaf F on $(\mathcal{X}, \mathcal{T})$, $f^*F := af^PF$ is called the (sheaf-theoretic) inverse image (or pull-back) of F (with respect to f).

(c) The functor

$$f^* = af^P : Sh(\mathcal{X}, \mathcal{T}) \to Sh(\mathcal{X}', \mathcal{T}')$$

is left adjoint to the functor

$$f_* : Sh(\mathcal{X}', \mathcal{T}') \to Sh(\mathcal{X}, \mathcal{T})\,.$$

Proof For sheaves F' on $(\mathcal{X}', \mathcal{T}')$ and F on $(\mathcal{X}, \mathcal{T})$ we have canonical isomorphisms

$$Hom_{Sh}(f^*F, F') \stackrel{\sim}{=} Hom_{Sh}(af^PF, F') \stackrel{\sim}{=} Hom_{Pr}(f^PF, iF')$$

$$\stackrel{\sim}{=} Hom_{Pr}(F, f_PiF') \stackrel{\sim}{=} Hom_{Sh}(F, f_*F')\,,$$

functorial in F and F'.

4 The abelian categories of sheaves and presheaves

Let $\mathcal{X}$ be a category.

Theorem 4.1 (a) The category $Pr(\mathcal{X})$ of abelian presheaves on $\mathcal{X}$ is an abelian category.

(b) A sequence of abelian presheaves

$$0 \to P' \to P \to P'' \to 0$$

is exact, if and only if for all $U \in ob(\mathcal{X})$ (:= objects of $\mathcal{X}$) the sequence

$$0 \to P'(U) \to P(U) \to P''(U) \to 0$$

is exact in $\underline{Ab}$.

Proof: Left to the readers!

Theorem 4.2 Let $(\mathcal{X}, \mathcal{T})$ be a site.

(a) The category $Sh(\mathcal{X}, \mathcal{T})$ of abelian sheaves on $\mathcal{X}$ with respect to $\mathcal{T}$ is an abelian category.

(b) The kernel of a morphism $\varphi : F_1 \to F_2$ of abelian sheaves is equal to the kernel of presheaves $\ker^P \varphi$ (i.e., $(\ker \varphi)(U) = \ker(\varphi_U : F_1(U) \to F_2(U))$ for all U in $\mathcal{X}$.

(c) The cokernel of a morphism $\varphi : F_1 \to F_2$ of abelian sheaves is equal to $a\, coker^P \varphi$, i.e., the sheaf associated to the presheaf cokernel $coker^P \varphi$ (defined by $(coker^P \varphi)(U) = coker(\varphi_U : F_1(U) \to F_2(U))$ for all U in $\mathcal{X}$).

(d) In particular, $\varphi : F_1 \to F_2$ is an epimorphism in $Sh(\mathcal{X}, \mathcal{T})$ if and only if for every U in $\mathcal{X}$ and every $s \in F_2(U)$ there is a covering $(U_i \to U)$ in $\mathcal{T}$ and there are sections $s_i \in F_1(U_i)$, mapped to $s|_{U_i}$ by φ.

Proof The properties (a) - (c) follow easily from 4.1 and the universal property of the associated sheaf. For (d) we note that φ is an epimorphism if and only if $coker \varphi = 0$, i.e., if $a(coker^P \varphi) = 0$. This means that there is a covering $(U_i \to U)$ for every U in $\mathcal{X}$ and every $\overline{s} \in (coker^P \varphi)(U)$ with $\overline{s}|_{U_i} = 0$ for all i. Since $(coker^P \varphi)(U_i) = coker(\varphi_{U_i} : F_1(U_i) \to F_2(U_i))$, the proposition follows.

Theorem 4.3 (a) There exist arbitrary limits (inverse limits) and colimits (direct limits) in $Pr(\mathcal{X})$ and $Sh(\mathcal{X}, \mathcal{T})$.

(b) The functor $i : Sh(\mathcal{X}, \mathcal{T}) \hookrightarrow Pr(\mathcal{X})$ is left exact.

(c) The functor $a : Pr(\mathcal{X}) \to Sh(\mathcal{X}, \mathcal{T})$ is exact.

Proof (a) In $Pr(\mathcal{X})$ we have

$$(\varprojlim^P P_i)(U) = \varprojlim_i P(U_i)\,,$$

and a similar formula for direct limits. If now $(F_i)_{i\in I}$ is a diagram of sheaves, then $\varprojlim\limits_i{}^P P_i$ is again a sheaf, because inverse limits commute with each other. Hence $\varprojlim\limits_i F_i = \varprojlim\limits_i{}^P F_i$. The direct limit is

$$\varinjlim_i F_i = a(\varinjlim_i{}^P F_i)\,,$$

since we have the universal property for every sheaf G

$$Hom_{Sh}(a\varinjlim{}^P F_i, G) \cong Hom_{Pr}(\varinjlim_i F_i, iG) \cong \varprojlim Hom_{Pr}(F_i, iG) \cong \varprojlim Hom_{Sh}(F_i, G)$$

(b) It follows from the adjunction of i and a that i is left exact and a is right exact (see Lemma 4.5 below).

(c) Since $aP = \tilde{\tilde{P}}$, it suffices to show that the functor $P \mapsto \tilde{P}$ is left exact. But we have

$$\tilde{P}(U) = \varinjlim_{\mathfrak{U}\in\mathfrak{T}(U)_0} \check{H}^0(\mathfrak{U}, P)\,,$$

the functor $P \mapsto \check{H}^0(\mathfrak{U}, P)$ is left exact, and forming an inductive limit of abelian groups is an exact functor (see also Annex 4.B).

Theorem 4.4 Let $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ be a morphism of sites.

(a) $f_P : Pr(\mathcal{X}') \to Pr(\mathcal{X})$ is exact and $f^P : Pr(\mathcal{X}) \to Pr(\mathcal{X}')$ is right exact.

(b) If finite limits exist in $\mathcal{X}$ and $\mathcal{X}'$, and if $f^0 : \mathcal{X} \to \mathcal{X}'$ commutes with these, then f^P is exact.

(c) $f_* : Sh(\mathcal{X}', \mathcal{T}') \to Sh(\mathcal{X}, \mathcal{T})$ is left exact and $f^* : Sh(\mathcal{X}, \mathcal{T}) \to (\mathcal{X}', \mathcal{T}')$ is right exact.

(d) If finite limits exist in $\mathcal{X}$ and $\mathcal{X}'$, and if $f^0 : \mathcal{X} \to \mathcal{X}'$ commutes with these, then f^* is exact.

Proof (a) The exactness of f_P follows from 4.1, and because of the adjunction, f^P is right exact (see below).

(b) It follows from the assumption that the category $I_{U'}$ (see Proof of 3.2) is cofiltered for every object U' in $\mathcal{X}'$ (see Annex 4.B.2). In fact, in $I_{U'}$ finite limits exist by assumption: For this one has to show the existence of finite products and difference kernels. But for objects $U' \to f^0(U_1)$ and $U' \to f^0(U_2)$ in $I_{U'}$ the product is the product morphism $U' \to f^0(U_1) \times f^0(U_2) = f^0(U_1 \times U_2)$, and for morphisms

$$\begin{tikzcd}
 & f^0(U_1) \arrow[dd, shift right, "f^0(\alpha)"'] \arrow[dd, shift left, "f^0(\beta)"] \\
U' \arrow[ur] \arrow[dr] & \\
 & f^0(U_1)
\end{tikzcd}$$

(with $U_1 \overset{\beta}{\underset{\alpha}{\rightrightarrows}} U_2$) the difference kernel is

$$\begin{tikzcd}
 & f^0(\ker(\alpha,\beta)) = \ker(f^0(\alpha), f^0(\beta)) \arrow[d] \\
U' \arrow[ur] \arrow[r] & f^0(U_1)
\end{tikzcd}$$

(check the universal properties!). If I'_U is cofiltered, $I^{op}_{U'}$ is filtered. Therefore forming the direct limit over $I^{op}_{U'}$ is exact (see Appendix 4.B.3).

(c) Since $f_*F' = f_P iF'$, the claim follows for f_*, because i is left exact (4.3(b)) and f_P is exact by 4.4 (a). Furthermore f^* is right exact, because f^* is left adjoint to f_* (see 4.5).

(d) By (b), f^P is exact, therefore $f^* = af^Pi$ is left exact, because a is exact and i is left exact.

Lemma 4.5 Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ be functors such that G is right adjoint to F ($\Leftrightarrow$ F is left adjoint to G). Then G is left exact and F right exact.

Proof By assumption we have bi-functorial isomorphisms

$$Hom_{\mathcal{A}}(A, GB) \cong Hom_{\mathcal{B}}(FA, B)$$

for $A \in ob(\mathcal{A})$ and $B \in ob(\mathcal{B})$.

(a) Let

$$0 \to B_1 \to B_2 \to B_3 \to 0 \tag{4.5.1}$$

be exact in $\mathcal{B}$. We have to show that

$$0 \to GB_1 \to GB_2 \to GB_3 \tag{4.5.2}$$

is exact. This means that the sequence

$$0 \to Hom_{\mathcal{A}}(A, GB_1) \to Hom_{\mathcal{A}}(A, GB_2) \to Hom_{\mathcal{A}}(A, GB_3) \tag{4.5.3}$$

is exact for all $A \in ob(\mathcal{A})$ ($\Leftrightarrow$ GB_1 is the kernel of $GB_2 \to GB_3$). By adjunction, (4.5.3) is isomorphic to the sequence

$$0 \to Hom_{\mathcal{B}}(FA, B_1) \to Hom_{\mathcal{B}}(FA, B_2) \to Hom_{\mathcal{B}}(FA, B_3)\,. \tag{4.5.4}$$

This sequence is exact by exactness of (4.5.1).

(b) The right exactness of F is shown in a similar (dual) way.

4.A Representable functors, limits, and colimits

Let $\mathcal{C}$ be a category.

Definition 4.A.1 (a) A contravariant functor

$$F \; : \; \mathcal{C} \; \rightarrow \; \underline{Sets}$$

is called **representable**, if there is an object X in $\mathcal{C}$ such that F is isomorphic to the contravariant Hom-functor

$$\begin{array}{rcrcl} h_X = Hom_{\mathcal{C}}(-,X) & : & \mathcal{C} & \rightarrow & \underline{Sets} \\ & & A & \mapsto & Hom_{\mathcal{C}}(A,X)\,, \end{array}$$

i.e., if there is a bijection, functorial in A,

$$F(A) \; = \; Hom_{\mathcal{C}}(A,X)\,.$$

(b) A covariant functor

$$G \; : \; \mathcal{C} \; \rightarrow \; \underline{Sets}$$

is called **representable**, if it is isomorphic to the covariant Hom-functor

$$\begin{array}{rcrcl} h^X = Hom_{\mathcal{C}}(X,-) & : & \mathcal{C} & \rightarrow & \underline{Sets} \\ & & A & \mapsto & Hom_{\mathcal{C}}(X,A) \end{array}$$

for an object X in $\mathcal{C}$, i.e., if there is a bijection, functorial in A,

$$G(A) \; = \; Hom_{\mathcal{C}}(X,A)\,.$$

By definition h_X and h^X are representable. In the situation 4.A.1(a) (resp. (b)) X is called **representing object** for F (resp. G). The object X – if it exists – is in each case unique up to canonical isomorphism: This follows from the famous

Lemma 4.A.2 (Yoneda-Lemma) (a) If

$$F \; : \; \mathcal{C} \; \rightarrow \; \underline{Sets}$$

is a contravariant functor, then one has a canonical bijection for every object X in $\mathcal{C}$

$$\begin{array}{rcrcl} e_X & : & Hom(h_X,F) & \xrightarrow{\sim} & F(X) \\ & & \varphi & \mapsto & \varphi_X(id_X)\,. \end{array}$$

(b) If

$$\mathcal{G} \; : \; \mathcal{C} \; \rightarrow \; \underline{Sets}$$

is a covariant functor, then for every object Y in $\mathcal{C}$ one has a canonical bijection

$$\begin{array}{rcrcl} e_Y & : & Hom(h^Y,G) & \xrightarrow{\sim} & G(Y) \\ & & \varphi & \mapsto & \varphi_Y(id_Y)\,. \end{array}$$

Proof (a): The inverse m_X of e_X assigns the following morphism $m_X(a) := \varphi^a$ of functors to an object $a \in F(X)$

$$\begin{array}{rcrcl} \varphi^a_A & : & h_X(A) = Hom_{\mathcal{C}}(A,X) & \to & F(A) \\ & & f & \mapsto & \varphi^a_A(f) := F(f)(a)\,. \end{array}$$

Note: For $f : A \to X$ we obtain $F(f) : F(X) \to F(A)$, because F is contravariant. Note that φ^a is indeed a morphism of functors: For a morphism $g : A \to A'$ in $\mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd} Hom_{\mathcal{C}}(A',X) \arrow[r, "\varphi^a_{A'}"] \arrow[d, "g^*"'] & F(A') \arrow[d, "F(g)"] \\ Hom_{\mathcal{C}}(A,X) \arrow[r, "\varphi^a_A"] & F(A) \end{tikzcd}$$

$$\begin{tikzcd} f' \arrow[r, mapsto] \arrow[dd, mapsto] & F(f')(a) \arrow[d, mapsto] \\ & F(g)(F(f')(a)) \arrow[d, equal] \\ f'g \arrow[r, mapsto] & F(f'g)(a)\,, \end{tikzcd}$$

since $F(f'g) = F(g) \circ F(f')$.

We have $e_X m_X = id$: For $a \in F(X)$ we have $e_X(\varphi^a) = \varphi^a_X(id_X) = a$, because $F(id_X) = id_{F(X)}$. Conversely we have $m_X e_X = id$: Let $\varphi : h_X \to F$ be given and let $e_X(\varphi) = \varphi_X(id_X) \in F(X)$, and let $\varphi^{e_X(\varphi)} : h_X \to F$ be constructed as above. For every $A \in ob(\mathcal{C})$ the maps

$$\varphi^{e_X(\varphi)}_A = \varphi_A \quad Hom_{\mathcal{C}}(A,X) \to F(A)$$

are equal, because

$$\varphi^{e_X(\varphi)}_A(f) = F(f)(\varphi_X(id_X)) = \varphi_A(f)\,,$$

since the diagram

$$\begin{tikzcd} id_X \in Hom_{\mathbb{C}}(X,X) \arrow[r, "\varphi_X"] \arrow[d, "f^*"] & F(X) \arrow[d, "F(f)"] \\ f \in Hom_{\mathcal{C}}(A,X) \arrow[r, "\varphi_A"] & F(A) \end{tikzcd}$$

commutes (φ is a morphism of functors).

The proof of (b) is analogous.

By applying 4.A.2 to $F = h_Y$ resp. $G = h^X$, we get the following:

Corollary 4.A.3 For objects X, Y in $\mathcal{C}$ one has canonical bijections

$$Hom_{\mathcal{C}}(X,Y) \ \tilde{\to}\ Hom(h_X, h_Y)$$

$$Hom_{\mathcal{C}}(X,Y) \ \tilde{\to}\ Hom(h^Y, h^X)$$

Hence the representing objects are unique up to canonical isomorphism: If

$$h_X \cong F \cong h_Y\,,$$

then by 4.A.2 we get a unique isomorphism $X \cong Y$; the same holds for the covariant functors. From 4.A.3 we get

Corollary 4.A.4 (Yoneda-embedding) The functor

$$\begin{array}{ccl} \mathcal{C} & \to & \mathcal{C}^{\sim} := (\text{contravariant functors } F : \mathcal{C} \to sets) \\ X & \mapsto & h_X \end{array}$$

is fully faithful and gives an embedding of $\mathcal{C}$ into $\mathcal{C}^{\sim}$. The essential image is the full subcategory of the representable functors.

Now we define fiber products and fiber sums.

Definition 4.A.5 Let

$$\begin{tikzcd} & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & S \end{tikzcd}$$

be morphisms in $\mathcal{C}$. The fiber product of α and β – or of X and Y over S, notation $X \times_S Y$, is characterized by the following properties:

(a) There is a commutative diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & S \end{tikzcd}$$

(pr_1 resp. pr_2 are called the first resp. second projection).

(b) If

$$\begin{tikzcd} W \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{\beta}"'] & Y \arrow[d, "\beta"] \\ X \arrow[r, "\alpha"] & S \end{tikzcd}$$

is another commutative diagram, then there is a unique morphism $\gamma : W \to X \times_S Y$ with $pr_1\gamma = \tilde{\alpha}$ and $pr_2\gamma = \tilde{\beta}$, i.e., such that the diagram

$$\begin{tikzcd} W \arrow[dr, dashed, "\exists!\gamma"] \arrow[drr, bend left, "\tilde{\alpha}"] \arrow[ddr, bend right, "\tilde{\beta}"'] & & \\ & X \times_S Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"] & S \end{tikzcd}$$

commutes.

Remark 4.A.6 (a) Fiber products do not always exist; but if they exist, they are unique up to canonical isomorphism (Exercise!).

(b) In the category *Sets* of sets fiber products exist: For maps $M \xrightarrow{\alpha} T \xleftarrow{\beta} N$ of sets one has

$$M \times_T N = \{(m,n) \in M \times N \mid \alpha(m) = \beta(n)\}.$$

(c) The universal property of a fiber product $X \times_S Y$ in a category $\mathcal{C}$ is equivalent to the property that for all other objects Z in $\mathcal{C}$ the map

$$\begin{array}{rcl} Hom_{\mathcal{C}}(Z, X \times_S Y) & \to & Hom_{\mathcal{C}}(Z,X) \times_{Hom_{\mathcal{C}}(Z,S)} Hom_{\mathcal{C}}(Z,Y) \\ \gamma & \mapsto & (pr_1\gamma, pr_2\gamma) \end{array}$$

is bijective. Here the fiber product on the right is taken for the maps

$$\begin{tikzcd}
 & Hom_{\mathcal{C}}(Z,Y) \arrow[d, "\beta_*"] & g \arrow[d, mapsto] \\
Hom_{\mathcal{C}}(Z,X) \arrow[r, "\alpha_*"] & Hom_{\mathcal{C}}(Z,S) & \beta g \\
f \arrow[r, mapsto] & \alpha f &
\end{tikzcd}$$

(d) A slightly different interpretation is given as follows: Let $\mathcal{C}/S$ be the category of **objects in $\mathcal{C}$ over** S: objects in $\mathcal{C}/S$ are objects X in $\mathcal{C}$ together with a morphism $\alpha : X \to S$; one can regard α itself as objects, because by α, X is already given. A morphism from $\alpha : X \to S$ to $\alpha' : X' \to S$ is a morphism $f : X \to X'$, for which the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & X' \arrow[dl, "\alpha'"] \\
 & S &
\end{tikzcd}$$

commutes.

Then a fiber product $X \times_S Y$ is the same as a product of $X \to S$ and $Y \to S$ in $\mathcal{C}/S$, because the universal properties correspond.

Lemma 4.A.7 The properties of Lemma 1.14 (commutativity, associativity, transitivity and functoriality) are valid for fiber products in any category $\mathcal{C}$ (if they exist).

Proof For example we show functoriality. We have a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
X \arrow[dd, "f"'] \arrow[dr, "\alpha"] & & Y \arrow[dl, "\beta"'] \arrow[dd, "g"] \\
 & S & \\
X' \arrow[ur, "\alpha'"'] & & Y' \arrow[ul, "\beta'"]
\end{tikzcd},$$

i.e., X, X', Y and Y' are objects over S (by α, α', β and β') and f and g are morphisms of objects over S (commutativity of triangles). If the fiber products exist, we obtain a diagram

$$\begin{tikzcd}
X \times_S Y \arrow[rr, "pr_2"] \arrow[dr, dashed, "\exists! h"] \arrow[dd, "pr_1"'] & & Y \arrow[d, "g"] \\
& X' \times_S Y' \arrow[r, "pr_2'"] \arrow[d, "pr_1'"'] & Y' \arrow[d, "\beta'"] \\
X \arrow[r, "f"] & X' \arrow[r, "\alpha'"] & S,
\end{tikzcd}$$

where the internal and external square are commutative (Note that $\alpha' f = \alpha$ and $\beta' g = \beta$). By the universal property of $X' \times_S Y'$ there exists a unique morphism $h : X \times_S Y \to X' \times_S Y'$ which makes the entire diagram commutative; we call this $f \times g$, and it fulfills the claim of Lemma 1.14 (d).

Remark 4.A.7 By reversing all arrows one obtains the notion of a fiber sum $X \coprod_S Y$ for a diagram

$$\begin{tikzcd}
S \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Y \\
X
\end{tikzcd}$$

where one has the dual universal properties and functorial properties.

Example 4.A.8 For every diagram of ring homomorphisms

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \ , \\
A
\end{tikzcd}$$

the fiber sum in the category of rings exists and is given by the tensor product: One has a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d] \\
A \arrow[r] & A \otimes_R B
\end{tikzcd}$$

and for every diagram of rings

$$\begin{tikzcd}
R \arrow[r, "\beta"] \arrow[d, "\alpha"'] & B \arrow[d] \arrow[ddr, bend left, "g"] \\
A \arrow[r] \arrow[drr, bend right, "f"'] & A \otimes_R B \arrow[dr, dashed, "\exists! h"] \\
& & C
\end{tikzcd}$$

with $f\alpha = g\beta$ there exists a uniquely determined ring homomorphism h as indicated above that makes the entire diagram commutative.

Now we get to the general theory of limits and colimits.

Definition 4.A.9 A category I is called small (or a diagram category), if the objects form a set.

Examples 4.A.10 Often, those small categories are "really small" in a sense that one can write all objects and morphisms in them.

(a) The **discrete category** $\underline{I}$ over a set I has the elements of I as objects and only the identities as morphisms.

(b) Let

$$\begin{tikzcd}
\bullet \arrow[dr] & \\
& \bullet \\
\bullet \arrow[ur] &
\end{tikzcd}$$

be the category with three objects (marked by points) and apart from for the identities only has the two indicated arrows (then all compositions are obvious!).

(c) For every group G one has the small category $\underline{G}$ with one object $*$ and all elements $\sigma \in G$ as morphisms, where the composition is given by the group law.

(d) For every ordered set $(I, \leq)$ one has the category with objects $i \in I$ and exactly one morphism $i \to j$, if $i \leq j$.

Definition 4.A.11 Let I be a small category, and let $\mathcal{C}$ be any category. A diagram in $\mathcal{C}$ over I (or a I-object in $\mathcal{C}$) is a (covariant) functor $X : I \to \mathcal{C}$.

The I-objects in $\mathcal{C}$ form a category

$$\mathcal{C}^I \,,$$

where the morphisms are the morphisms of functors. Often, one describes the objects of I with small letters i, j,... and writes X_i for $X(i)$.

Example 4.A.12 Let $\mathcal{C}$ be a category.

(a) For the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ from 4.A.6 (b) a corresponding diagram in $\mathcal{C}$ is given by a diagram

$$X \xrightarrow{g} Y \xleftarrow{\beta} Z$$

with morphisms α and β in $\mathcal{C}$. Morphisms of such diagrams are commutative diagrams

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d] & Z \arrow[l, "\beta"'] \arrow[d] \\
X' \arrow[r, "\alpha'"] & Y' & Z' \arrow[l, "\beta'"'] .
\end{tikzcd}$$

(b) Consider the small category $\bullet \longrightarrow \bullet$ (2 objects, apart from the identities only one arrow). Diagrams over this category in $\mathcal{C}$ are simply morphisms

$$A \xrightarrow{f} B$$

in $\mathcal{C}$, where morphisms of these are commutative diagrams

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

This is called the **category of arrows in** $\mathcal{C}$, notation $Ar(\mathcal{C})$.

(c) If $(I, \leq)$ is an inductively ordered set, regarded as a category by 4.A.6 (d), then a covariant functor

$$X \;:\; (I, \leq) \;\to\; \mathcal{C}$$

is the same as an inductive system over I in $\mathcal{C}$. A contravariant functor

$$X' \;:\; (I, \leq) \;\to\; \mathcal{C}$$

is the same as a projective system in $\mathcal{C}$ over I.

Now let I be a small category and $\mathcal{C}$ any category.

Definition 4.A.13 For a object $A \in \mathcal{C}$ define the **constant I-object** $\underline{A}$ as the functor

$$\begin{array}{rrcl} \underline{A} \;: & I & \to & \mathcal{C} \\ & i & \mapsto & A \\ & i \to j & \mapsto & id_A \,. \end{array}$$

Example 4.A.14 In Example 4.A.12 (a) $I = \bullet \longrightarrow \bullet \longleftarrow \bullet$ the constant object A is

$$A \xrightarrow{id_A} A \xleftarrow{id_A} A \,.$$

Definition 4.A.15 (a) One says that the limit (or inverse limit) of an I-object $(A_i)_{i\in I}$ exists in $\mathcal{C}^I$, if the contravariant functor

$$\begin{array}{rcl} \mathcal{C} & \to & \underline{Sets} \\ X & \mapsto & Hom_{\mathcal{C}^I}(\underline{X}, (A_i)_{i\in I}) \end{array}$$

is representable. The representing object is called the limit of $(A_i)_{i\in I}$, notation

$$\lim\,(A_i)_{i\in I} \quad \text{oder} \quad \lim_{i\in I}\, A_i \,.$$

(b) One says that the colimit (or direct limit) of $(A_i)_{i\in I}$ exists, if the covariant functor

$$\begin{array}{rcl} \mathcal{C} & \to & \underline{Sets} \\ X & \mapsto & Hom_{\mathcal{C}^I}((A_i)_{i\in I}, \underline{X}) \end{array}$$

is representable. The representing object is called the colimit of $(A_i)_{i\in I}$, notation

$$\operatorname{colim}\,(A_i)_{i\in I} \;=\; \operatorname*{colim}_{i\in I}\, A_i \,.$$

Now we make this elegant definition more explicit.

Remark 4.A.16 (explicit description) (a) An element of $Hom_{\mathcal{C}^I}(\underline{X}, (A_i)_{i\in I})$ is obviously given by the following:
(i) For every $i \in I$ one has a morphism in $\mathcal{C}$

$$\varphi_i \;:\; X \;\to\; A_i\,.$$

(ii) For every morphism $i \to j$ in I the diagram

$$\begin{tikzcd} & A_i \arrow[dd] \\ X \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\ & A_j \end{tikzcd}$$

commutes, where the vertical morphism belongs to $i \to j$ (one has a functor $a : I \to \mathcal{C}$, we write $a(i) = A_i$, and then the morphism on the right is $a(i \to j)$).
(b) If $\lim\,(A_i)_{i\in I}$ exists (other notation $\lim\limits_{I} a$), and if we fix isomorphisms functorial in X

$$(4.A.16.1) \qquad \alpha : Hom_{\mathcal{C}}(X, \lim_{I}\,(A_i)) \xrightarrow{\sim} Hom_{\mathcal{C}^I}(\underline{X}, (A_i))\,,$$

then for $X = \lim\,(A_i)_{i\in I}$, the image of $id_{\lim(A_i)}$ gives an element $\varphi^{univ} \in Hom_{\mathcal{C}^I}(\underline{\lim(A_i)}, (A_i))$, by (a), therefore morphisms $p_i : \lim(A_i) \to A_i$ for all $i \in I$ and commutative triangles

$$(4.A.16.2) \qquad \begin{tikzcd} & A_i \arrow[dd] \\ \lim(A_i)_{i\in I} \arrow[ur, "p_i"] \arrow[dr, "p_j"'] & \\ & A_j \end{tikzcd}$$

for every morphism $i \to j$ in I. The morphism p_i is called i-th projection. If we have an element $\alpha \in Hom_{\mathcal{C}^I}(\underline{X}, (A_i))$, i.e. morphisms $\varphi_i : X \to A_i$ for all $i \in I$ and commutative diagrams

$$(4.A.16.3) \qquad \begin{tikzcd} & A_i \arrow[dd] \\ X \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\ & A_j \end{tikzcd}$$

for all morphisms $i \to j$, then there exists a uniquely determined morphism

$$\varphi \;:\; X \;\to\; \lim\,(A_i)$$

(the preimage of α under (4.A.16.1)) with

$$\varphi_i \;=\; p_i\varphi$$

for all i (this follows from the choice of φ^{univ} and the functoriality of (4.A.16.1)).

(c) For $\operatorname{colim}(A_i)$ one obtains an analogous conclusion, by reversing all arrows.

Example 4.A.17 (a) If I is a set and $\underline{I}$ the discrete category associated to I (4.A.10 (a)), then an $\underline{I}$-object in $\mathcal{C}$ is simply given by a family $(A_i)_{i\in I}$ in $\mathcal{C}$ (there are no morphisms between $i \neq j$), and one has

$$\lim_{\underline{I}}(A_i) \;=\; \prod_{i\in I} A_i\,,$$

the product of the A_i if this exists in $\mathcal{C}$, because the universal properties of $\lim\limits_{\underline{I}}(A_i)$ (4.A.16 (b)) and $\prod\limits_{i\in I} A_i$ are identical. Similarly,

$$\operatorname*{colim}_{\underline{I}}(A_i) \;=\; \coprod_{i\in I} A_i\,,$$

is the sum or the co-product of the A_i, if this exist in $\mathcal{C}$.

(b) Similarly one can show: If $(I,\leq)$ is an filtered ordered set (regarded as category), then for every I-object in $\mathcal{C}$, hence every inductive system $(A_i)_{i\in I}$ over I in $\mathcal{C}$,

$$\operatorname{colim}(A_i)_{i\in I} \;=\; \varinjlim_{i\in I} A_i$$

is the inductive limit of the system, if it exists. Dually, for every I°-object $(A_i)_{i\in I}$ in $\mathcal{C}$ (where I° notes the dual category to I), hence every projective system over I in $\mathcal{C}$, one sees that

$$\lim\,(A_i)_{i\in I^\circ} \;=\; \varprojlim_{i\in I} A_i$$

is the projective limit of the system, if it exists.

(c) Consider the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ of 4.A.10 (b). For a corresponding diagram

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{C}$, it follows from the universal properties that the limit is the fiber product

$$X \times_Y Z$$

in $\mathcal{C}$ – if it exists.

We now look at a special, but very important example. Consider the following small category

$$\bullet \rightrightarrows \bullet$$

(two objects, and apart from both identities only the two indicated arrows). A diagram in $\mathcal{C}$ is

$$A \underset{\beta}{\overset{\alpha}{\rightrightarrows}} B\,.$$

Definition 4.A.18 (a) If it exists, the limit of the above diagram is called the difference kernel of α and β, notation

$$\ker(\alpha,\beta)\,.$$

(b) If it exists, the colimit of this diagram is called the difference kernel α and β, notation

$$\operatorname{coker}(\alpha,\beta)\,.$$

Now we describe the universal properties:

Lemma 4.A.19 (a) One has a morphism

$$\ker(\alpha,\beta) \xrightarrow{i} A$$

with $\alpha i = \beta i$. If

$$X \xrightarrow{\gamma} A$$

is another morphism with $\alpha\gamma = \beta\gamma$, then there is a unique morphism $\gamma' : X \to \ker(\alpha,\beta)$ which makes the diagram

$$\begin{tikzcd} \ker(\alpha,\beta) \arrow[rr, "i"] & & A \\ & X \arrow[ul, dashed, "\exists\gamma'"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

commutative.

(b) One has a morphism $B \xrightarrow{\pi} \operatorname{coker}(\alpha,\beta)$ with $\pi\alpha = \pi\beta$. If

$$B \xrightarrow{\rho} X$$

is another morphism with $\rho\alpha = \rho\beta$, then there is a unique morphism $\rho' : \operatorname{coker}(\alpha,\beta) \to X$, which makes the diagram

$$\begin{tikzcd} B \arrow[rr] \arrow[dr, "\rho"'] & & \operatorname{coker}(\alpha,\beta) \arrow[dl, dashed, "\rho'"] \\ & X & \end{tikzcd}$$

commutative.

Proof: This follows immediately from the explicit description in 4.A.16.

Differential kernels and cokernels are non-additive analogues of kernel and cokernel in additive categories. Like there we have:

Lemma 4.A.20 Let $\alpha, \beta : A \to B$ be morphisms.

(a) If $\ker(\alpha,\beta)$ exists, then
$$i : \ker(\alpha,\beta) \to A$$
is a monomorphism.

(b) If $\operatorname{coker}(\alpha,\beta)$ exists, then
$$\pi \; : \; B \;\to\; \operatorname{coker}(\alpha,\beta)$$
is an epimorphism.

Proof (a): Let $f, g : Z \to \ker(\alpha,\beta)$ be two morphisms with $if = ig$. Since $\alpha i = \beta i$ we have
$$\alpha i f \;=\; \beta i f$$
as well. By the universal property of the difference kernel (4.A.19 (a)) there is a unique morphism $h : Z \to \ker(\alpha,\beta)$ with
$$ih \;=\; if\,.$$
Since $if = ig$ by assumption, we deduce that $h = f = g$, and in particular $f = g$.

The proof of (b) is dual.

Because of the following result, difference kernels and cokernels play a special role for limits and colimits.

Theorem 4.A.21 (a) In $\mathcal{C}$, there exist arbitrary (resp. arbitrary finite) limits if and only if all difference kernels exist and all (resp. all finite) products exist in $\mathcal{C}$.

(b) In $\mathcal{C}$ there exist all (resp. all finite) colimits if and only if all difference cokernels and all (resp. all finite) sums exist in $\mathcal{C}$.

Here one speaks of finite limits or colimits, if the underlying index category I is finite, i.e., if it has finitely many objects and only finite sets of morphisms.

Proof We only show (a), then (b) follows by passing to the dual category, where colimits turn to limits, sums to products and difference cokernels to difference kernels.

Let I be a small (resp. finite) category. Let $ob(I)$ be a set of all objects in I and let $mor(I)$ be the set of all morphisms in I. For a morphism $f : A \to B$ in $\mathcal{C}$ let $s(f) := A$ be the source and $t(f) := B$ the target of f. For a I-diagram $a : I \to \mathcal{C}$ consider the morphisms

$$(4.A.21.1) \qquad \prod_{i\in ob(I)} a(i) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{f\in mor(I)} a(t(f))$$

defined as follows (by assumption the considered products exist): The "f-component" of α (according to the universal property of the product) is the morphism
$$\prod_{i\in ob(i)} a(i) \overset{pr_{t(f)}}{\longrightarrow} a(t(f))\,,$$
the f-component of β is the morphism
$$\prod_{i\in ob(i)} a(i) \overset{pr_{s(f)}}{\longrightarrow} a(s(f)) \overset{a(f)}{\longrightarrow} a(t(f))$$

Claim: $\ker(\alpha,\beta) \;=\; \lim\limits_{i\in I} a(i)$.

Proof: By the universal property of $\ker(\alpha,\beta)$, a morphism $Z \to \ker(\alpha,\beta)$ corresponds to a morphism

$$\varphi \;:\; Z \;\to\; \prod_{i\in ob(I)} a(i)$$

with $\alpha\varphi = \beta\varphi$. By the universal property of the product, φ is given by giving morphisms

$$\varphi_i \;:\; Z \;\to\; a(i)$$

for all $i \in I$, and $\alpha\varphi = \beta\varphi$ means that for every $f : i \to j$ in I the diagram

(4.A.21.2)

$$\begin{tikzcd} & a(i) \arrow[dd, "a(f)"] \\ Z \arrow[ur, "\varphi_i"] \arrow[dr, "\varphi_j"'] & \\ & a(j) \end{tikzcd}$$

commutes. This just gives the universal property of $\lim\limits_{i\in I} a(i)$ (see 4.A.16) for $Z = \ker(\alpha,\beta)$, because the argument above shows that all diagrams (4.A.21.2) factorize through the diagram

$$\begin{tikzcd} & a(i) \arrow[dd, "a(f)"] \\ \ker(\alpha,\beta) \arrow[ur] \arrow[dr] & \\ & a(j)\,. \end{tikzcd}$$

Theorem 4.A.22 Let $\mathcal{C}$ be category with finite products. The following properties are equivalent:

(a) $\mathcal{C}$ possesses fiber products.

(b) $\mathcal{C}$ possesses difference kernels.

Proof (a) $\Rightarrow$ (b): For $A \underset{g}{\overset{f}{\rightrightarrows}} B$ consider the fiber product diagram

$$\begin{tikzcd} K \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "(id,g)"] \\ A \arrow[r, "(id,f)"] & A \times B \end{tikzcd}$$

Then $K \overset{p_2}{\to} A$ is the difference kernel of f and g. In fact, consider $h : C \to A$ with $fh = gh$. If one has

$$(id,f)h = (h,fh) = (h,gh) = (id,g)h\,,$$

then there exists a uniquely determined morphism

$$\tilde{h} : C \to K \quad \text{ with } \quad p_1 \tilde{h} = h = p_2 \tilde{h} \,.$$

Furthermore, if $q_1 : A \times B \to A$ is the first projection, then $p_1 = q_1(id, f)p_1 = q_1(id, g)p_2 = p_2$. Hence we obtain a *uniquely* determined morphism $K \to A$ with the desired property.

(b) $\Rightarrow$ (a): Consider a diagram

(1)
$$\begin{tikzcd}
A \arrow[dr, "f"] & \\
& C \\
B \arrow[ur, "g"'] &
\end{tikzcd} .$$

In the associated diagram

(2)
$$\begin{tikzcd}
& A \arrow[drr, "f"] \arrow[dl, phantom, "1"] \arrow[dr, phantom, "2"] & & \\
K \arrow[ur, "q_A"] \arrow[r, "q"] \arrow[dr, "q_B"'] & A \times B \arrow[u, "p_A"'] \arrow[d, "p_B"] \arrow[rr, shift left, "fp_A"] \arrow[rr, shift right, "gp_B"'] & & C \\
& B \arrow[urr, "g"'] \arrow[ul, phantom, "3"] \arrow[ur, phantom, "4"] & &
\end{tikzcd}$$

let K be the difference kernel of fp_A and gp_B, and let $q_A = p_A q$ and $q_B = q_B q$. Then all triangles 1 to 4 are commutative. We claim that K forms a fiber product of diagram (1) via q_A and q_B.

Since $fp_A q = gp_B q$ (by assumption!), we have $fq_A = fp_A q = gp_B q = gq_B$. If one has further morphisms $h : D \to A$ and $k : D \to B$ with $fh = gk$, then for the morphism $(h, k) : D \to A \times B$ one has

$$fp_A(h, k) = fh = gk = gp_B(h, k)$$

by definition. Hence, by the definition of K, there is a unique morphism

$$\ell : D \to K$$

with $q\ell = (h, k)$. Then we have

$$q_A \ell = p_A q \ell = h \text{ and } q_B \ell = p_B q \ell = k$$

by definition. Furthermore the diagram

$$\begin{tikzcd}
& & A \arrow[drr, "f"] & & \\
D \arrow[urr, bend left, "h"] \arrow[r, "\ell"] \arrow[drr, bend right, "k"'] & K \arrow[ur, "q_A"] \arrow[r, "q"] \arrow[dr, "q_B"'] & A \times B \arrow[u] \arrow[d] & & C \\
& & B \arrow[urr, "g"'] & &
\end{tikzcd}$$

is commutative, and the uniqueness of ℓ follows, because q is a monomorphism (universal property of the difference kernel).

4.B Filtered categories

The following definition generalizes the notion of an inductively ordered set and an inductive limit. The dual terms are treated accordingly.

Definition 4.B.1 (a) A category I is called pseudo-filtered, if the following holds

(1) Every diagram of the form

$$\begin{tikzcd} & j \\ i \arrow[ur] \arrow[dr] & \\ & j' \end{tikzcd}$$

can be extended to a commutative diagram

$$\begin{tikzcd} & j \arrow[dr] & \\ i \arrow[ur] \arrow[dr] & & k \\ & j' \arrow[ur] & \end{tikzcd} .$$

(2) Every diagram of the form

$$i \underset{\alpha}{\overset{\beta}{\rightrightarrows}} j$$

can be extended to a commutative diagram

$$i \underset{\alpha}{\overset{\beta}{\rightrightarrows}} j \xrightarrow{\gamma} k$$

(i.e., such that $\gamma\alpha = \gamma\beta$).

(b) I is called connected, if, for any objects i and j in I, there is a finite chain of morphisms

$$i \to i_1 \leftarrow j_1 \to i_2 \leftarrow \ldots \leftarrow j\,.$$

(c) I is called filtered, if I is pseudo-filtered and connected.

(d) I is called (pseudo-)cofiltered, if I^{op} is (pseudo-)filtered.

Examples 4.B.2 (a) I is filtered, if I has finite colimits: In (1) one can take the fiber sum of j and j' over i, in (2) the difference cokernel, and in (b) the sum $i \to i\coprod j \leftarrow j$. Accordingly I is cofiltered, if I has finite limits.

(b) Let M be an ordered set, considered as category with $\leq$ as morphisms. Then M is filtered if and only if M is ordered inductively (every morphism $x \leq y$ is unique).

Theorem 4.B.3 Let I be a filtered category and $f : I \to \mathcal{C}$ a covariant functor, where $\mathcal{C} = \underline{Set}$ or $\mathcal{C} = \underline{Ab}$. Then the direct limit

$$\varinjlim_{i \in I} f(i)$$

exists in $\mathcal{C}$, and the formation of the direct limit is exact (exchanges with finite limits and colimits).

Proof Explicitly, we have

$$\varinjlim_{i \in I} f(i) = (\coprod_{i \in I} f(i))/\sim\,,$$

where for $x \in f(i)$ and $y \in f(j)$ the following holds.

$$\begin{array}{rcl} x \sim y & \Leftrightarrow & i \to k, j \to k, \text{ exist, then } x \text{ and } y \\ & & \text{have the same image under } f(i) \to f(k) \text{ and } f(j) \to f(k). \end{array}$$

The exactness follows easily from this.

5 Cohomology on sites

Let $(\mathcal{X}, \mathcal{T})$ be a site.

Proposition 5.1 The abelian categories $Pr(\mathcal{X})$ and $Sh(\mathcal{X}, \mathcal{T})$ have enough injectives, i.e., every object P in $Pr(\mathcal{X})$ has a monomorphism $P \hookrightarrow I$ into an injective presheaf I, and the analogous fact holds for $Sh(\mathcal{X}, \mathcal{T})$.

For this we use a method of Grothendieck. Let $\mathcal{A}$ be an abelian category.

Lemma/Definition 5.2 A family $(E_i)_{i \in I}$ of objects of $\mathcal{A}$ is called a **family of generators**, if the following equivalent conditions hold:

(a) The functor

$$\begin{array}{rcl} \mathcal{A} & \to & \underline{Ab} \\ A & \mapsto & \prod\limits_{i \in I} Hom_{\mathcal{A}}(E_i, A) \end{array}$$

is faithful, i.e., for all objects A, A' in $\mathcal{A}$ the map

$$Hom_{\mathcal{A}}(A, A') \to Hom(\prod_{i \in I} Hom_{\mathcal{A}}(E_i, A), \prod_{i \in I} Hom_{\mathcal{A}}(E_i, A'))$$

is injective.

(b) For every object A in $\mathcal{A}$ and every subobject $B \subsetneqq A$, there exists a morphism $E_i \to A$ which does not factorize over B.

Proof of the equivalence of (a) and (b):

(a) $\Rightarrow$ (b): We have the exact sequence

$$0 \to B \xrightarrow{i} A \xrightarrow{\pi} A' = A/B \to 0\,,$$

and by assumption $A' \neq 0$. Then $\pi \neq 0$, and by (a) there is an $i \in I$, for which the induced map

$$Hom_{\mathcal{A}}(E_i, A) \to Hom_{\mathcal{A}}(E_i, A')$$

is not zero. If $\varphi : E_i \to A$ is not in the kernel, then φ does not factorize over B.

(b) $\Rightarrow$ (a): Left to the readers!

Examples 5.3 If R is a ring with unit, then $E = R$ is a generator for Mod_R, since we have a canonical isomorphism

$$\begin{array}{rcl} Hom_R(R, M) & \xrightarrow{\sim} & M \\ f & \mapsto & f(1) \end{array}$$

for every R-module M.

Definition 5.4 We say that the abelian category $\mathcal{A}$ has the property
$(AB3)$, if any direct sums $\bigoplus\limits_{i \in I} A_i$ exist in $\mathcal{A}$ (Since cokernels exist, it follows that arbitrary direct limits (colimits) exist in A, see 4.A.21 above)

$(AB4)$, if $(AB3)$ holds, and forming direct sums is an exact functor,
$(AB5)$, if $(AB3)$ holds, and forming inductive limits is an exact functor.
We define the properties $(AB3^*), (AB4^*)$ and $(AB5^*)$ dually.

Definition 5.5 An abelian category is called a Grothendieck category, if $(AB5)$ holds and if it has a family of generators.

The category of R-modules in Example 5.3 is a Grothendieck category, because it has the property $(AB5)$, as one can see easily, and the second property applies by 5.3.

Theorem 5.6 A Grothendieck category has enough injectives.

Idea of Proof: Because of $(AB3)$, $\mathcal{A}$ has a generator E (for a family $(E_i)_{i\in I}$ let $E = \bigoplus_{i\in I} E_i$). For the ring with unit $R = Hom_{\mathcal{A}}(E, E)$ then consider the faithful functor

$$\begin{array}{rcl} \mathcal{A} & \to & Mod_R \\ A & \mapsto & Hom_{\mathcal{A}}(E, A)\,. \end{array}$$

For the Proof of 5.1 it thus suffices to show:

Theorem 5.7 $Pr(\mathcal{X})$ and $Sh(\mathcal{X}, \mathcal{T})$ are Grothendieck categories.

Proof: First we consider the generators.

Lemma/Definition 5.8 (a) For an object U in $\mathcal{X}$ define the abelian presheaf $\mathbb{Z}_U^P$ by

$$\mathbb{Z}_U^P(V) = \bigoplus_{Hom(V,U)} \mathbb{Z} \;=\; \bigoplus_{f\in Hom(V,U)} \mathbb{Z}f \qquad \text{for } V \text{ in } \mathcal{X}.$$

For every abelian presheaf Q on $\mathcal{X}$, we then have isomorphisms

$$Hom_{Pr(\mathcal{X})}(\mathbb{Z}_U^P, Q) \cong Hom_{\mathbb{Z}}(\mathbb{Z}, Q(U)) = Q(U)\,,$$

functorially in Q (i.e., $\mathbb{Z}_U^P$ represents the functor $Q \mapsto Q(U)$).
(b) We define the abelian sheaf $\mathbb{Z}_U$ as the associated sheaf

$$\mathbb{Z}_U = a\mathbb{Z}_U^P\,.$$

Then for sheaves F on $(\mathcal{X}, \mathcal{T})$, we have

$$Hom_{Sh(\mathcal{X},\mathcal{T})}(\mathbb{Z}_U, F) = F(U)\,,$$

functorially in F.

Proof (a): Every morphism $f : \mathbb{Z}_U^P \to Q$ is uniquely determined by $f_U(1_{id_U}) \in Q(U)$.
(b) follows from (a) by adjointness of a and i.

Corollary 5.9 $Pr(\mathcal{X})$ and $Sh(\mathcal{X}, \mathcal{T})$ have a family of generators.

Proof: (a) If $P \to P'$ is non-zero in $Pr(\mathcal{X})$, then $P(U) \to P'(U)$ is non-zero for some object U in $\mathcal{X}$.

(b)The same applies for $Sh(\mathcal{X}, \mathcal{T})$.

Lemma 5.10 $Pr(\mathcal{X})$ and $Sh(\mathcal{X}, \mathcal{T})$ satisfy $(AB5)$.

Proof $(AB3)$ and $(AB5)$ are obvious for $Pr(\mathcal{X})$ by (the proof of) Theorem 4.3 (a), (since these properties hold for Ab), and it follows from 5.8 (a) that $(\mathbb{Z}_U)_{U\in\mathcal{X}}$ is a family of generators.

$(AB3)$ holds for $Sh(\mathcal{X}, \mathcal{T})$ (arbitrary limits and colimits exist by 4.3(a)), and $(AB5)$ follows from the explicit description of the colimits of the proof of 4.3 (a) and the exactness of the functor $P \mapsto aP$ (see 4.3 (c)). This proves Proposition 5.1.

Definition 5.11 Let $(\mathcal{X}, \mathcal{T})$ be a site and U an object in $\mathcal{X}$. The functor

$$\begin{array}{rcccl} H^i(U,-) := H^i(U,\mathcal{T};-) & : & Sh(\mathcal{X},\mathcal{T}) & \to & Ab \\ & & F & \mapsto & H^i(U,F) \end{array}$$

is the i-th right derivative of the left exact functor

$$F \mapsto F(U) =: \Gamma(U,F) =: H^0(U,F)\,.$$

$H^i(U,F)$ is called i-th cohomology of F on U (or i-th cohomology group on U with coefficients in F).

By construction, $H^i(U,F) = H^i(I^\cdot(U))$, where $F \hookrightarrow I^\cdot$ is an injective resolution F in $Sh(\mathcal{X}, \mathcal{T})$.

Example 5.12 If X is a topological space and F a sheaf on X, then $H^i(X,F)$ is the usual cohomology of sheaves on X.

Definition 5.13 Let $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ a morphism of sites. Then R^if_* is the i-th right derivative of the left exact functor

$$f_* : Sh(\mathcal{X}', \mathcal{T}') \to (Sh(\mathcal{X}, \mathcal{T})\,.$$

R^if_*F is called the i-th higher direct image of F under f.

Hence $R^if_*F = \mathcal{H}^i(f_*I^\cdot)$, where $F \hookrightarrow I^\cdot$ is an injective resolution in $Sh(\mathcal{X}', \mathcal{T}')$.

Remark 5.14 (a) By the general properties of right derived functors, for each short exact sequence

$$0 \to F' \to F \to F'' \to 0 \tag{5.14.1}$$

of sheaves on $(\mathcal{X}, \mathcal{T})$ and for every object U in $\mathcal{X}$ one has a long exact sequence of cohomology

$$\begin{array}{ccccccccccc} 0 & \to & H^0(U,F') & \to & H^0(U,F) & \to & H^0(U,F'') & \stackrel{\delta}{\to} & H^1(U,F') & & \\ \ldots & \to & H^n(U,F') & \to & H^n(U,F) & \to & H^n(U,F'') & \stackrel{\delta}{\to} & H^{n+1}(U,F') & \to & \ldots \end{array}$$

This is functorial in U and functorial in short exact sequences (5.14.1).

(b) Similarly, for every morphism of sites $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ and every short exact sequence of sheaves on $(\mathcal{X}', \mathcal{T}')$

$$0 \to G' \to G \to G'' \to 0 \tag{5.14.2}$$

we get a long exact sequence

$$\ldots \to R^n f_* G' \to R^n f_* G \to R^n f_* G'' \xrightarrow{\delta} R^{n+1} f_* G' \to \ldots ,$$

which is functorial in the short exact sequences (5.14.2).

Theorem 5.15 (a) Let F be an abelian sheaf on $(\mathcal{X}, \mathcal{T})$. For every morphism $\alpha : V \to U$ in $\mathcal{X}$, one has canonical restriction homomorphisms

$$\alpha^* : H^i(U, F) \to H^i(V, F) \qquad (i \geq 0), \tag{5.15.1}$$

which coincide with the restriction $F(U) \to F(V)$ for $i = 0$, are functorial in F, and are compatible with the exact sequences of 5.14 (a) (i.e., also compatible with the connecting morphisms).

(b) By this we obtain an abelian presheaf

$$\underline{H}^i(F) : U \mapsto H^i(U, F)$$

for all $i \geq 0$.

Proof If $F \hookrightarrow I^{\cdot}$ is an injective resolution, then we have a homomorphism of complexes

$$I^{\cdot}(U) \to I^{\cdot}(V)\,,$$

and the maps (5.15.1) are obtained by passing to the cohomology. By transivity of the restrictions of $I^{\cdot}$, it follows that one has the relation $(\alpha\beta)^* = \beta^*\alpha^*$ for each further morphism $\beta : W \to V$. We obtain (b), since $id_U^* = id$. The other functorialities in (a) are again obvious by construction: For a morphism of sheaves $F \to G$, we obtain a morphism

$$\begin{tikzcd} G \arrow[r, hook] & J^{\cdot} \\ F \arrow[u] \arrow[r, hook] & I^{\cdot} \arrow[u] \end{tikzcd}$$

of injective resolutions that, by definition, provides the functoriality of the cohomology, i.e. the canonical morphisms $H^i(U, F) = H^i(I^{\cdot}(U)) \to H^i(J^{\cdot}(U)) = H^i(U, G)$. This shows the compatibility with α^*. The case of exact sequences follows from an exact diagram of injective resolutions

$$\begin{tikzcd} 0 \arrow[r] & F' \arrow[r] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & F'' \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & I^{\cdot} \arrow[r] & J^{\cdot} \arrow[r] & K^{\cdot} \arrow[r] & 0\,, \end{tikzcd}$$

by passing to the sections over U in the bottom line.

Theorem 5.16 Let $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ be a morphism of sites. For every sheaf $F \in Sh(\mathcal{X}', \mathcal{T}')$ and every $i \geq 0$, $R^i f_* F$ is the sheaf associated to the presheaf

$$U \mapsto H^i(f^0(U), F)$$

on $\mathcal{X}'$.

Proof Let $F \hookrightarrow I^{\cdot}$ be an injective resolution in $Sh(\mathcal{X}', \mathcal{T}')$. Then $R^i f_* F = \mathcal{H}^i(f_* I^{\cdot})$ is the sheaf associated to the presheaf quotient

$$
\begin{array}{rcc}
U & \mapsto & \ker^P(f_* I^i \to f_* I^{i+1})(U)/im^P(f_* I^{i-1} \to f_* I)(U) \\
 & & \shortparallel \\
 & & \ker(I^i(f^0(U)) \to I^{i+1}(f^0(U)))/im(I^{i-1}(f^0(U))/I^i(f^0(U))) \\
 & & \shortparallel \\
 & & H^i(f^0(U), F)\,.
\end{array}
$$

6 Spectral sequences

Let $\mathcal{A}$ be an abelian category.

Definition 6.1 A spectral sequence in $\mathcal{A}$

$$E_1^{p,q} \Rightarrow E^{p+q}$$

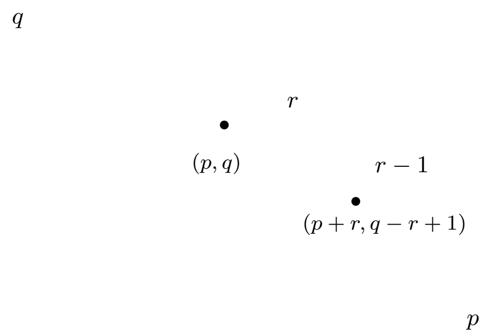
consists of

(a) objects $E_1^{p,q}$ in $\mathcal{A}$ for all $p, q \in \mathbb{Z}$,

(b) subquotients $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$ of $E_1^{p,q}$ for all $r \geq 2$,

(c) morphisms (called the **differentials** of the spectral sequence)

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\,,$$



such that

$$E_{r+1}^{p,q} = \frac{\ker(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-1})}{im(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q})}\,,$$

(d) subquotients $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$ of $E_1^{p,q}$, such that

$$B_r^{p,q} \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq Z_r^{p,q}$$

for all $r \geq 1$ (hence $E_\infty^{p,q}$ "is smaller than $E_r^{p,q}$ for all $r \geq 1$"),

(e) objects $(E^n)_{n\in\mathbb{Z}}$ in $\mathcal{A}$ with descending filtrations

$$E^n \supseteq \ldots \supseteq F^pE^n \supseteq F^{p+1}E^n \supseteq \ldots$$

and isomorphisms

$$E_\infty^{p,q} \xrightarrow{\sim} F^pE^{p+q}/F^{p+1}E^{p+q}$$

for all $p, q \in \mathbb{Z}$.

Definition 6.2 The spectral sequence is called finitely convergent, if

$$E_r^{p,q} = E_\infty^{p,q} \quad \text{for} \quad r >> 0, \quad \text{for all} \quad (p,q) \in \mathbb{Z}^2\,,$$

and if for every $n \in \mathbb{Z}$ the filtration F^pE^n is finite, i.e.,

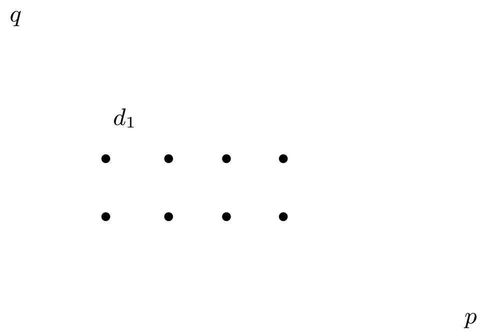
$$F^pE^n = \begin{cases} 0 & \text{for} \quad p >> 0, \\ E^n & \text{for} \quad p << 0. \end{cases}$$

Some spectral sequences begin with $E_2^{p,q}$; then $E_1^{p,q}$ does not exist, and all $E_r^{p,q}$ are subquotients of $E_2^{p,q}$.

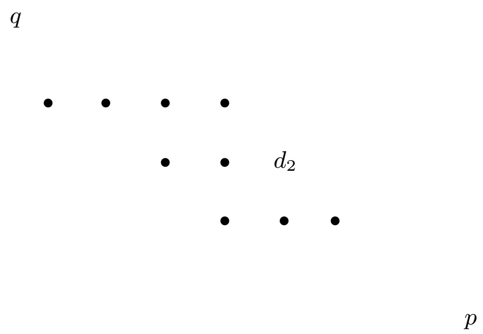
We now explain how to work with spectral sequences.

1) The **layers**: For each r one considers the E_r-layer of all terms $E_r^{p,q}$ and their differentials

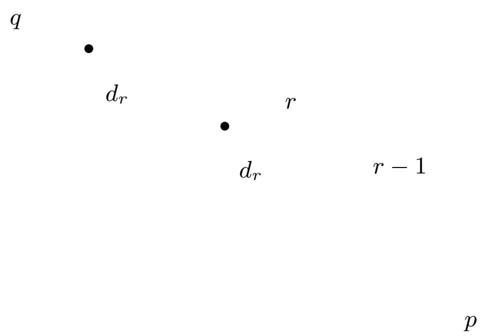
E_1-layer:



E_2-layer:



E_r-layer:

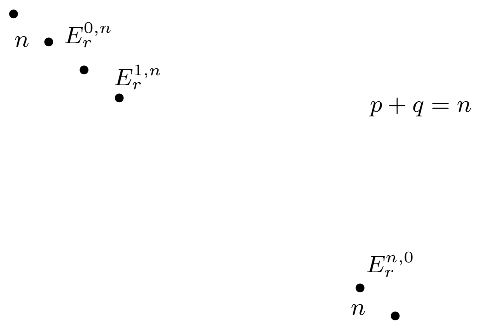


One has $d_rd_r = 0$ and $E_{r+1} = \ker d_r / im\, d_r$.

2) Limit/convergence: Let

$$E_1^{p,q} \Rightarrow E^{p+q} \qquad (\text{or } E_2^{p,q} \Rightarrow E^{p+q}\ )$$

be a finitely convergent spectral sequence. We obtain the following picture:



The terms which contribute to E^n are on the line $p+q=n$. We have a (finite) filtration

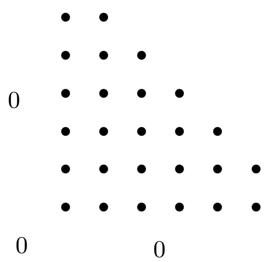
$$E^n \supseteq \ldots \supseteq F^pE^n \supseteq F^{p+1}E^n \supseteq \ldots$$

and

$$F^pE^n/F^{p+1}E^n = E_\infty^{p,q} \quad , \quad p+q=n\,.$$

The term on the left is a subquotient of E^n, the term on the right is a subquotient of $E_1^{p,q}$; furthermore we have $E_\infty^{p,q} = E_r^{p,q}$ for $r >> 0$.

3) Spectral sequences in the first quadrant: Let $E_1^{p,q} \Rightarrow E^{p+q}$ (resp. $E_2^{p,q} \Rightarrow E^{p+q}$) be a finitely convergent spectral sequence with $E_1^{p,q} = 0$ (resp. $E_2^{p,q} = 0$) for $p < 0$ or $q < 0$.



Lemma/Definition 6.3 (a) One has $E_\infty^{p,q} = E_r^{p,q}$ for $r > \max(p, q+1)$.

(b) For $E_1^{p,q} \Rightarrow E^{p+q}$, there are canonical morphisms

$$\begin{array}{rcl} E^n & \xrightarrow{e} & E_1^{0,n} \\ E_1^{n,0} & \xrightarrow{e} & E^n\,. \end{array}$$

These are called the **edge morphisms**.

(c) For $E_2^{p,q} \Rightarrow E^{p+q}$, there are edge morphisms

$$\begin{array}{rcl} E^n & \xrightarrow{e} & E_2^{0,n} \\ E_2^{n,0} & \xrightarrow{e} & E^n \end{array}$$

Proof (a):

p

$p-1$

(p,q) $q+1$

$\bullet$

q

If $r > q+1$, the differential $d_r^{p,q}$ starting from $E_r^{p,q}$ is zero (since it ends in a zero object). If $r > p$, the differential arriving in $E_r^{p,q}$ ($d_r^{p-r,q+r-1}$) is zero. If both properties hold, then we have $E_{r+1}^{p,q} = \ker d_r/im\, d_2 = E_r^{p,q}/0 = E_r^{p,q}$. Since this holds for all higher r (and the spectral sequence converges), we have $E_r^{p,q} = E_\infty^{p,q}$.

(b): If $E_\infty^{p,q} = 0$ for $p < 0$, then we have $E^n = F^0E^n$ (because of the convergence), and we have

$$E_{r+1}^{0,n} = \ker(E_r^{0,n} \xrightarrow{d_r^{0,n}} E_r^{r,n-r+1}) \subseteq E_r^{0,n}$$

for all r, i.e. $E_\infty^{0,n} \subseteq E_1^{0,n}$. We obtain

$$e : E^n \twoheadrightarrow F^0E^n/F^1E^n \cong E_\infty^{0,n} \hookrightarrow E_1^{0,n}\,.$$

If $E_\infty^{p,q} = 0$ for $q < 0$, then we have $F^{n+1}E^n = 0$ (because of the convergence), and $E_{r+1}^{n,0} = coker(E_r^{n-r,r-1} \xrightarrow{d_r} E_r^{n,0})$ is a quotient of E_r^n. Then we have $E_1^{n,0} \twoheadrightarrow E_\infty^{n,0}$ and a morphism

$$e : E_1^{n,0} \twoheadrightarrow E_\infty^{n,0} \cong F^nE^n/F^{n+1}E \hookrightarrow E^n\,.$$

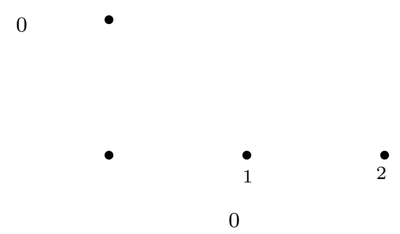
(c) is analogous.

Lemma 6.4 (Exact sequence of the lower terms) Let $E_1^{p,q} \Rightarrow E^{p+q}$ (or $E_2^{p,q} \Rightarrow E^{p+q}$) be a finitely convergent spectral sequence in the first quadrant. Then one has an exact sequence

$$0 \to E_2^{1,0} \xrightarrow{e} E^1 \xrightarrow{e} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{e} E^2\,,$$

where e always denotes the edge morphism.

Proof The picture



(or the Proof of 6.5 (a)) shows

$$E_\infty^{1,0} = E_2^{0,1}, \quad E_\infty^{0,1} = \ker d_2^{0,1}, \quad E_\infty^{2,0} = \operatorname{coker} d_2^{0,1}\,.$$

From this we obtain exact sequences

$$0 \to E_2^{0,1} \xrightarrow{e} E^1 \to \ker d_2^{0,1} \to 0$$

$$0 \to \ker d_2^{0,1} \to E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \to E_\infty^{2,0}\,,$$

and by splicing together these sequences and composing with $E_\infty^{2,0} \subseteq E^2$ we obtain the claimed sequence.

Theorem 6.5 (a) Let A^* be a complex in $\mathcal{A}$, and let F^pA^* be a descending filtration by subcomplexes. Then we have a spectral sequence

$$E_1^{p,q} = H^{p+q}(F^pA^*/F^{p+1}A^*) \Rightarrow E^{p+q} = H^n(A^*)\,.$$

(b) If the filtration F^p is biregular, i.e.,

$$F^pA^n = \begin{cases} 0 & \text{for} \quad p >> 0\,, \\ A^n & \text{for} \quad p << 0 \end{cases}$$

for every $n \in \mathbb{Z}$, then the spectral sequence is finitely convergent.
(c) The E_1-differential

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(F^pA^*/F^{p+1}A^*) \to H^{p+q+1}(F^{p+1}A^*/F^{p+2}A^*) = E_1^{p+1,q}$$

is the connecting homomorphism for the exact sequence of complexes

$$0 \to F^{p+1}A^*/F^{p+2}A^* \to F^pA^*/F^{p+2}A^* \to F^pA^*/F^{p+1}A^* \to 0\,.$$

Proof (a): For $r \geq 1$, $p \in \mathbb{Z}$ and $q := n - p$ let

$$F^pH^n(A^*) = im(H^n(F^pA^*) \to H^n(A^*))$$

as well as

$$\begin{tikzcd}
Z_r^{p,q} = & im(H^n(F^pA^*/F^{p+r}A^*) \arrow[r, "\alpha^{p,r}"] & H^n(F^pA^*/F^{p+1}A^*) = E_1^{p,q}) \\
Z_\infty^{p,q} = \arrow[u, hook] & im(H^n(F^pA^*) \arrow[u] \arrow[r, "\alpha^p"] & H^n(F^pA^*/F^{p+1}A^*)) \arrow[u, equal] \\
B_\infty^{p,q} = \arrow[u, hook] & im(H^{n-1}(A^*/F^pA^*) \arrow[u, "\delta"] \arrow[r, "\delta^p"] & H^n(F^pA^*/F^{p+1}A^*)) \arrow[u, equal] \\
B_r^{p,q} = \arrow[u, hook] & im(H^{n-1}(F^{p-r+1}A^*/F^pA^*) \arrow[u] \arrow[r, "\delta^{p,r}"] & H^n(F^pA^*/F^{p+1}A^*))\,. \arrow[u, equal]
\end{tikzcd}$$

Here, the morphisms $\alpha, \alpha^{p,r}$ and the commutative diagram above are induced by the commutative diagram

$$\begin{tikzcd}
F^p/F^{p+r} \arrow[r, "\alpha^{p,r}"] & F^p/F^{p+1} \ , \\
F^p \arrow[u] \arrow[r, "\alpha^p"] & F^p/F^{p+1} \arrow[u, equal]
\end{tikzcd}$$

where we write F^m for F^mA^*.

The connecting homomorphisms δ, δ^p and $\delta^{p,r}$ as well as the other commutative diagrams are induced by the following commutative diagrams and the corresponding long exact cohomology sequences:

$$\begin{tikzcd}
\delta : & 0 \arrow[r] & F^p \arrow[r] \arrow[d, "\alpha^p"] & A^* \arrow[r] \arrow[d] & A^*/F^p \arrow[r] \arrow[d, equal] & 0 \\
\delta^p : & 0 \arrow[r] & F^p/F^{p+1} \arrow[r] & A^*/F^{p+1} \arrow[r] & A^*/F^p \arrow[r] & 0 \\
\delta^{p,r} : & 0 \arrow[r] & F^p/F^{p+1} \arrow[r] \arrow[u, equal] & F^{p-r+1}/F^{p+1} \arrow[r] \arrow[u, hook] & F^{p-r+1}/F^p \arrow[r] \arrow[u, hook] & 0\,.
\end{tikzcd}$$

We obtain a commutative diagram with exact rows

(1)
$$\begin{tikzcd}
H^n(F^p/F^{p+r+1}) \arrow[r] \arrow[d, equal] & H^n(F^p/F^{p+r}) \arrow[r, "\delta^{p+r,r+1}"] \arrow[d, "\alpha^{p,r}"] \arrow[dr, "\beta"] & H^{n+1}(F^{p+r}/F^{p+r+1}) \arrow[d, "\gamma"] \\
H^n(F^p/F^{p+r+1}) \arrow[r, "\alpha^{p,r+1}"] & H^n(F^p/F^{p+1}) \arrow[r] & H^{n+1}(F^{p+1}/F^{p+r+1})
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
H^n(F^{p+1}/F^{p+r}) \arrow[r] \arrow[d, equal] & H^n(F^p/F^{p+r})) \arrow[r, "\alpha^{p,r}"] \arrow[d, "\delta^{p+r,r+1}"] \arrow[dr, "\beta"] & H^n(F^p/F^{p+1}) \arrow[d] \\
H^n(F^{p+1}/F^{p+r}) \arrow[r, "\delta^{p+r,r}"] & H^{n+1}(F^{p+r}/F^{p+r+1}) \arrow[r, "\gamma"] & H^{n+1}(F^{p+1}/F^{p+r+1})
\end{tikzcd}$$

From (1) we get $Z_{r+1}^{p,q} \subseteq Z_r^{p,q}$, and from (2) we get (by renumbering $p + r \rightsquigarrow p$) $B_r^{p,q} \subseteq B_{r+1}^{p,q}$. This gives the following inclusions

$$0 = B_1^{p,q} \subseteq \ldots \subseteq B_r^{p,q} \subseteq B_{r+1}^{p,q} \subseteq \ldots \subseteq B_\infty^{p,q} \subseteq Z_\infty^{p,q} \subseteq \ldots \subseteq Z_{r+1}^{p,q} \subseteq Z_r^{p,q} \subseteq \ldots \subseteq Z_1^{p,q} = E_1^{p,q}\,.$$

We now use the following Lemma

Lemma 6.6 If

$$\begin{tikzcd} & C \arrow[d, "\varphi"] \arrow[dr, "\psi"] & \\ A' \arrow[ur] \arrow[r, "\varphi'"] & A \arrow[r, "\eta"] & A'' \end{tikzcd}$$

is a commutative diagram in $\mathcal{A}$ with exact line, then we have $im(\varphi') \subseteq im(\varphi)$, as well as canonically

$$im(\varphi)/im(\varphi') \cong im(\psi)\,.$$

Proof: The fist claim is obvious, and since $\ker\eta = im\,\varphi' \subseteq im\,\varphi$, η induces an isomorphism

$$im\,\varphi/im\,\varphi' = im\,\varphi/\ker\eta \overset{\sim}{\to} \eta(im\,\varphi) = im(\eta\varphi) = im\,\psi\,.$$

By applying 6.6 we get an isomorphism

$$\delta_r^{p,q} : Z_r^{p,q}/Z_{r+1}^{p,q} = im\,\alpha^{p,r}/im\,\alpha^{p,r+1} \overset{(1)}{\cong} im\,\beta \overset{(2)}{\cong} im\,\delta^{p+r,r+1}/im\,\delta^{p+r,r} = B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}\,.$$

With this, we define the differential

$$d_r^{p,q} : E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} \twoheadrightarrow Z_r^{p,q}/Z_{r+1}^{p,q} \underset{\sim}{\overset{\delta_r^{p,q}}{\to}} B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1} \hookrightarrow Z_r^{p+r,q-r+1}/B_r^{p+r,q-r+1}\,.$$

Then we have

$$\ker d_r^{p,q} = Z_{r+1}^{p,q}/B_r^{p,q} \quad , \quad im\,d_r^{p,q} = B_{r+1}^{p+r,q-r+1}/B_r^{p+r,q-r+1}$$

and therefore

$$\frac{\ker d_r^{p,q}}{im\,d_r^{p-r,q+r-1}} = \frac{Z_{r+1}^{p,q}}{B_{r+1}^{p,q}} = E_{r+1}^{p,q}\,.$$

Finally, the commutative diagrams with exact rows

(3)

$$\begin{tikzcd} & H^n(F^p) \arrow[d] \arrow[dr, "\rho"] & \\ H^n(F^{p+1}) \arrow[ur] \arrow[r] & H^n(A^*) \arrow[r] & H^n(A^*/F^{p+1}) \end{tikzcd}$$

(4)

$$\begin{tikzcd} H^{n-1}(A^*/F^p) \arrow[r, "\delta"] \arrow[d, equal] & H^n(F^p) \arrow[r] \arrow[d, "\alpha^p"] \arrow[dr, "\rho"] & H^n(A^*) \arrow[d] \\ H^{n-1}(A^*/F^p) \arrow[r, "\delta^p"] & H^n(F^p/F^{p+1}) \arrow[r] & H^n(A/F^{p+1}) \end{tikzcd}$$

together with Lemma 6.6 produce the relations $F^{p+1}H^n(A^*) \subseteq F^pH^n(A^*)$, as well as

$$E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q} = im\,\alpha^p/im\,\delta^p \overset{(4)}{\cong} im\,\rho \overset{(3)}{\cong} F^pH^n(A^*)/F^{p+1}H^n(A^*)\,.$$

This shows all the properties of a spectral sequence.

(b): The additional claim about convergence is obvious, since the n-th cohomology of a complex C^* depends only on $C^{n-1} \to C^n \to C^{n+1}$.

(c): For $r = 1$, the diagrams (1) and (2) are

(1)
$$\begin{tikzcd}
H^n(F^p/F^{p+2}) \arrow[r] \arrow[d, equal] & H^n(F^p/F^{p+1}) \arrow[r, "\delta^{p+1,2}"] \arrow[d, "\alpha^{p,1}", "\shortparallel"'] \arrow[dr, "\beta"] & H^{n+1}(F^{p+1}/F^{p+2}) \arrow[d, equal] \\
H^n(F^p/F^{p+2}) \arrow[r, "\alpha^{p,2}"] & H^n(F^p/F^{p+1}) \arrow[r, "\delta^{p+1,2}"] & H^{n+1}(F^{p+1}/F^{p+2})
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
0 = & H^n(F^{p+1}/F^{p+1}) \arrow[r] \arrow[d, equal] & H^n(F^p/F^{p+1}) \arrow[r, "\alpha^{p,1}", "="'] \arrow[d, "\delta^{p+1,2}"] \arrow[dr, "\beta"] & H^n(F^p/F^{p+1}) \arrow[d] \\
0 = & H^n(F^{p+1}/F^{p+1}) \arrow[r, "\delta^{p+1,1}"] & H^{n+1}(F^{p+1}/F^{p+2}) \arrow[r, "="'] & H^{n+1}(F^{p+1}/F^{p+2})
\end{tikzcd}$$

From the definition of $d_1^{p,q}$ (with $p + q = n$) we get that $d_1^{p,q} = \delta^{p+1,2}$, the connecting homomorphism for the exact sequence

$$0 \to F^{p+1}/F^{p+2} \to F^p/F^{p+2} \to F^p/F^{p+1} \to 0\,,$$

since $d_1^{p,q} = \beta = \delta^{p+1,2}$ by the equalities in the diagrams (1) and (2).

An important example for spectral sequences is the following.

Theorem 6.7 (Grothendieck-Leray-spectral sequence) Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories, where $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and F maps injectives to G-acyclic objects. Then for every object X in $\mathcal{A}$ we have a finitely convergent spectral sequence

$$E_2^{p,q} = R^pG(R^qFX) \Rightarrow R^{p+q}(G \circ F)X$$

The proof needs some preliminary considerations

Definition 6.8 (a) A naive double complex $C^{*,*}$ in $\mathcal{A}$ is a *commutative* diagram of objects $C^{p,q} \in \mathcal{A}$

$$\begin{tikzcd}
\ldots \arrow[r] & C^{p-1,q+1} \arrow[r] & C^{p,q+1} \arrow[r, "d_I^{p,q+1}"] & C^{p+1,q+1} \arrow[r] & \ldots \\
\ldots \arrow[r] & C^{p-1,q} \arrow[r, "d_I^{p-1,q}"] \arrow[u] & C^{p,q} \arrow[r, "d_I^{p,q}"] \arrow[u, "d_{II}^{p,q}"'] & C^{p+1,q} \arrow[r] \arrow[u, "d_{II}^{p+1,q}"'] & \ldots \\
\ldots \arrow[r] & C^{p-1,q-1} \arrow[r] \arrow[u] & C^{p,q-1} \arrow[r] \arrow[u, "d_{II}^{p,q-1}"'] & C^{p+1,q-1} \arrow[r] \arrow[u] & \ldots \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

(b) A double complex is a corresponding diagram, in which all squares are *anticommutative*, i.e. $d_I^{p,q+1}d_{II}^{p,q} + d_{II}^{p+1,q}d_I^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.

(c) The double complex associated to a naive double complex as in (a) is the double complex where $d_{II}^{p,q}$ is replaced by $(-1)^p d_{II}^{p,q}$.

(d) If there is an $N \in \mathbb{Z}$ with $C^{p,q} = 0$ for $p < N$ or $q < N$, then the total complex associated to a double complex is the complex $Tot(C^{*,*})$ with components

$$Tot(C^{*,*})^n = \bigoplus_{p+q=n} C^{p,q}$$

and differential

$$d = d_I + d_{II}\,,$$

i.e. $d|_{C^{p,q}} = d_I^{p,q} + d_{II}^{p,q}$.

The following construction is very important for the treatment and definition of spectral sequences.

Construction 6.9 Let $Tot(I^{*,*})$ be the total complex associated to the double complex, which is associated to the naive double complex $I^{*,*}$. This has two descending filtrations:

(6.9.1)
$$F_I^p Tot(I^{*,*})^n = \bigoplus_{\substack{r+s=n \\ r \geq p}} I^{r,s} \quad , \quad \text{and}$$

(6.9.2)
$$F_{II}^p Tot(I^{*,*})^n = \bigoplus_{\substack{r+s=n \\ s \geq q}} I^{r,s}\,.$$

Since

$$F_I^p Tot(I^{*,*})/F_I^{p+1} Tot(I^{*,*}) = I^{p,*}[-p]$$

and

$$F_{II}^p Tot(I^{*,*})/F_{II}^{q+1} Tot(I^{*,*}) = I^{*,q}[-q]$$

the corresponding spectral sequences from Theorem 6.5 are

$${}_I E_1^{p,q} = H^{p+q}(I^{p,*}[-p]) = H^q(I^{p,*}) \Rightarrow E^{p+q} = H^{p+q}(Tot(I^{*,*}))\,,$$

and for the first filtration, and

$${}_{II} E_1^{p,q} = H^{p+q}(I^{*,q}[-q]) = H^p(I^{*,q}) \Rightarrow E^{p+q} = H^{p+q}(Tot(I^{*,*}))$$

for the second filtration. The d_1-differential of the spectral sequence ${}_I E_1^{p,q} \Rightarrow E^{p+q}$ is the morphism

$${}_I E_1^{p,q} = H^q(I^{p,*}) \to H^q(I^{p+1,*}) = {}_I E_1^{p+1,q}\,,$$

which is induced by the morphism of complexes

$$d_I^{p,*} : I^{p,*} \to I^{p+1,*}\,.$$

By 6.5 (c), for a filtered complex $(A^*, F^p A^*)$ as in 6.5, the d_1-differential

$$d_1^{p,q} : E_1^{p,q} = H^{p+q}(F^p A^*/F^{p+1} A^*) \to H^{p+q+1}(F^{p+1} A^*/F^{p+2} A^*) = E_1^{p+1,q}$$

is the connecting homomorphism for the exact sequence of complexes

$$0 \to F^{p+1}/F^{p+2} \to F^p/F^{p+2} \to F^p/F^{p+1} \to 0\,.$$

In this situation, we have the sequence

$$\begin{tikzcd}
0 \arrow[r] & F^{p+1}/F^{p+2} \arrow[r] \arrow[d, equal] & F^p/F^{p+2} \arrow[r] \arrow[d, equal] & F^p/F^{p+1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I^{p+1,*}[-p-1] \arrow[r, "\alpha"] & Tot\begin{pmatrix} I^{p+1,*}[-p-1] \\ \uparrow d_{II}^{p,*} \\ I^{p,*}[-p-1] \end{pmatrix} \arrow[r, "\beta"] & I^{p,*}[-p] \arrow[r] & 0\,,
\end{tikzcd}$$

where α and β are the obvious morphisms.

We obtain an exact sequence of complexes

$$\begin{tikzcd}
0 \arrow[r] & I^{p+1,q+1} \arrow[r] & I^{p+1,q+1} \oplus I^{p,q+2} \arrow[r] & I^{p,q+2} \arrow[r] & 0 \\
0 \arrow[r] & I^{p+1,q} \arrow[r] \arrow[u, "(-1)^{p+1}d_{II}"] & I^{p+1,q} \oplus I^{p,q+1} \arrow[r] \arrow[u, "d"'] & I^{p,q+1} \arrow[r] \arrow[u, "(-1)^p d_{II}"'] & 0\,,
\end{tikzcd}$$

$$a \longmapsto (a,0), (a,b) \longmapsto b$$

where the arrow in the middle, expressed in elements, is given by

$$(a,b) \mapsto ((-1)^{p+1}d_{II}a + d_I b, (-1)^p d_{II}b)\,.$$

By the standard description of the connecting morphism ($b \in I^{p,q+1}$ with $d_{II}b = 0$ is lifted to $(0,b) \in I^{p+1,q} \oplus I^{p,q+1}$, mapped on $(d_I b, 0)$ below D, which is the image $d_I b \in I^{p+1,q+1}$), we see that this maps the class of b to the class $d_I b$, hence is induced by d_I as claimed. The analogous claims hold for the second spectral sequence.

Now we remember the well-known

Lemma 6.10 Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{A}$. If $\mathcal{A}$ has enough injectives, then, for any given injective resolutions

$$A \hookrightarrow I^* \text{ and } C \hookrightarrow K^*$$

there exists an exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & I^* \arrow[r] & J^* \arrow[r] & K^* \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, hook] & B \arrow[r] \arrow[u, hook] & C \arrow[r] \arrow[u, hook] & 0\,,
\end{tikzcd}$$

which is split in each degree.

Proof: One starts with a diagram

$$\begin{tikzcd}
0 \arrow[r] & I^0 \arrow[r] & I^0 \oplus K^0 \arrow[r] & K^0 \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, hook, "\alpha"] & B \arrow[r] \arrow[u, "(\beta,\gamma)"'] \arrow[ul, "\beta"'] & C \arrow[r] \arrow[u, "\gamma"'] & 0\,,
\end{tikzcd}$$

where β is an extension of α to B (this exists, since I^0 is injective). Then one can easily see that the middle arrow (β,γ) is a monomorphism. By the snake lemma, the sequence

$$0 \to A^1 \to B^1 \to C^1 \to 0$$

of the cokernels of $\alpha, (\beta,\gamma)$ and γ is exact, and one proceeds with $A^1 \hookrightarrow I^1$ and $C^1 \hookrightarrow K^1$, etc.

By this we obtain inductively:

Theorem 6.11 (Cartan-Eilenberg-resolution) If A^* is a complex in $\mathcal{A}$ which is bounded below, e.g. $A^n = 0$ for $n < N$, then we have a naive double complex $(I^{*,*}, d_I, d_{II})$ with $I^{p,q} = 0$ for $p < N$ and $q < 0$ and a morphism of complexes

$$\begin{tikzcd}
I^{N,0} \arrow[r, "d^{N,0}"] & I^{N+1,0} \arrow[r, "d^{N+1,0}"] & A^{N+2,0} \arrow[r] & \dots \\
A^N \arrow[u] \arrow[r, "d^N"] & A^{N+1} \arrow[u] \arrow[r, "d^{N+1}"] & A^{N+2} \arrow[u] \arrow[r] & \dots ,
\end{tikzcd}$$

such that the following holds:

(a) For all p, $A^p \hookrightarrow I^{p,*}$ is an injective resolution.

(b) For all p, $ZA^p \hookrightarrow ZI^{p,*}, BA^p \hookrightarrow BI^{p,*}$ and $H^p(A^*) \to H^p_I(I^{*,*})$ are injective resolutions. Here let $H^p_I(I^{*,*}) = H^p((I^{*,*}, d_I))$ be the cohomology of $I^{p-1,*} \overset{d_I}{\to} I^{p,*} \overset{d_I}{\to} I^{p+1,*}$, $ZA^p = \ker(A^p \to A^{p+1}), BA^p = im(A^{p-1} \to A^p), ZI^{p,*} = \ker(I^{p,*} \overset{d_I}{\to} I^{p+1,*})$ and $BI^{p,*} = im(I^{p-1,*} \overset{d_I}{\to} I^{p,*})$, so that $H^p_I(I^{*,*}) = ZI^{p,*}/BI^{p,*}$.

Proof Without restriction, let $A^n = 0$ be for $n < 0$. Then we have a chain of morphisms

$$ZA^0 \hookrightarrow A^0 \twoheadrightarrow BA^1 \hookrightarrow ZA^1 \hookrightarrow A^1 \twoheadrightarrow BA^2 \hookrightarrow \dots$$

If we choose injective resolutions $ZA^0 \hookrightarrow ZI^{0,*}$ and $BA^1 \hookrightarrow BI^{1,*}$, then by 6.9 we obtain an exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & ZI^{0,*} \arrow[r, hook] & I^{0,*} \arrow[r, two heads] & BI^{1,*} \arrow[r] & 0 \\
0 \arrow[r] & ZA^0 \arrow[u, hook] \arrow[r, hook] & A^0 \arrow[u, hook] \arrow[r, two heads] & BA^1 \arrow[u, hook] \arrow[r] & 0 .
\end{tikzcd}$$

If, moreover, we choose an injective resolution $H^1(A^*) \hookrightarrow HI^{1,*}$, then, by 6.9, we get an exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & BI^{1,*} \arrow[r] & ZI^{1,*} \arrow[r] & HI^{1,*} \arrow[r] & 0 \\
0 \arrow[r] & BA^1 \arrow[u, hook] \arrow[r, hook] & ZA^1 \arrow[u, hook] \arrow[r] & H^1(A^*) \arrow[u, hook] \arrow[r] & 0 .
\end{tikzcd}$$

Now we choose an injective resolution $B^2A^* \hookrightarrow BI^{2,*}$ and we obtain an exact sequence of injective resolutions

$$\begin{tikzcd}
0 \arrow[r] & ZI^{1,*} \arrow[r] & I^{1,*} \arrow[r] & BI^{2,*} \arrow[r] & 0 \\
0 \arrow[r] & ZA^1 \arrow[u, hook] \arrow[r, hook] & A^1 \arrow[u, hook] \arrow[r] & BA^2 \arrow[u, hook] \arrow[r] & 0 .
\end{tikzcd}$$

If we continue like this, we obtain a naive double complex $I^{*,*}$ with a co-argumentation

$$A^* \to I^{*,*}$$

which is a resolution of A^* in the category of complexes, and where the first differential d_I is given by the composition

$$d_I^{p,*} : I^{p,*} \twoheadrightarrow BI^{p+1,*} \hookrightarrow ZI^{p+1,*} \hookrightarrow I^{p+1,*},$$

so that $BI^{p+1,*}$ is really the image of $d_I^{p_i}$ and $ZI^{p,*}$ is the kernel of $d_I^{p+1,*}$, and moreover $HI^{p,*} = ZI^{p,*}/BI^{p,*} = H^p(I^{*,*}, d_I)$.

Now we come to the **Proof of theorem 6.7**. Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories with enough injectives, where F maps injectives to G-acyclic objects.

Let X be an object in $\mathcal{A}$ and let $X \hookrightarrow I^*$ an injective resolution. For $A^* = FI^*$ we have $H^n(A^*) = H^n(FI^*) = R^nF(A)$ by definition. Let

$$\begin{tikzcd} J^{*,*} \\ A^* \arrow[u, hook] \end{tikzcd}$$

be a Cartan-Eilenberg-resolution as in Theorem 6.11.

We consider the first spectral sequence of the double complex $GJ^{*,*}$

$${}_IE_1^{p,q} = H^q(GJ^{p,*}) \Rightarrow E^{p+q} = H^{p+q}(Tot(J^{*,*}))\,.$$

Since $A^p = FI^p$ is G-acyclic, $H^q(GJ^{p,*}) = 0$ for $q > 0$ and

$$H^0(GJ^{p,*}) = \ker(GJ^{p,0} \xrightarrow{d_{II}} GJ^{p,1}) = GA^p\,.$$

This implies that the edge morphisms

$$R^n(GF)(X) = H^n(GFI^*) = H^n(GA^*) \to H^n(Tot(GJ^{*,*}))$$

are all isomorphisms.

We consider the second spectral sequence of the double complex $GJ^{*,*}$,

$${}_{II}E_1^{p,q} = H^p(GJ^{*,q}) \Rightarrow E^{p,q} = H^{p,q}(Tot(GJ^{*,*}))$$

We have

$${}_{II}E_1^{p,q} = Z_IGJ^{p,q}/B_IGJ^{p,q} = G(Z_IJ^{p,q}/B_IJ^{p,q})$$

and by assumption

$$R^nFX = H^n(A^*) \hookrightarrow H_I(J^{n,*})$$

is an injective resolution. Therefore we have

$${}_{II}E_2^{p,q} = H_{II}^q(GH_I(J^{n,*})) = R^pG(R^qFA)$$

and we obtain the desired Grothendieck-spectral sequence.

7 The étale site

Definition 7.1 (a) A class E of morphisms of schemes is called **admissible**, if the following holds

(M1) all isomorphisms are in E,

(M2) E is closed under compositions (if $\varphi : Y \to X$ and $\psi : Z \to Y$ are in E, then so is $\psi \circ \varphi : Z \to X$),

(M3) E is closed under base change (If $\varphi : Y \to X$ is in E and $\psi : X' \to X$ any morphism, then the base change $\varphi' : Y' = Y \times_X X' \to X'$ is in E).

(b) Let E be admissible. An E-covering $(U_i \xrightarrow{g_i} X)_{i \in I}$ of a scheme X is a family of E-morphisms (morphisms in E) with $\bigcup\limits_i g_i(U_i) = X$.

Example 7.2 Important examples of admissible classes are

(a) the class (Zar) of all open immersions,

(b) the class (ét) of all étale morphisms,

(c) the class (fl) of all flat morphisms which are locally of finite type.

Remark 7.3 If the considered schemes are not locally noetherian, one should replace "of finite type" by "of finite presentation", and the same should be done in the following recollection.

Recollection 7.4 (a) A morphism $f : Y \to X$ of schemes is called **unramified**, if it has the following properties for all $y \in Y$ and $x = f(y) \in X$.

(i) f is locally of finite type, $\mathfrak{m}_x \mathcal{O}_{Y,y} = \mathfrak{m}_y$, where $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ are the local rings at y and x and $\mathfrak{m}_y$ and $\mathfrak{m}_x$ are their maximal ideals, and for all $y \in Y$, $k(y)/k(x)$ is a finite separable field extension.

(ii) f is locally of finite type, and $\Omega^1_{Y/X} = 0$.

(iii) f is locally of finite type, and the diagonal $\Delta_{Y/X} : Y \hookrightarrow Y \times_X Y$ is an open immersion.

(iv) f is locally of finite type and formally unramified.

(b) A morphism $f : Y \to X$ is called **étale**, if it has the following equivalent conditions:

(i) f is flat and unramified,

(ii) f is locally of finite type and formally étale.

Lemma/Definition 7.5 (a) Let $\mathcal{C}$ be a category of schemes and let E be an admissible class of morphisms. Then, with the E-coverings, $\mathcal{C}$ forms a site, which is denoted by $\mathcal{C}_E$.

(b) The **small** E**-site** of a scheme X consists of all X-schemes $Y \to X$ for which the structural morphism $Y \to X$ lies in E, equipped with the E-coverings, and is denoted by X_E.

Definition 7.6 In particular, this defines the small étale site $X_{\text{ét}}$ of a scheme X. In general, one understands an étale sheaf on X as a sheaf F on the small site $X_{\text{ét}}$ and the étale cohomology

$$H^i_{\text{ét}}(X, F) := H^i(X_{\text{ét}}, F)$$

is the cohomology on this site, defined by Definition 5.3.

Remark 7.7 All morphisms in $X_{\text{ét}}$ are étale: If

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "\varphi_1"'] & & Y_2 \arrow[dl, "\varphi_2"] \\ & X & \end{tikzcd}$$

is a morphism of étale X-schemes (i.e., a commutative diagram with étale φ_1 and φ_2), then f is étale.

Lemma/Definition 7.8 Let E and E' be two admissible classes of morphisms. A morphism

$$f : X' \to X$$

of schemes defines a morphism of sites

$$f : X'_{E'} \to X_E$$

via

$$\begin{array}{rccc} f^0 : & X_E & \to & X'_{E'} \\ & V & \mapsto & V \times_X X' \,, \end{array}$$

if the base change $V_1 \times_X X' \to V_2 \times_X X'$ is in E' for any $V_1 \to V_2$ in X_E. In this case, we get functors

$$\begin{array}{rccc} f_P : & Pr(X'_E) & \to & Pr(X_E) \\ f^P : & Pr(X_E) & \to & Pr(X'_{E'}) \\ f_* : & Sh(X'_{E'}) & \to & Sh(X_E) \\ f^* : & Sh(X_E) & \to & Sh(X'_{E'}) \,, \end{array}$$

where f^P is left adjoint to f_P and f^* left adjoint to f_*. The functors f_P, f^P and f^* are exact, the functor f_* is left exact.

In particular, this holds for $E' = E$, for example for $E' = E = \text{ét}$. Hence we have adjoint pairs

$$\begin{array}{rcccrccc} f_P : & Pr(X') & \to & Pr(X) & f^P : & Pr(X) & \to & Pr(X') \,, \\ f_* : & Sh(X'_{\text{ét}}) & \to & Sh(X_{\text{ét}}), & f^* : & Sh(X_{\text{ét}}) & \to & Sh(X'_{\text{ét}}) \,, \end{array}$$

where f_* is left exact and the other functors are exact.

Proof of the claims: f^0 defines a morphism of sites: If $Y \to X$ in X_E (hence an E-morphism) and $(U_i \to Y)$ is an E-covering (hence a surjective family of E-morphisms), then, by assumption, $Y \times_X X' \to X'$ is in $X'_{E'}$. Furthermore,

$$(U_i \times_X X' \to Y \times_X X')$$

is an E'-covering: by assumption, the morphisms are E'-morphisms, and for *every* surjective family

$$(Y_i \xrightarrow{\pi_i} Y)$$

of morphisms of schemes and *every* morphism of schemes $X' \to X$, the family

$$(Y_i \times_X X' \xrightarrow{\pi'_i} Y \times_X X' =: Y')$$

is again a surjective family: If $y' \in Y'$, with image y in Y, then there exists an i for which $\pi_i^{-1}(y) = (Y_i)_y = Y_i \times_Y k(y)$ is non-empty. Then we have $(\pi_i')^{-1}(y') = (Y_i \times_X X') \times_{Y \times_X X'} y' = Y_i \times_Y y' = (Y_i \times_Y y) \times_y k(y') = (Y_i \times_Y k(y)) \times_{k(y)} k(y') \neq \emptyset$. This shows the property (S1) from 2.7 for f^0. Furthermore, for any $Z \to X$ in X_E and any X-morphism $Z \to Y$ (i.e. any morphism in X_E) the canonical morphism

$$(U_i \times_Y Z) \times_X X' \to (U_i \times_X X') \times_{Y \times_X X'} (Z \times_X X')$$

is an isomorphism. This shows 2.7 (S2).

The claims on exactness follow from Theorem 4.4, since finite products and fiber products, and therefore finite limits exist in X_E and $X'_{E'}$ (see 4.A.22).

Corollary 7.9 By assumption of Lemma/Definition 7.8, the functor $f_* : Sh(X'_{E'}) \to Sh(X_E)$ maps injective sheaves to injective sheaves.

Proof Since f_* is left exact, and has the exact left adjoint f^*, this follows from the next lemma.

Lemma 7.10 Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. If F has a left exact left adjoint functor $G : \mathcal{B} \to \mathcal{A}$, then F maps injective objects to injective objects.

Proof: Let I be an injective object in $\mathcal{A}$. Then FI is injective if and only if for every monomorphism $B' \hookrightarrow B$ the morphism

$$Hom_{\mathcal{B}}(B, FI) \to Hom_{\mathcal{B}}(B', FI)$$

is surjective. By the functoriality of the adjunction, we obtain a commutative diagram

$$\begin{tikzcd}
Hom_{\mathcal{B}}(B, FI) \arrow[r] \arrow[d, equal, "\wr"] & Hom_{\mathcal{B}}(B', FI) \arrow[d, equal, "\wr"] \\
Hom_{\mathcal{A}}(GB, I) \arrow[r] & Hom_{\mathcal{A}}(GB', I)
\end{tikzcd}$$

where $GB' \hookrightarrow GB$ is a monomorphism, because G is left exact. Since I is injective, the bottom map is a monomorphism, and hence the top map, too.

Corollary 7.11 In the setting of Lemma 7.8, for every sheaf $\mathcal{F}'$ on X_E, we have a Grothendieck-Leray-spectral sequence

$$E_2^{p,q} = H^p(X_E, R^q f_* \mathcal{F}') \Rightarrow H^{p+q}(X_{E'}, \mathcal{F}) \,.$$

Proof: This follows from Theorem 6.7, since

$$H^0(X_E, f_* \mathcal{F}') = H^0(X_{E'}, \mathcal{F}')$$

and since the left exact functor f_* maps injectives to injectives, which then are $H^0(X_{E'}, -)$-acyclic objects.

Definition 7.12 Let X be a scheme.

(a) A geometric point of X is a morphism

$$i_{\overline{x}} : \overline{x} = \mathrm{Spec}(\Omega) \to X\,,$$

where Ω is a separably closed (e.g. an algebraically closed) field. If $x = i_{\overline{x}}(\overline{x}) \in X$, then we say that $\overline{x}$ is a geometric point over x.

(b) For an étale presheaf P on X and a geometric point as above,

$$P_{\overline{x}} := (i_{\overline{x}}^P P)(\overline{x}) \in \underline{Ab}$$

is called the stalk of P at $\overline{x}$.

Remark 7.13 (a) The functor

$$\begin{array}{ccc} Pr(X_{\text{ét}}) & \to & \underline{Ab} \\ P & \mapsto & P_{\overline{x}} \end{array}$$

is exact. In fact, by 7.8, $i_{\overline{x}}^P : Pr(X_{\text{ét}}) \to Pr(\overline{x}_{\text{ét}})$ is exact. Furthermore, the functor

$$\begin{array}{ccc} Pr(\overline{x}_{\text{ét}}) & \to & \underline{Ab} \\ Q & \mapsto & Q(\overline{x}) \end{array}$$

is exact.

(b) The adjunction morphism $P \to (i_{\overline{x}})_P (i_{\overline{x}}^P) P$ induces a homomorphism of abelian groups

$$P(X) \to ((i_{\overline{x}})_P (i_{\overline{x}})^P P)(X) = ((i_{\overline{x}})^P P)(\overline{x}) = P_{\overline{x}}\,.$$

Definition 7.14 Let $F \in Sh(X_{\text{ét}})$, $s \in F(X)$, and $\overline{x}$ be a geometric point.

(a) The image of s under $F(X) \to F_{\overline{x}}$ is denoted by $s_{\overline{x}}$ and is called the germ of s at $\overline{x}$.

(b) If $U \to X$ is étale, then, in general, there is *no* canonical map $F(U) \to F_{\overline{x}}$, but every lift $\overline{x} \to X$ to U defines a canonical map, and we denote the image of $s \in F(U)$ in $F_{\overline{x}}$ again by $s_{\overline{x}}$ (Obviously, there is always a lift if there exists a point $u \in U$, which is mapped to the image $x \in X$ of $\overline{x} \to X$).

Definition 7.15 An **étale neighborhood** of a geometric point $\overline{x} \to X$ is a commutative diagram

$$\begin{tikzcd} \overline{x} \arrow[r] \arrow[dr] & U \arrow[d, "\text{étale}"] \\ & X \end{tikzcd}$$

where $U \to X$ is an étale map as indicated. A morphism of étale neighborhoods is a commutative diagram

$$\begin{tikzcd} & U_2 \arrow[d] \\ \overline{x} \arrow[ur] \arrow[r] \arrow[dr] & U_1 \arrow[d] \\ & X \end{tikzcd}$$

where U_2 and U_1 are étale over X.

Remark 7.16 A commutative diagram of schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}$$

corresponds to an X'-morphism $X' \to Y \times_X X'$, i.e., to a commutative diagram

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "id"'] & & Y \times_X X' \arrow[dl, "pr_2"] \\ & X' & \end{tikzcd}$$

and hence to a section of $pr_2 : Y \times_X X' \to X'$ (in short: $Hom_X(X', Y) \xrightarrow{\sim} Hom_{X'}(X', Y \times_X X'))$. This shows that an étale neighborhood of $\overline{x} \to X$ can be identified with a morphism

$$\overline{x} \to U \times_X \overline{x}$$

in $\overline{x}_{\text{ét}}$, where $U \to X$ is étale, and therefore with an object in the category $I_{\overline{x}}$ for the morphism of sites $\overline{x}_{\text{ét}} \to X_{\text{ét}}$, $U \mapsto U \times_X \overline{x}$ (see 7.8), which is used for the definition of $i^P_{\overline{x}}$. Furthermore, the morphisms of étale neighborhoods of $\overline{x} \to X$ correspond to the morphisms in $I_{\overline{x}}$, to wit, the X-morphisms $U_1 \to U_2$, for which the diagram

$$\begin{tikzcd} & U_1 \times_X \overline{x} \arrow[dd] \\ \overline{x} \arrow[ur] \arrow[dr] & \\ & U_2 \times_X \overline{x} \end{tikzcd}$$

is commutative. Combining this with the definition of $i^P_{\overline{x}}$, we see that for an étale presheaf P on X we have

$$P_{\overline{x}} = \varinjlim P(U)\,, \tag{7.16.1}$$

where the inductive limits runs over all étale neighborhoods of $\overline{x} \to X$.

Lemma 7.17 (a) If $U \to X$ is étale and $s \in F(U)$ non trivial, then there is a geometric point $\overline{x}$ of U with $s_{\overline{x}} \neq 0$ in $F_{\overline{x}}$.

(b) In particular we have $F = 0$ if and only if $F_{\overline{x}} = 0$ holds for all geometric points $\overline{x}$ von X.

Proof (a): If there are no such geometric points, then every point $u \in U$ has an étale neighborhood $V_u \to U$ with $s|_{V_u} = 0$, and by the separateness of F we get $s = 0$ (the V_u form a covering of U).

(b) is obvious from this.

Lemma 7.18 Let $f : \mathcal{S}' \to \mathcal{S}$ be a morphism of sites. If f^P maps sheaves to sheaves, then canonically $f^P a = af^P$ (more precisely: $f^P ia = iaf^P$ for the embeddings $i : Sh(\mathcal{S}) \to Pr(\mathcal{S})$ and $i : Sh(\mathcal{S}') \to Pr(\mathcal{S}')$).

Proof: Let P be a presheaf on $\mathcal{S}$ and let F be a sheaf on $\mathcal{S}'$. Then we have isomorphisms

$$\begin{aligned}
&Hom_{Pr(\mathcal{S}')}(f^P iaP, iF) \cong Hom_{Pr(\mathcal{S})}(iaP, f_P iF) \cong Hom_{Pr(\mathcal{S})}(iaP, if_*F)\\
&\cong Hom_{Sh(\mathcal{S})}(aP, f_*F) \cong Hom_{Pr(\mathcal{S})}(P, if_*F) \cong Hom_{Pr(\mathcal{S})}(P, f_P iF)\\
&\cong Hom_{Pr(\mathcal{S}')}(f^P P, iF) \cong Hom_{Sh(\mathcal{S}')}(af^P P, F)
\end{aligned}$$

This implies the claim: By assumption, $f^P iaP = iG$ for a sheaf G, and the first group is isomorphic to $Hom_{Sh(\mathcal{S}')}(G, F)$. The Yoneda-Lemma implies $G \cong af^P P$, hence $f^P iaP \cong iaf^P P$.

Corollary 7.19 For an étale presheaf P on X and a geometric point $\overline{x}$ of X one has $P_{\overline{x}} = (aP)_{\overline{x}}$.

Proof: For $i_{\overline{x}} : \overline{x} \to X$, one has $P_{\overline{x}} = (i^P_{\overline{x}} P)(\overline{x}) = (ai^P_{\overline{x}} P)(\overline{x}) \overset{7.15}{=} (i^P_{\overline{x}} aP)(\overline{x}) \overset{\text{Def.}}{=} (aP)_{\overline{x}}$, because $i^P_{\overline{x}}$ maps sheaves to sheaves: For a sheaf F on X we have

$$(i^P_{\overline{x}} F)(\coprod_{i\in I} \overline{x}) = \prod_{i\in I} (i^P_{\overline{x}} F)(\overline{x}),$$

since for a diagram

$$\begin{tikzcd}
V = \coprod_{i\in I} \overline{x} \arrow[r, "(f_i)"] \arrow[dr] & U \arrow[d] \\
 & X
\end{tikzcd}$$

one has the factorization

$$\begin{tikzcd}
 & \coprod_{i\in I} U \arrow[d] \\
V = \coprod_{i\in I} \overline{x} \arrow[ur, "\coprod_i f_i"] \arrow[r] & U .
\end{tikzcd}$$

Hence the morphisms above are cofinite in the category I_V, and for $(f^P F)(V)$ we can form the limit over these; furthermore $F(\coprod_i U_i) = \prod_i F(U_i)$.

Corollary 7.20 A sequence

$$0 \to F' \to F \to F'' \to 0 \tag{7.20.1}$$

of étale sheaves on X is exact if and only if the sequences of the stalks

$$0 \to F'_{\overline{x}} \to F_{\overline{x}} \to F''_{\overline{x}} \to 0 \tag{7.20.2}$$

are exact for all geometric points $\overline{x}$ of X.

Proof (a) If (7.20.1) is exact, then

$$0 \to F' \to F \to F''$$

is exact as an sequence of presheaves, hence

$$0 \to F'_{\overline{x}} \to F_{\overline{x}} \to F''_{\overline{x}}$$

is exact by Remark 7.13 (a).

(b) Conversely, let $0 \to F'_{\overline{x}} \to F_{\overline{x}} \to F''_{\overline{x}}$ be exact for all geometric points $\overline{x}$ of X.

(i) Then $F' \to F$ is a monomorphism: If $U \to X$ is étale and $s \in F'(U)$ is in the kernel of $F'(U) \to F(U)$, then for every geometric point $\overline{x}$ of U, the germ $s_{\overline{x}}$ of s is in the kernel of $F'_{\overline{x}} \to F_{\overline{x}}$, therefore zero, as this map is injective. But this implies $s = 0$ by 7.17 (a).

(ii) Let s be in the kernel of $F(U) \to F''(U)$ for $U \to X$ étale. By assumption, for every geometric point $\overline{x}$ of U, the germ $s_{\overline{x}}$ lies in the stalk $F'_{\overline{x}} \hookrightarrow F_{\overline{x}}$. Then for every $u \in U$ there is an étale morphism $V_u \to U$, such that $s|_{V_u}$ lies in the subgroup $F'(V_u) \subseteq F(V_u)$. As $(V_u)_{u\in U}$ is an étale covering of U and F' and F are sheaves, s lies in $F'(U) \subseteq F(U)$. By (i) and (ii),

$$0 \to F' \to F \to F''$$

is exact.

(c) Let P be the presheaf cokernel of $F \to F''$, i.e., let

$$F(U) \to F''(U) \to P(U) \to 0$$

be exact for all étale morphisms $U \to X$. Then

$$F \to F'' \to aP \to 0$$

is an exact sequence of sheaves, as the functor a (associated sheaf) is exact and $aF = F$, $aF' = F'$. Then we have the equivalences

$$\begin{array}{ll} & F \to F' \text{ epimorphism of sheaves} \\ \Leftrightarrow & aP = 0 \\ \Leftrightarrow & (aP)_{\overline{x}} = 0 \text{ for all geometric points } \overline{x} \text{ (by 7.17 (b))} \\ \Leftrightarrow & P_{\overline{x}} = 0 \text{ for all geometric points } \overline{x} \text{ (by 7.19)} \\ \Leftrightarrow & F_{\overline{x}} \to F'_{\overline{x}} \text{ surjective for all geometric points } \overline{x}, \end{array}$$

because forming stalks is exact on the exact sequence of presheaves

$$F \to F' \to P \to 0$$

(see 7.13 (a)).

Corollary 7.21 A morphism

(7.21.1) $$\varphi : F_1 \to F_2$$

of étale sheaves on X is zero if and only if the maps of stalks

(7.21.2) $$\varphi_{\overline{x}} : (F_1)_{\overline{x}} \to (F_2)_{\overline{x}}$$

are zero for all geometric points $\overline{x}$ of X.

Proof: For the non trivial direction, let F_0 be the kernel of φ. For the exact sequence

$$0 \to F_0 \to F_1 \to F_2 \,,$$

the sequence of stalks

$$0 \to (F_0)_{\overline{x}} \to (F_1)_{\overline{x}} \xrightarrow{0} (F_2)_{\overline{x}}$$

is then exact for all geometric points $\overline{x}$. It follows as in (ii) that for every étale morphism $U \to X$ every section $s \in F_1(U)$ already lies in $F_0(U)$, hence $\varphi_U : F_1(U) \to F_2(U)$ is the zero map.

8 The étale site of a field

The following theorem is Grothendieck's version of (infinite) Galois theory.

Theorem 8.1 Let k be a field, let k_s be a separable closure of k and let $G_k := Gal(k_s/k)$ be the absolute Galois group of k. Then the functor

$$\begin{array}{rcl} \phi = Hom_k(\mathrm{Spec}(k_s), -) : \mathrm{Spec}(k)_{\text{ét}} & \to & \mathcal{C}(G_k) = \text{ category of discrete } G_k\text{-sets} \\ Y & \mapsto & \phi(Y) := Y(k_s) := Hom_k(\mathrm{Spec}(k_s), Y) \end{array}$$

is an equivalence of sites, where the Grothendieck topology on $\mathcal{C}(G_k)$ is given by the surjective families $(M_i \to M)_i$.

Remarks 8.2 (a) With the Krull topology (where the subgroups $Gal(k_s/L)$, for finite extensions L/k, form a basis of the neighborhoods of 1 in G_k), G_k is a profinite group, i.e., a projective limit of finite groups

$$G_k = \varprojlim_{L/k \text{ fin. gal., } L \subseteq k_s} Gal(L/k)\,.$$

(b) G_k operates from the left on k_s, thus from the right on $\mathrm{Spec}\, k_s$, thus from the left on $\phi(Y)$.

(c) A G_k-set M is called discrete, if for every $m \in M$ the stabilizer

$$Stab(m) := \{g \in G_k \mid gm = m\}$$

is open (and thus of finite index) in G_k.

We use the following lemma.

Lemma 8.3 If $Y \to \mathrm{Spec}(k)$ is étale, then one has $Y = \coprod_{i \in I} \mathrm{Spec}(L_i)$, where L_i/k are finite separable field extensions. If Y is of finite type over k, then I is finite.

Proof: The last claim is well-known. In general, every $y \in Y$ has an open neighborhood $U = \coprod_{i=1}^{r} \mathrm{Spec}(L_i)$. This shows that every point is open, so that Y carries the discrete topology. This implies the first claim.

Proof of Theorem 8.1: 1) A quasi-inverse functor to ϕ is the functor

$$\psi : M = \coprod_{j \in J} M_j \quad \mapsto \quad \coprod_{j \in J} \mathrm{Spec}(Hom_{G_k}(M_j, k_s))\,.$$

Here, let M_j be connected, i.e., let G_k operate transitively on M_j, and $Hom_{G_k}(M_j, k_s)$ becomes a k-algebra by the k-algebra structure of k_s. Obviously, every connected discrete G_k-set is of the form G_k/U with $U \leq G_k$ open, and the assignment is

$$G_k/U \mapsto \mathrm{Spec}(k_s^U)\,.$$

To show that ψ is quasi inverse to ϕ it suffices to check this for connected G_k-sets or, respectively, for finite separable field extensions. Let M be a connected discrete G_k-set, without restriction $M = G_k/U$ for an open subgroup $U \subseteq G_k$. Then

$$\begin{array}{rcl} \psi(M) = Hom_{G_k}(G_k/U, k_s) & \tilde{\rightarrow} & k_s^U =: L \subseteq k_s \\ \alpha & \mapsto & \alpha(1) \end{array}$$

is an isomorphism of k-algebras. Conversely

$$\phi(\mathrm{Spec}(L)) = Hom_k(\mathrm{Spec}(k_s), \mathrm{Spec}(L)) \cong Hom_k(L, k_s) \cong G_k/U\,,$$

is an isomorphism of discrete G_k-sets, by mapping the embedding $L \hookrightarrow k_s$ to $1 \in G_k$.

By choosing a k-embedding $L \hookrightarrow k_s$ for every finite separable field extension L/k, we obtain an equivalence of categories between étale k-schemes Y and discrete G_k-sets. From these facts it follows that ϕ is also an equivalence of sites:

2) One has $\phi(Y' \times_Y Y'') = (Y' \times_Y Y'')(k_s) = Y'(K_s) \times_{Y(k_s)} Y''(k_s)$

3) If $\mathrm{Spec}\, L' \to \mathrm{Spec}\, L$ is étale, then $L \subseteq L'$ is a separable field extension of k in k_s and the map $Hom_k(L', k_s) \twoheadrightarrow Hom_k(L, k_s)$ is surjective, as known from classical algebra.

Corollary 8.4 We have an equivalence of categories

$$\begin{array}{rcl} Sh(\mathrm{Spec}(k)_{\text{ét}}) & \tilde{\rightarrow} & (\text{discrete } G_k\text{-modules})\,, \\ F & \mapsto & F_{\overline{x}} \end{array}$$

where $F_{\overline{x}}$ is the stalk in the geometric point $\overline{x} = \mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$.

Proof: We show a more general fact.

Definition 8.5 For a site $\mathcal{S} = (\mathcal{X}, \mathcal{T})$, let $(\mathcal{X}, \mathcal{T})^\sim$ be the category of sheaves of sets on $\mathcal{S}$; this is also called the **topos** to $\mathcal{S}$.

Theorem 8.6 For a field k, the functor

$$\begin{array}{rcl} (\mathrm{Spec}(k)_{\text{ét}})^\sim & \tilde{\rightarrow} & (\text{discrete } G_k\text{-sets})\,, \\ F & \mapsto & F_{\overline{x}} \end{array}$$

is an equivalence of categories.

Corollary 8.4 follows, because in 8.6 the abelian sheaves correspond to the discrete G_k-modules (as the abelian group objects in these categories).

Proof of Theorem 8.6: We have functorial isomorphisms for every sheaf F of sets on $\mathrm{Spec}(k)_{\text{ét}}$:

$$F_{\overline{x}} = \varinjlim_{\substack{k \subseteq L \subseteq k_s \\ L/k \text{ fin. separable}}} F(\mathrm{Spec}(L))\,.$$

Since the equivalence of categories in Theorem 8.1 assigns

$$\mathrm{Spec}(L) \mapsto G_k/U\,,$$

where $U = Gal(k_s/L) \leq G_k$, and, as above, the $k \subseteq L \subseteq k_s$ correspond to all open subgroups $U \leq G_k$, the claim follows from the following theorem.

Theorem 8.7 (a) Let G be a group and $\mathcal{M}(G)$ the category of the left G-sets. Then the surjective families $(M_i \to M)$ of G-sets form a Grothendieck topology $\mathcal{T}_G$ on $\mathcal{M}(G)$, the so-called canonical topology. The functor

$$\begin{array}{rcl} \Phi : (\mathcal{M}(G), \mathcal{T}_G)^{\sim} & \to & \mathcal{M}(G) \\ F & \mapsto & M_F = F(G) \end{array}$$

is an equivalence of categories with quasi-inverse

$$F_M = Hom_G(-, M) \overset{\Psi}{\longleftarrow\!\shortmid} M\,.$$

(b) Let G be a profinite group (i.e., a projective limit of finite groups, equipped with the profinite topology) and let $\mathcal{C}(G)$ be the category of the continuous G-sets (with respect to the discrete topology on this sets). Then the surjective families $(M_i \to M)$ of discrete G-sets form a Grothendieck topology, the so-called canonical topology, which again is denoted by $\mathcal{T}_G$. The functor

$$\begin{array}{rcl} \Phi : (\mathcal{C}(G), \mathcal{T}_G)^{\sim} & \to & \mathcal{C}(G) \\ F & \mapsto & M_F = F(G) := \varinjlim\limits_{U \leq G \text{ open}} F(G/U) \end{array}$$

is an equivalence of categories with quasi inverse

$$F_M = Hom_G(-, M) \overset{\Psi}{\longleftarrow\!\shortmid} M$$

Proof (a) Let G be a discrete group.
(i) Then $F(G)$ is a left G-set: for $g \in G$, the right translation with g

$$\begin{array}{rccc} R_g : & G & \to & G \\ & g' & \mapsto & g'g \end{array}$$

is a morphism of left G-sets and we define a left G-operation on $F(G)$ by

$$gx = F(R_g)(x)\,.$$

We have $(F(R_{gg'}) = F(R_{g'} \circ R_g) = F(R_g) \circ F(R_{g'})$, as F is contravariant). Obviously, here the assignment $F \leadsto F(G)$ is functorial.
(ii) F_M is a sheaf: easy.
(iii) We have a functorial isomorphism $M_{F_M} \xrightarrow{\sim} M$, since the map

$$\begin{array}{rcl} Hom_G(G, M) & \xrightarrow{\sim} & M \\ f & \mapsto & f(1) \end{array}$$

is a bijection of G-sets: $gf \mapsto gf(1) = f(1g) = f(g) = gf(1)$.

(iv) We have a functorial isomorphism $F_{M_F} \xrightarrow{\sim} F$, i.e.,

$$Hom_G(N, F(G)) \xrightarrow{\sim} F(N)\,.$$

In fact, for $N = \coprod_{i\in I} N_i$ we have $F(N) = \prod_{i\in I} F(N_i)$, and $Hom_G(N, F(G)) = \prod_{i\in I} Hom_G(N_i, F(G))$. By considering the orbits, we may thus assume that $N = G/U$ for a subgroup $U \subset G$. Now we consider the sheaf condition for the covering $G \twoheadrightarrow G/U$. We have a bijection of G-sets

$$\begin{array}{rcl} \coprod_{u\in U} G & \to & G \times_{G/U} G \\ g_u & \mapsto & (g, gu)\,; \end{array}$$

and hence the diagram

$$\begin{array}{rcl} F(G/U) \to F(G) & \rightrightarrows & F(G \times_{G/U} G) \xrightarrow{\sim} \prod_{u\in U} F(G) \\ f & \rightrightarrows & \begin{array}{l}(\dots, f, \dots)\\ (\dots, uf, \dots)_{u\in U}\end{array} \end{array}$$

is exakt. This gives canonical bijections

$$\begin{array}{rcl} F(G/U) \xrightarrow{\sim} F(G)^U = \{f \in F(U), uf = f \text{ for all } u \in U\} & \cong & Hom_G(G/U, F(G)) \\ \varphi(1) & \leftarrow\!\shortmid & \varphi \end{array}$$

as wanted.

(b) Let G be profinite.

(i) We have

$$M_F = \varinjlim_{\substack{U \trianglelefteq G \text{ open} \\ \text{normal factor}}} F(G/U)\,,$$

and M_F it becomes a discrete G-Modul, since, by the first case, $F(G/U)$ is a G/U-module.

(ii) It follows easily again that F_M is a sheaf.

(iii) For M in $\mathcal{C}(G)$ we have isomorphisms

$$M_{F_M} = \varinjlim_{V \leq G \text{ open}} Hom_G(G/U, M) \xrightarrow{\sim} \varinjlim_{V \leq G \text{ open}} M^U \xrightarrow{\sim} M\,.$$

(iv) By the first case, for every open subgroup $U < G$ and every open normal subgroup $U' \trianglelefteq G$ with $U' \subseteq U$ we have

$$F(G/U) \xrightarrow{\sim} \{f \in F(G/U') \mid \overline{u}f = f \text{ for alle } \overline{u} \in U/U'\}\,,$$

so that

$$F(G/U) \cong (M_F)^U\,,$$

and from the above we obtain

$$F_{M_F}(N) = Hom_G(N, M_F) \xrightarrow{\sim} F(N)\,,$$

since, for $N = G/U$ with U open in G, we have

$$\begin{array}{rcl} F_{M_F}(G/U) = Hom_G(G/U, M_F) & \cong & (M_F)^U \cong F(G/U) \\ f & \mapsto & f(1)\,. \end{array}$$

Remark 8.8 From 8.1 and (the proof of) 8.4 we get an equivalence of categories

$$\begin{array}{rcl} Sh(\mathrm{Spec}(k)_{\text{ét}}) & \to & \mathcal{C}(G_k) = (\text{discrete } G_k\text{-modules}) \\ F & \mapsto & F_{\overline{x}} \end{array}$$

with quasi-inverse

$$M \mapsto F \text{ with } F(\mathrm{Spec}(L)) = M^{G_L} \text{ for } L/K \text{ finitely separable.}$$

Corollary 8.9 Let k be a field with separable closure k_s, and let $G_k = Gal(k_s/k)$ be the absolute Galois group of k and $\overline{x} : \mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$. Then we have functorial isomorphisms for all étale abelian sheaves F on $\mathrm{Spec}(k)$ and all $i \geq 0$

$$H^i_{\text{ét}}(\mathrm{Spec}(k), F) \xrightarrow{\sim} H^i(G_k, F_{\overline{x}}) \,,$$

which are compatible with long exact sequences of cohomology.

Proof This follows from the equivalence of categories

$$\begin{array}{rcl} Sh(\mathrm{Spec}(k)_{\text{ét}}) & \to & \mathcal{C}(G_k) = (\text{discrete } G_k\text{-modules}) \\ F & \mapsto & F_{\overline{x}} \end{array}$$

and the following facts:
(i) We have canonical functorial isomorphisms (see 8.8)

$$F(k) := F(\mathrm{Spec}(k)) \xrightarrow{\sim} F_{\overline{x}}^{G_k} \,.$$

(ii) By definition, $H^i(G_k, -)$ is the i-th right derived functor of

$$M \mapsto H^0(G_k, M) = M^{G_k} \,.$$

Hence étale cohomology of fields is Galois cohomology.

9 Henselian rings

Henselian rings, and in particular the strictly henselian rings, play the same role in the étale topology as the local rings do in the Zariski topology.

Let A be a local ring with maximal ideal $\mathfrak{m}$ and factor field $k = A/\mathfrak{m}$.

Lemma/Definition 9.1 Let x be the closed point of $X = \mathrm{Spec}(A)$.

A is called henselian, if the following equivalent conditions hold.

(a) If $f \in A[X]$ is monic and $\overline{f} = g_0 \cdot h_0$ with $g_0.h_0 \in k[X]$ monic and coprime (i.e., $\langle g_0, h_0\rangle = k[X]$), then there are monic polynomials $g, h \in A[X]$ with $f = g \cdot h$, $\overline{g} = g_0$ and $\overline{h} = h_0$. Here let $\overline{f} = f$ mod $\mathfrak{m}$ in $k[X]$; similarly for g and h). The polynomials g and h are strictly coprime (i.e., $\langle g, h\rangle = A[X]$).

(a′) If $f \in A[X]$ and $\overline{f} = g_0 \cdot h_0$ where g_0 is monic and g_0 and h_0 are coprime, then there exist $g, h \in A[X]$ with monic g, $f = g \cdot h$, $\overline{g} = g_0$ and $\overline{h} = h_0$.

(b) Any finite A-algebra is a direct product of local rings B_i.

(b′) If B is a finite A-algebra, then every idempotent $e_0 \in B/\mathfrak{m}B$ (i.e., $e_0^2 = e_0$) can be lifted to an idempotent $e \in B$.

(c) If $f : Y \to X$ quasi-finite (see below) and separated, then we have a disjoint decomposition

$$Y = Y_0 \amalg Y_1 \amalg \ldots \amalg Y_r \,,$$

where $x \notin f(Y_0)$ and where, for $i \geq 1$, Y_i is finite over X and $Y_i = \mathrm{Spec}(B_i)$ for a local ring B_i.

(d) If $f : Y \to X$ is étale and if Y has a point y with $f(y) = x$ and $k(x) \xrightarrow{\sim} k(y)$, then f has a section $s : X \to Y$ (i.e., $fs = id_X$).

(d′) Let $f_1, \ldots, f_n \in A[X_1, \ldots, X_n]$ and let $a = (a_1, \ldots, a_n) \in k^n$ with $\overline{f}_i(a) = 0$ for all $i = 1, \ldots, n$ and $\det(\partial\overline{f}_i/\partial x_j(a)) \neq 0$. Then there exists an element $c \in A^n$ with $\overline{c} = a$ and $f_i(c) = 0$ for $i = 1, \ldots, n$.

Definition 9.2 A morphism $f : Y \to X$ of schemes is called **quasi-finite**, if it is finitely presented (for noetherian schemes: of finite type) and has finite fibers (i.e., $f^{-1}(x)$ is finite for all $x \in X$).

If f is étale and finitely presented (noetherian schemes: of finite type), then f is quasi-finite.

Proof of the equivalence of the conditions in 9.1:

(a′) $\Rightarrow$ (a) is trivial, except for the last sentence in (a). But if f is monic, then $A[x]/\langle f\rangle$ is finite over A. Since we have $f \in \langle g, h\rangle$, i.e., $\langle f\rangle \subseteq \langle g, h\rangle$, $D = A[X]/\langle g, h\rangle$ is finite over A as well, and by the Nakayama-Lemma we have $D = 0$, since $D/\mathfrak{m}D = k[x]/\langle g_0, h_0\rangle = 0$.

(a) $\Rightarrow$ (b): Let B be a finite A-algebra. By the going-up-theorem, every maximal ideal of B lies over $\mathfrak{m}$; thus B is local if and only if $B/\mathfrak{m}B$ is local.

Special case: Let $B = A[X]/\langle f\rangle$ be with a monic polynomial f.

If $\overline{f}$ is a power of an irreducible polynomial, then $B/\mathfrak{m}B = k[X]/\langle\overline{f}\rangle$ is local, hence B is local. Otherwise, by (a) we obtain that $f = g \cdot h$ with g, h monic of degrees ≥ 1 and strictly coprime, and with the Chinese remainder theorem we get

$$B = A[X]/\langle f\rangle \;\cong\; A/\langle g\rangle \times A/\langle h\rangle\,.$$

The claim now follows by induction over the number of prime factors of $\overline{f}$.

General case: Assume B is not local. Then there is an element $b \in B$, such that $\overline{b}$ is a non-trivial idempotent in $B/\mathfrak{m}B$ ($B/\mathfrak{m}B$ is an artinian k-algebra, hence a product of local rings). Since b is integral over A, there is a monic polynomial $f \in A[X]$ with $f(b) = 0$. Then we have a ring homomorphism by evaluating $g \in A[X]$ at b

$$\varphi : C = A[X]/\langle f\rangle \to B \quad , \quad X \mapsto b\,.$$

Consider the reduction mod $\mathfrak{m}$

$$\overline{\varphi} : C/\mathfrak{m}C = k[X]/\langle\overline{f}\rangle \to B/\mathfrak{m}B\,.$$

If $\overline{f} = \prod\limits_i p_i^{n_i}$, with irreducible polynomials p_i, then

$$k[X]/\langle\overline{f}\rangle \cong \prod_i k[X]/\langle p_i^{n_i}\rangle\,,$$

and for the quotient im $(\overline{\varphi})$ we have

$$im(\overline{\varphi}) = k[X]/\langle g_0\rangle = \prod_i k[X]/\langle p_i^{m_i}\rangle$$

with $g_0 = \prod\limits_i p_i^{m_i} \mid \overline{f}$. This shows that the idempotent $\overline{b} \in im(\overline{\varphi})$ lifts to an idempotent $\overline{e} \in C/\mathfrak{m}C$ (the decomposition is unique). By the first case there is an idempotent $e \in C$ with e mod $\mathfrak{m} = \overline{e}$, hence $\overline{\varphi(e)} = \overline{b}$. Therefore, $\varphi(e)$ is a non trivial idempotent in B.

This gives a decomposition of rings $B = Be \times B(1-e)$ in two non trivial rings, and the claim follows by induction over the (finite) number of components of $B/\mathfrak{m}B$.

Note: There is a bijective correspondence for commutative rings with unit R:

decomposition into a product		idempotents
$R = R_1 \times R_2$	$\mapsto$	$(1,0)$ and $(0,1)$
$R = Re \times R(1-e)$	$\leftarrow\!\shortmid$	e and $1-e$

(Note also: e idempotent $\Rightarrow$ $1-e$ idempotent).

This shows (b) $\Leftrightarrow$ (b$'$).

(b) $\Rightarrow$ (c): We need:

Theorem 9.3 (Stein-factorisation/Zariskis main theorem) Let $f : Y \to X$ be a quasi-finite, separated morphism of schemes, where X is quasi-compact. Then there is a factorization

$$f : Y \overset{j}{\hookrightarrow} Y' \overset{f'}{\to} X\,,$$

where f' is finite and j is an open immersion.

Remark 9.4 Let $f : Y \to X$ be a morphism of schemes.

(a) f is called affine, if for any open set $U \subseteq X$, $f^{-1}(U)$ is affine as well.

(b) f is called finite, if f is affine, and for any affine open set $U \subseteq X$ the ringhomomorphism $\Gamma(U, \mathcal{O}_X) \to \Gamma(f^{-1}(U), \mathcal{O}_Y)$ is finite.

Proof of Theorem 9.3: Without! See the references in Milne's 'Étale Cohomology', page 6.

Now consider $f : Y \to X = \mathrm{Spec}(A)$, quasi-finite and separated, with a local ring A as above. Let

$$Y \overset{j}{\hookrightarrow} Y' \overset{f'}{\to} X$$

be a Stein factorization as in Theorem 9.3. Since f' is affine, $Y' = \mathrm{Spec}(B')$ is affine. By (b), we get

$$Y' = \coprod_{i=1}^{r} Y_i \,,$$

where $Y_i = \mathrm{Spec}(B_i)$ for a local finite A-algebra B_i. Let

$$Y_* = \coprod_{i \in I} Y_i$$

be the product of those Y_i, whose closed point y_i lies in Y. Then Y_* is open and closed in Y' and lies in Y, because Y_i is the smallest open neighborhood of $y_i \in Y_i$. Therefore Y_* is open and closed in Y as well, and we have

$$Y = Y_0 \amalg Y_* \,,$$

where $x \notin f(Y_0)$, hence (c).

(c) $\Rightarrow$ (d): Let $f : Y \to X = \mathrm{Spec}(A)$ be étale and let $y \in Y$ be a point with $f(y) = x$ and $k(x) \overset{\sim}{\to} k(y)$. By replacing Y with an affine open neighborhood of y, f is quasi-finite and separated without restriction. Then, by (c), we may assume that $Y = \mathrm{Spec}(B)$, where B is local and finite over A. Since f is étale, we have

$$B/\mathfrak{m}B = B/\mathfrak{m}_B = k(y) = k(x) = A/\mathfrak{m} \,.$$

Hence, by the Nakayama-Lemma, B is generated by $1 \in B$ as an A-module. By this we obtain an exact sequence

$$0 \to \mathfrak{a} \to A \to B \to 0 \,,$$

with an ideal $\mathfrak{a} \subseteq A$. Since B is flat over A,

$$0 \to \mathfrak{a} \otimes_A B \to A \otimes_A B \overset{\beta}{\to} B \otimes_A B \to 0$$

is exact. The homomorphism β can be identified with the map

$$i_2 : B \to B \otimes_A B \quad , \quad b \mapsto 1 \otimes b \,.$$

This map is injective, since the composition with $\mu : B \otimes_A B \to B$, $b_1 \otimes b_2 \mapsto b_1 b_2$ is the identity. It follows that $\mathfrak{a} \otimes_A B = 0$, therefore $\mathfrak{a} = 0$, since $A \to B$ is faithfully flat, as a flat homomorphism of local rings (see Corollary 10.4 below). Hence $A \xrightarrow{\sim} B$ is an isomorphism, and this gives the wanted section.

(d) $\Rightarrow$ (d′): Let $B = A[X_1, \ldots, X_n]/\langle f_1, \ldots, f_n\rangle$ and $a = (a_1, \ldots, a_n) \in k^n$ with $\overline{f}_i(a) = 0$ $(i = 1, \ldots, n)$ and $\det(\partial f_i/\partial X_j(a)) \neq 0$ in k. Then a corresponds to a maximal ideal of ($B/\mathfrak{m}B$ hence also of) B; let this be denoted by $\mathfrak{n}$. Then $\det(\partial f_i/\partial X_j)$ is a unit in $B_{\mathfrak{n}}$, thus there is an element $b \in B \smallsetminus \mathfrak{n}$ such that $\det(\partial f_i/\partial X_j)$ is a unit in B_b. But we have

$$B_b \cong A[X_1, \ldots, X_n, T]/\langle f_1, \ldots, f_n, bT - 1\rangle$$

and $\det(\partial f_i/\partial X_j)b$ is the corresponding Jacobian determinant, hence a unit. By the Jacobian criterion, B_b is étale over A. Furthermore, $\mathfrak{n}$ gives a maximal ideal of B_b over $\mathfrak{m}$ with residue field isomorphic to $k = A/\mathfrak{m}$. By (d), a section $s : \mathrm{Spec}(A) \to \mathrm{Spec}(B_b)$ exists, i.e., an A-homomorphism $B_b \to A$ and this gives an element $c \in A^n$ with $f_i(c) = 0$ for $i = 1, \ldots, n$ and $\overline{c} = a$ (since $\mathfrak{n}$ lies over $\mathfrak{m}$).

(d′) $\Rightarrow$ (a′): Let $f(X) = a_nX^n + \ldots + a_1X + a_0 \in A[X]$ and let $\overline{f} = g_0 \cdot h_0$ monic with g_0 of degree ≥ 1. Then we have

$$f(X) = g(X) \cdot h(X) = (X^r + b_{r-1}X^{r-1} + \ldots + b_0)(c_sX^s + \ldots + c_0)$$

with $r + s = n$, if and only if $(b_0, \ldots, b_{r-1}, c_0, \ldots, c_s) \in A^{n+1}$ solves the following system of equations in the $n + 1$ variables $(X_0, \ldots, X_{r-1}, Y_0, \ldots, Y_s)$:

$$\begin{array}{rcl} X_0Y_0 & = & a_0 \\ X_0Y_1 + X_1Y_0 & = & a_1 \\ X_0Y_2 + X_1Y_1 + X_2Y_0 & = & a_2 \\ & \vdots & \\ X_{r-1}Y_s + Y_{s-1} & = & a_{n-1} \\ Y_s & = & a_n \end{array} \tag{9.1.1}$$

$(n + 1$ equations). The corresponding Jacobian is

$$J = \det \begin{array}{c} \quad\overbrace{\qquad\qquad\qquad\qquad}^{r} \quad \overbrace{\qquad\qquad\qquad}^{s} \\ \begin{pmatrix} Y_0 & 0 & & & X_0 & 0 & \\ Y_1 & Y_0 & & & X_1 & X_0 & \\ Y_2 & Y_1 & Y_0 & & X_2 & & X_0 \\ \vdots & \vdots & \vdots & & \vdots & & \\ Y_s & \vdots & & & & & \\ & Y_s & \vdots & & \vdots & \vdots & \\ & & \vdots & & 1 & \vdots & \\ & & & & \vdots & 1 & \end{pmatrix} \end{array}$$

This is just $res(G, H)$, the resultant of the two polynomials

$$\begin{array}{rcl} G & = & x^r + X_{r-1}x^{r-1} + \ldots + X_0 \\ H & = & Y_sx^s + Y_{s-1}x^{s-1} + \ldots + Y_0 \end{array}$$

By (d′) there is a solution of (9.1.1), if $Res(g_0, h_0) \neq 0$ in k (because then the vector $(\overline{b}, \overline{c})$ of the coefficients of g_0 resp. h_0 solves the system (9.1.1) modulo $\mathfrak{m}$ and we have $J(\overline{b}, \overline{c}) \neq 0$).

But by classical algebra, $Res(g_0, h_0)$ is 0 if and only if $\deg(g_0) < r$ and $\deg(h_0) < s$, or if g_0 and h_0 have a common factor; by assumption, this is not the case.

Corollary 9.4 If A is henselian, then any local ring B finite over A is henselian. In particular, any factor ring A/J is henselian.

Proof: This follows with criterion 9.1 (b), since a finite B-algebra is finite over A.

Corollary 9.5 If A is henselian, then the functor

$$B \leadsto B \otimes_A k = B/\mathfrak{m}B$$

gives an equivalence of categories

$$\begin{pmatrix} \text{finite étale} \\ A\text{-algebras} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \text{finite étale} \\ k\text{-algebras} \end{pmatrix} .$$

Proof: This follows with the criteria of 9.1(b), (b′) and (d). Details: left to the readers!

Every field is a henselian ring, as well as every artinian ring (because every artinian ring is a product of local rings). Furthermore we have:

Proposition 9.6 Every complete local ring is henselian.

Proof We use criterion 9.1 (d). Let B be an étale A-algebra, and let $s_0 : B/\mathfrak{m}B \to k$ be a section of $k \to B/\mathfrak{m}B$. To find a section

$$s : B \to A \cong \varprojlim_r A/\mathfrak{m}^r$$

of $A \to B$, it suffices to find compatible A-linear maps for all $r \geq 1$

$$s_r : B \to A/\mathfrak{m}^r .$$

For $r = 1$ we take $s_1 : B \xrightarrow{can} B/\mathfrak{m} \xrightarrow{s_0} A/\mathfrak{m} = k$. If s_r is found for some $r \geq 1$, then the existence of s_{r+1} follows from the formal smoothness of B over A: The lift s_{r+1} of s_r exits in the diagram

$$\begin{tikzcd} B \arrow[r, "s_r"] \arrow[dr, dashed, "s_{r+1}"] & A/\mathfrak{m}^r \\ A \arrow[u] \arrow[r] & A/\mathfrak{m}^{r+1} \arrow[u, "\varphi_r"'] \end{tikzcd} ,$$

because $\ker(\varphi_r) = \mathfrak{m}^r/\mathfrak{m}^{r+1}$ is a nilpotent ideal.

Lemma/Definition 9.7 Let A be a local ring. There is a henselian ring A^h with the following universal properties: There is a local homomorphism $i : A \to A^h$, and any local homomorphism $\varphi : A \to B$ into a henselian local ring B factorizes uniquely over i (we have a unique homomorphism $\tilde{\varphi}$, which makes the following diagram commutative):

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & A^h \arrow[dl, dashed, "\exists !\, \tilde{\varphi}"] \\ & B & \end{tikzcd}$$

The ring A^h is called the henselization of A.

To construct A^h we need

Definition 9.8 Let A be local with maximal ideal $\mathfrak{m}$. An étale (resp. essentially étale) neighborhood of $(A, \mathfrak{m})$ is a pair $(B, \mathfrak{n})$, such that B is an étale A-algebra (resp. a localization of an étale A-algebra) and $\mathfrak{n} \subseteq B$ is an ideal over $\mathfrak{m}$, such that the induced map $k = A/\mathfrak{m} \to B/\mathfrak{n} = k(\mathfrak{n})$ is an isomorphism (then $\mathfrak{n}$ is a maximal ideal as well).

Lemma 9.9 (a) If $(B, \mathfrak{n})$ and $(B', \mathfrak{n}')$ are (essentially) étale neighborhoods of $(A, \mathfrak{m})$ with $\mathrm{Spec}(B')$ connected, then there is at most one an A-Homomorphism $f : B \to B'$ with $f^{-1}(\mathfrak{n}') = \mathfrak{n}$.

(b) If $(B, \mathfrak{n})$ and $(B', \mathfrak{n}')$ are (essentially) étale neighborhoods of $(A, \mathfrak{m})$, then there is an (essentially) étale neighborhood $(B'', \mathfrak{n}'')$ of A and A-homomorphisms

$$\begin{tikzcd} B \arrow[dr, "f"] & \\ & B'' \\ B' \arrow[ur, "f'"'] & \end{tikzcd}$$

with $f^{-1}(\mathfrak{n}'') = \mathfrak{n}$ and $(f')^{-1}(\mathfrak{n}'') = \mathfrak{n}'$.

Proof: (a) follows from the following, more general result:

Lemma 9.10 Let $f, g : Y' \to Y$ be morphisms of X-schemes, where Y' is connected and Y is étale and separated over X. If there is a point $y' \in Y'$ with $f(y') = g(y') = y$, and such that the maps $k(y) \to k(y')$ induced by f and g are equal, then we have $f = g$.

Proof Let $\Gamma_f, \Gamma_g : Y' \to Y' \times_X Y$ be the graphs of f and g, respectively ($\Gamma_f = (id_{Y'}, f)$ similarly for g). These are sections of $pr_1 : Y' \times_X Y \to Y'$, and pr_1 is étale and separated as a base change of $Y \to X$. The assumption implies that $\Gamma_f(y') = \Gamma_g(y')$. Then $\Gamma_f = \Gamma_g$ (see Milne, Étale cohomology, I Cor. 3.12), and the claim follows, since $f = pr_2 \circ \Gamma_f$ and $g = pr_2 \circ \Gamma_g$.

For (b) consider $B'' = B \otimes_A B'$. The homomorphisms $B \to k(\mathfrak{n}) = k$ and $B' \to k(\mathfrak{n}') = k$ induce a homomorphism $\alpha : B'' \twoheadrightarrow k$. If $\mathfrak{n}'' = \ker \alpha$, then $(B'', \mathfrak{n}'')$ has the required property.

The above implies that the connected étale (resp. essential étale) neighborhoods of A form an inductive system, and we define

$$(A^h, \mathfrak{m}^h) = \varinjlim_{\substack{(B,\mathfrak{n})\text{ étale} \\ \text{neighb. of } (A,\mathfrak{m})}} (B, \mathfrak{n}) = \varinjlim_{\substack{(B,\mathfrak{n})\text{ ess. étale} \\ \text{neighb. of } (A,\mathfrak{m})}} (B, \mathfrak{n}) .$$

Note: the étale A-algebras B are of finite presentation and therefore form an index *set* without restriction.

1) A^h is local with maximal ideal $\mathfrak{m}^h$: It suffices to show that $A^h \smallsetminus \mathfrak{m}^h$ consists of units. Let $x \in A^h$, represented by $y \in B$, $(B, \mathfrak{n})$ étale neighborhood of $(A, \mathfrak{m})$. If $x \notin \mathfrak{m}^h$, then $y \notin \mathfrak{n}$,

hence y is a unit in $B_{\mathfrak{n}}$. Therefore there is an element $b \in B - \mathfrak{n}$ such that y is a unit in B_b. Then $(B_b, \mathfrak{n}_b)$ is an étale neighborhood of $(A, \mathfrak{m})$, and the image of the inverse of y in B_b is an inverse of x, i.e., x is a unit.

2) Obviously we have $k \xrightarrow{\sim} \varinjlim B/\mathfrak{n}_B = A^h/\mathfrak{m}^h$.

3) $A \to A^h$ ist a local homomorphism, since $\mathfrak{m}$ is mapped into $\mathfrak{m}^h$.

4) A^h is henselian: We use the section criterion 9.1 (d): Let $A^h \to C$ be étale, $\mathfrak{c} \subseteq C$ be an ideal over $\mathfrak{m}^h$ with $k = k(\mathfrak{m}^h) \xrightarrow{\sim} k(\mathfrak{c})$. Since C is of finite type over $A^h = \varinjlim B$, there exists an étale neighborhood $(B_0, \mathfrak{n}_0)$ of $(A, \mathfrak{m})$ with $C = B_0 \otimes_A A^h$ (Consider a presentation $C = A^h[X_1, \ldots, X_n]/\langle f_1, \ldots, f_m\rangle$ and a B_0 such that the finitely many coefficients of the f_i lie in the image of $B_0 \to A^h$). Then we obtain a section

$$C = B_0 \otimes_A \varinjlim B \to \varinjlim B$$

of $A^h \to C$: Without restriction, we consider the cofinite family of the étale neighborhoods $(B, \mathfrak{n})$ with a (uniquely determined!) morphism $(B_0, \mathfrak{n}_0) \to (B, \mathfrak{n})$, and then the homomorphism above is induced by the homomorphisms

$$B_0 \otimes_A B \to B \quad , \quad b_0 \otimes b \mapsto b_0 b\,.$$

5) Universal property: Let $(C, \mathfrak{n}_C)$ be henselian and $\varphi : A \to C$ a local morphism. We look for the homomorphism $\tilde{\varphi}$, which makes the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & A^h \arrow[dl, dashed, "\exists!\, \tilde{\varphi}\,?"] \\ & C & \end{tikzcd}$$

commutative in a unique way. It suffices to show: For all étale neighborhoods $(B, \mathfrak{n})$ of $(A, \mathfrak{m})$, there exists a unique homomorphism φ_B, which makes the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\varphi"'] & & B \arrow[dl, "\exists!\, \varphi_B"] \\ & C & \end{tikzcd}$$

commutative. Equivalent: In the commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\varphi"'] & & B \otimes_A C \\ & C \arrow[ur, "\psi"'] & \end{tikzcd}$$

there exists a unique A-linear section $B \otimes_A C \to C$ of ψ. But from $k = A/\mathfrak{m} \xrightarrow{\sim} B/\mathfrak{n}$ we get a surjective homomorphism $\psi : B \otimes_A C \twoheadrightarrow C/\mathfrak{n}_C$ and an isomorphism $C/\mathfrak{n}_C \xrightarrow{\sim} (B \otimes_A C)/\ker(\psi)$. Since C is henselian, by 9.1 (d) we obtain a section of $C \to B \otimes_A C$ as wanted.

Proposition 9.11 (a) A^h is flat over A.

(b) Let

$$\begin{aligned}\hat{A} &= \varprojlim_r A/\mathfrak{m}^r \\ \hat{A^h} &= \varprojlim_r A^h/(\mathfrak{m}^h)^r\end{aligned}$$

be the completions of A and A^h, respectively. Then $\hat{A} \xrightarrow{\sim} \hat{A^h}$ is an isomorphism.

Proof (a): A^h is flat as direct limit of flat A-algebras (the tensor product commutes with direct limits).

(b) It suffices to show that $A/\mathfrak{m}^r \xrightarrow{\sim} B/\mathfrak{n}^r$ for all étale neighborhoods $(B, \mathfrak{n})$ of $(A, \mathfrak{m})$. But we have $A/\mathfrak{m} \xrightarrow{\sim} B/\mathfrak{n}$ and $\mathfrak{m}B = \mathfrak{n}$ by assumption; hence $\mathfrak{m}^r B = \mathfrak{n}^r$ for all r and

$$\mathfrak{m}^r/\mathfrak{m}^{r+1} \cong \mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_{A/\mathfrak{m}} B/\mathfrak{n} \cong \mathfrak{m}^r B/\mathfrak{m}^{r+1}B = \mathfrak{n}^r/\mathfrak{n}^{r+1}\,. \tag{9.11.1}$$

The isomorphism in the middle of (9.11.1) follows from the flatness of B over A: By this, the exact sequence

$$0 \to \mathfrak{m}^{r+1} \to \mathfrak{m}^r \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \to 0$$

induces an exact top row in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^{r+1} \otimes_A B \arrow[r] \arrow[d, "\wr"] & \mathfrak{m}^r \otimes_A B \arrow[r] \arrow[d, "\wr"] & \mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_A B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{m}^{r+1}B \arrow[r] & \mathfrak{m}^r B \arrow[r] & \mathfrak{m}^r B/\mathfrak{m}^{r+1}B \arrow[r] & 0\,, \end{tikzcd} \tag{9.11.2}$$

and the indicated vertical isomorphisms, since, by flatness of B over A, the injection $\mathfrak{m}^r \hookrightarrow A$ induces an injection $\mathfrak{m}^r \otimes_A B \to A \otimes_A B = B$ with image $\mathfrak{m}^r B$ (similarly for $r+1$). Therefore the vertical map on the right is an isomorphism.

From (9.11.1) the isomorphisms $A/\mathfrak{m}^r \xrightarrow{\sim} B/\mathfrak{m}^{r+1}$ follows inductively, via the commutative diagram with exakt rows

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{m}^r/\mathfrak{m}^{r+1} \arrow[r] \arrow[d, "\wr"] & A/\mathfrak{m}^{r+1} \arrow[r] \arrow[d] & A/\mathfrak{m}^r \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathfrak{n}^r/\mathfrak{n}^{r+1} \arrow[r] & B/\mathfrak{n}^{r+1} \arrow[r] & B/\mathfrak{n}^r \arrow[r] & 0\,. \end{tikzcd}$$

Remark 9.12 One can show:

(a) If A is noetherian, then A^h is noetherian, too.

(b) Let A be integral and normal, with fraction field K. Let K_s be a separable closure of K, A_s the integral closure of A in K_s, $\overline{m} \subset A_s$ a maximal ideal over $\mathfrak{m}$ and $Z_{\overline{\mathfrak{m}}} \subset Gal(K_s/K) = G_K$ the decomposing group. Then we have $A^h = A_s^{Z_{\overline{\mathfrak{m}}}}$.

Definition 9.13 Let X be a scheme and let $x \in X$ be a point. An étale neighborhood of x is a pair (U, y), with $f : U \to X$ étale, $f(y) = x$ and $k(x) \xrightarrow{\sim} k(y)$.

Lemma 9.14 (a) With the obvious morphisms $(U, y) \to (U', y')$, i.e.

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U' \arrow[dl] \\ & X & \end{tikzcd} \quad , \qquad y \longmapsto y' \, ,$$

these form a cofiltered category.

(b) We have

$$\mathcal{O}^h_{X,x} = \varinjlim_{\substack{U \text{ étale} \\ \text{neighb. of } x}} \Gamma(U, \mathcal{O}_U) = \varinjlim_{\substack{(U,y) \text{ étale} \\ \text{neighb. of } x}} \mathcal{O}_{U,y} \, .$$

Proof Analogous to the proof of 9.7.

Definition 9.15 A **strictly henselian ring** is a henselian ring whose residue field is separably closed.

Lemma 9.16 (a) For every local ring $(A, \mathfrak{m})$ there exists a strictly henselian ring $(A^{sh}, \mathfrak{m}^{sh})$ and a local morphism $i : A \to A^{sh}$, which satisfies the following universal properties: If $\varphi : A \to C$ is a local morphism in a strict henselian ring $(C, \mathfrak{m}_C)$, and if a $k = A/\mathfrak{m}$-embedding

$$\varphi_0 : A^{sh}/\mathfrak{m}^{sh} \to C/\mathfrak{n}_C$$

is given, then there exists a unique local morphism $\tilde{\varphi} : A^{sh} \to C$, which makes the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] \arrow[dr, "\varphi"'] & & A^{sh} \arrow[dl, dashed, "\exists!\, \tilde{\varphi}"] \\ & C & \end{tikzcd}$$

commutative and induces the embedding φ_0 of the residue fields.

(b) For a scheme X, a point $x \in X$ and a geometric point $\overline{x}$ of X over x we have

$$\mathcal{O}^{sh}_{X,x} \cong \mathcal{O}_{X,\overline{x}} := \varinjlim_{\substack{U \text{ étale} \\ \text{neighb. of } \overline{x}}} \Gamma(U, \mathcal{O}_U) = \varinjlim_{\substack{U \text{ étale} \\ \text{neighb. of } \overline{x}}} \mathcal{O}_{U, \text{image of } \overline{x}} \, .$$

Proof Analogous to the proof of 9.7.

Lemma 9.17 Let X be a strictly local scheme, i.e., the spectrum of a strictly local ring. Let x be the closed point, considered as geometric point of X. Then, for sheaves $\mathcal{F}$ on X there exists a functorial isomorphism

$$F_x \cong H^0(X, \mathcal{F}) \, ,$$

and we have $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

Proof: Let X' be an étale neighborhood of x. By 9.16 there exists exactly one local section of $X' \to X$ so that X is initial in the set of all étale neighborhoods of x. By 9.14 (b),

$F_x \cong H^0(X, \mathcal{F})$, functorial in $\mathcal{F}$. Since the stalk functor is exact, we have $H^q(X, \mathcal{F}) = 0$ for $q > 0$.

Lemma 9.18 Let $f : X \to Y$ be a finite morphism, where Y is a strictly local scheme. For an étale sheaf $\mathcal{F}$ on X we then have

$$\begin{array}{rcll} H^0(X, \mathcal{F}) & \cong & \prod\limits_{x \in X_y} F_x & \\ H^q(X, \mathcal{F}) & = & 0 & \text{for } q > 0\,. \end{array}$$

Proof Since $X \to Y$ is finite and Y is strictly local, we have

$$X = \coprod_x \mathrm{Spec}(\mathcal{O}_{X,x})\,,$$

where x runs over the finitely many closed points of X, which are the points over the closed points y of Y (see Lemma 9.1). Every ring $\mathcal{O}_{X,x}$ ist henselian (see Lemma 9.1), and even strictly henselian, since $k(x)$, as finite extension of the spectral closed field $k(y)$, is again separably closed.

10 Examples of étale sheaves

First we consider sheaves (of sets), represented by schemes.

Remark 10.1 In a category $\mathcal{X}$, a morphism $f : Y \to X$ is an epimorphism if and only if for every object Z in $\mathcal{X}$ the morphism

$$f^* : Hom(X, Z) \to Hom(Y, Z)$$

is injective.

Definition 10.2 Let $\mathcal{X}$ be a category with fiber products. A morphism $f : Y \to X$ is called a **strict epimorphism**, if for all objects Z in $\mathcal{X}$ the sequence

$$Y \times_X Y \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} Y \xrightarrow{f} X$$

is exact, i.e., if f is the difference cokernel of pr_1 and pr_2, i.e., if for all objects Z in X the sequence

$$Hom(X, Z) \xrightarrow{f^*} Hom(Y, Z) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} Hom(Y \times_X Y, Z)$$

is exact, i.e., if f^* is the difference kernel of pr_1^* and pr_2^* (In particular, this implies that f^* is injective for all Z, i.e., that f is an epimorphism).

Definition 10.3 A morphism of schemes $f : Y \to X$ is called **faithfully flat**, if f is flat and surjective.

The following lemma shows that this corresponds to the usual notions for affine schemes.

Lemma 10.4 Let $\varphi : A \to B$ be a flat ring homomorphism. Then the following conditions are equivalent

(a) φ is faithfully flat, i.e., for an A-module M we have $M = 0$ if $M \otimes_A B = 0$.

(b) A sequence $M' \to M \to M''$ of A-modules is exact, if $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$ is exact.

(c) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

(d) For any maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m}B \neq B$.

Proof (a) $\Rightarrow$ (b): Assume that $M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$ becomes exact after tensoring with B. By the flatness of B over A, we have $im(\psi\varphi) \otimes_A B = im((\psi \otimes id) \circ (\varphi \otimes id)) = 0$, hence, by (a), we have $im(\psi\varphi) = 0$, i.e., $\psi\varphi = 0$. Furthermore we have $\ker(\psi)/im(\varphi) \otimes_A B = \ker(\psi \otimes id)/im(\varphi \circ id) = 0$, hence $ker(\psi)/im(\varphi) = 0$.

(b) $\Rightarrow$ (a): $0 \to M \to 0$ is exact if and only if $M = 0$.

(a) $\Rightarrow$ (c): For any prime ideal $\mathfrak{p} \subseteq A$, we have $B \otimes_A k(\mathfrak{p}) \neq 0$, hence

$$\mathrm{Spec}(\varphi)^{-1}(\mathfrak{p}) = \mathrm{Spec}(B \otimes_A k(\mathfrak{p})) \neq \emptyset\,.$$

(c) $\Rightarrow$ (d) is obvious, since the prime ideals over $\mathfrak{m}$ correspond to the prime ideals of $B/\mathfrak{m}B$.

(d) $\Rightarrow$ (a): Let $x \in M, x \neq 0$, and $N = Ax \subseteq M$. By the flatness of B over A it suffices to show $B \otimes_A N \neq 0$ (then we also have $B \otimes_A M \neq 0$). But $N \cong A/J$ for an ideal $J \subsetneqq A$, so that $B \otimes_A N \cong B/JB$. If $\mathfrak{m} \subseteq A$ is a maximal ideal with $J \subseteq \mathfrak{m}$, then we have $JB \subseteq \mathfrak{m}B \neq B$, therefore $B/JB \neq 0$.

Corollary 10.5 A flat morphism of local rings $\varphi : A \to B$ is faithfully flat.

Proof This follows from 10.4 (d), since, by assumption, $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ for the maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$ of A and B, respectively.

Theorem 10.6 Let $f : Y \to X$ be a morphism of schemes. If f is faithfully flat and of finite type, then f is a strict epimorphism.

Proof For any scheme Z we have to show the exactness of

$$Hom(X, Z) \to Hom(Y, Z) \rightrightarrows Hom(Y \times_X Y, Z)\,.$$

First case: Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and $Z = \mathrm{Spec}(C)$ be affine. In this case, the sequence above can be identified with the sequence

$$Hom(C, A) \to Hom(C, B) \rightrightarrows Hom(C, B \otimes_A B)\,,$$

and the claim follows from the exactness of the sequence

$$\begin{array}{ccccc} A & \to & B & \to & B \otimes_A B \quad , \\ & & b & \mapsto & b \otimes 1 - 1 \otimes b\,. \end{array}$$

Second case: Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be affine and let Z be arbitrary. Let $h \in Hom(Y, Z)$ be given, with $h\,pr_1 = h\,pr_2$. We have to show that there is a unique $g \in Hom(X, Z)$ with $gf = h$.

First we show the uniqueness of g (if it exists). Let $g_1, g_2 : X \to Z$ be given with $g_1 f = g_2 f$. Then g_1 and g_2 have to coincide as maps of topological spaces, since f is surjective. Let $x \in X$, and let U be an affine open neighborhood of $g_1(x) = g_2(x)$ in Z. Then there is an $a \in A$ with $g_1(D(a)) = g_2(D(a)) \subseteq U$. Furthermore, B_a is faithfully flat over A_a. From the first case we get that $g_1|_{D(a)} = g_2|_{D(a)}$ as scheme morphisms, hence the uniqueness of g.

Now let $h : Y \to Z$ be given with $h\,pr_1 = h\,pr_2$. Because of the proven uniqueness of g it suffices to define g locally. Consider $x \in X$, $y \in Y$ with $f(y) = x$, and let $U \subseteq Z$ be an affine open neighborhood of $h(y)$ in Z. We now use the following lemma.

Lemma 10.7 Let $f : Y \to X$ be flat and of finite type. Then f is an open map.

Proof See Milne, 'Etale Cohomology', p. 14, Th. 2.1.

In our setting, we deduce that $f(h^{-1}(U))$ is open in X. We have a diagram

$$\begin{tikzcd}
& y & x \\
Y \times_X Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r, "f"] \arrow[d, "h"'] & X = \mathrm{Spec}(A) \\
& h(y) \in U \subseteq Z &
\end{tikzcd}$$

By 10.7, there exists an $a \in A$ with $x \in D(a) \subseteq f(h^{-1}(U))$. Then $f^{-1}(D(a)) \subseteq h^{-1}(U)$. In fact, if we have $x_1 \in D(a)$, hence $x_1 = f(y_1)$ with $h(y_1) \in U$, and if we have $y_2 \in Y$ with $f(y_2) = x_1$, then, since $f(y_1) = f(y_2)$, there is an element $y' \in Y \times_X Y$ with $pr_1(y') = y_1$ and $pr_2(y') = y_2$ (consider a point in $y_1 \times_{x_1} y_2$ and its image in $Y \times_X Y$). We get

$$h(y_2) = h\,pr_2(y') = h\,pr_1(y') = h(y_1) \in U\,,$$

hence $y_2 \in h^{-1}(U)$.

If $b \in B$ is the image of a, then $D(b) = f^{-1}(D(a))$, hence $h(D(b)) \subseteq U$, and by the first case we obtain $g|_{D(a)}$. As stated previously, these local solutions glue together to a global g.

Third case: Let X, Y and Z be arbitrary. We easily reduce to the case where X is affine (choose an affine open covering and its inverse image in Y; because of the uniqueness, the morphisms g glue on the covering). Since f is quasi-compact, Y is a finite union $Y = Y_1 \cup \ldots \cup Y_n$ of affine open subsets. Let

$$Y^* = \coprod_{i=1}^{n} Y_i\,.$$

Then Y^* is affine and $Y^* \to Y$ is faithfully flat. We obtain a commutative diagram

$$\begin{tikzcd}
Hom(X,Z) \arrow[r] \arrow[d, equal] & Hom(Y,Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & Hom(Y \times_X Y, Z) \arrow[d] \\
Hom(X,Z) \arrow[r] & Hom(Y^*,Z) \arrow[r, shift left] \arrow[r, shift right] & Hom(Y^* \times_X Y^*, Z)\,,
\end{tikzcd}$$

where, by the second case, the bottom row is exact. Furthermore, the middle vertical map is obviously injective ($Hom(-,Z)$ is a Zariski sheaf on Y). The exactness of the top row now follows by a diagram chase.

Before we use Theorem 10.6 for the construction of sheaves, we introduce a useful criterion for a presheaf to be a sheaf.

Proposition 10.8 A presheaf P (of sets or abelian groups) on $X_{\text{ét}}$ (resp. X_{fl}) is a sheaf if and only if the following two conditions hold:

(a) For any $U \in X_{\text{ét}}$ (resp. X_{fl}), the restriction of P to U is a Zariski sheaf.

(b) For any étale covering $(U' \to U)$, where U and U' are affine,

$$P(U) \to P(U') \rightrightarrows P(U' \times_U U')$$

is exact.

Proof Obviously, these properties are necessary. Conversely, if (a) holds, then for any disjoint sum $V = \coprod_j V_j$ of schemes we have

$$P(V) = \prod_j P(V_j)\,.$$

For a covering $(U_i \to U)$ the sequence

$$P(U) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j) \tag{10.8.1}$$

is isomorphic to the sequence

$$P(U) \to P(U') \rightrightarrows P(U' \times_U U') \tag{10.8.2}$$

for the covering $U' \to U$ with $U' = \coprod_i U_i$, since

$$(\coprod U_i) \times_U (\coprod U_i) \cong \coprod_{i,j} U_i \times_U U_j\,.$$

If (b) holds, the sequence (10.8.1) is exact for every covering $(U_i \to U)_{i\in I}$ with finite I and affine U_i and U, because then $\coprod U_i$ is affine: $\mathrm{Spec}(A) \coprod \mathrm{Spec}(B) \cong \mathrm{Spec}(A \times B)$. For any $(U'_j \to U)$ write U as the union of affine open sets U_i, $U = \underset{i}{\cup} U_i$, and write $f : U' = \coprod_j U'_j \to U$. Then we have $f^{-1}(U_i) = \underset{k\in K_i}{\cup} U'_{ik}$ with affine open subsets $U'_{ik} \subseteq f^{-1}(U_i)$. Since $U' \to U$ is flat and locally of finite type, $U'_{ik} \to U_i$ is of finite type (since both schemes are affine), therefore $f(U'_{ik})$ is open in U_i by Lemma 10.7. Since U_i (as an affine scheme) is quasi-compact, there is a finite index set $E_i \subseteq K_i$ with $U_i = \underset{k\in E_i}{\cup} f(U'_{ik})$, i.e., $(U'_{ik} \to U_i)_{k\in E_i}$ is a covering. By adding all morphisms of the form $U'_{ik} \to f(U'_{ik})$ for $k \in K_i - E_i$, we can assume that all K_i are finite, and $U' = \cup U'_{ik}$. Now we consider the commutative diagram

$$\begin{tikzcd}
P(U) \arrow[r] \arrow[d] & P(U') \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P(U' \times_U U') \arrow[d] \\
\prod_i P(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_k P(U'_{ik}) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \prod_{k,\ell} P(U'_{ik} \times_U U'_{i\ell}) \\
\prod_{i,j} P(U_i \times_U U_j) \arrow[r] & \prod_{i,j} \prod_{k,\ell} P(U'_{ik} \cap U'_{j\ell})\,. &
\end{tikzcd}$$

By 10.8 (a), the first two columns are exact, and by the first remark about (b), the second row is exact (k runs in the finite set K_i for U'_{ik}). Therefore $P(U) \hookrightarrow P(U')$ is injective, hence the presheaf P is separated (since $(U_i \to U)$ was arbitrary). This in turn implies the injectivity of the bottom arrow. Now an easy diagram chase shows the exactness of the top row.

Corollary 10.9 For every X-scheme Z, the functor represented by Z,

$$Hom_X(-, Z) : U \mapsto Z(U) := Hom_X(U, Z)$$

is a sheaf of sets on $X_{\text{ét}}$ and X_{fl}.

Proof Condition 10.8 (a) is well-known (glueing of morphisms). Condition 10.8 (b) follows from Theorem 10.6: If $f : U' \to U$ is a surjective étale (or flat) X-morphism, with U and U' affine, then f is faithfully flat and of finite type, hence the diagram

$$Hom(U, Z) \to Hom(U', Z) \rightrightarrows Hom(U' \times_U U', Z)$$

is exact by 10.6. Let $\pi_U : U \to X, \pi_{U'} : U' \to X$ and $\pi_Z : Z \to X$ be the structure morphisms. We then have a commutative diagram

$$\begin{tikzcd}
Hom_X(U,Z) \arrow[r] \arrow[d, hook] & Hom_X(U',Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & Hom_X(U' \times_U U', Z) \arrow[d, hook] \\
Hom(U,Z) \arrow[r, hook] \arrow[d, "(\pi_Z)_*"] & Hom(U',Z) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "(\pi_Z)_*"] & Hom(U \times_U U', Z) \\
Hom(U,X) \arrow[r, hook, "f^*"] & Hom(U',X) & \\
\pi_U \arrow[r, mapsto] & \pi_{U'} &
\end{tikzcd}$$

where the middle row is exact for all schemes Z, so that the bottom morphism f^* is injective as well. Furthermore we have

$$Hom_X(U, Z) = \{g \in Hom(U, Z) \mid (\pi_Z)_*(g) = \pi_Z g = \pi_U\} ,$$

and the same for U' in place of U. This implies the exactness of the top row in the diagram.

Corollary 10.10 For every (abelian) group scheme G over X, the functor represented by G is a sheaf of (abelian) groups on $X_{\text{ét}}$ and X_{fl}.

Examples 10.11 (a) The sheaf $\mathbb{G}_{a,X} = X \times_{\text{Spec}(\mathbb{Z})} \mathbb{G}_{a,\mathbb{Z}} = X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[T])$ satisfies

$$\mathbb{G}_{a,X}(U) = \Gamma(U, \mathcal{O}_U) ,$$

for every X-scheme U and is called the additive group over X.
(b) The sheaf $\mathbb{G}_{m,X} = X \times_{\text{Spec}(\mathbb{Z})} G_m$, with $\mathbb{G}_m = \text{Spec}(\mathbb{Z}[T, T^{-1}])$ has the value

$$\mathbb{G}_{m,X}(U) = \Gamma(U, \mathcal{O}_U)^\times = \Gamma(U, \mathcal{O}_U^\times) ,$$

for every X-scheme U and is called the multiplicative group over X.
(c) For every $n \in \mathbb{N}$ let μ_n be the sheaf $X \times_{\text{Spec}(\mathbb{Z})} \text{Spec}(\mathbb{Z}[T]/\langle T^n - 1\rangle)$. It satiesfies

$$\mu_n(U) = \{a \in \Gamma(U, \mathcal{O}_U)^\times | a^n = 1\} ,$$

for every X-scheme U and is called the sheaf of the n-th unit roots.

Lemma 10.12 The sequence of sheaves

$$\begin{array}{rcl} 1 \to \mu_n \to \mathbb{G}_m & \xrightarrow{n} & \mathbb{G}_m \\ a & \mapsto & a^n \end{array}$$

is exact.

Proof: For every X-scheme U,

$$\begin{tikzcd}
1 \arrow[r] & \mu_n(U) \arrow[r] \arrow[d, equal] & \mathbb{G}_m(U) \arrow[r, "n"] \arrow[d, equal] & \mathbb{G}_m(U) \arrow[d, equal] \\
1 \arrow[r] & \mu_n(U) \arrow[r] & \Gamma(U, \mathcal{O}_U)^\times \arrow[r] & \Gamma(U, \mathcal{O}_U)^\times \\
& & a \arrow[r, mapsto] & a^n
\end{tikzcd}$$

is exact.

Proposition 10.13 Let n be invertible on the scheme X ($\Leftrightarrow$ for all $x \in X$, n is not divisible by $char(k(x))$). Then the sequence of étale sheaves on X

$$1 \to \mu_n \to \mathbb{G}_m \overset{n}{\to} \mathbb{G}_m \to 1$$

is exact. For any n, the sequence

$$1 \to \mu_n \to \mathbb{G}_m \overset{n}{\to} \mathbb{G}_m \to 1$$

of flat sheaves is exact. The sequences above are called the Kummer sequence.

Proof One only has to show that, by the assumptions, $\mathbb{G}_m \overset{n}{\to} \mathbb{G}_m$ is an epimorphism. We use the criterion 4.2 (d). Let U be a scheme and $s \in \mathbb{G}_m(U)$. By passing to an affine open covering, we may assume that $U = \mathrm{Spec}(A)$ is affine and $s = a \in \Gamma(U, \mathcal{O}_U)^\times = A^\times$. Then $B = A[T]/\langle T^n - a\rangle$ is a faithfully flat A-algebra of finite type, hence $V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U$ is a flat covering, and for $a \in A^\times \subseteq B^\times$ there is an element $b \in B^\times$ (the image of T in B) with $b^n = a$. Therefore, by the exactness criterion of 4.2 (d), $\mathbb{G}_m \overset{n}{\to} \mathbb{G}_m$ is an epimorphism of flat sheaves. If n is invertible on X, then $n \in A^\times$ and therefore B is an étale A-algebra, so that $\mathbb{G}_m \overset{n}{\to} \mathbb{G}_m$ is an epimorphism of étale sheaves.

Lemma 10.14 Let X be a scheme. For every group G, the corresponding constant Zariski, étale or flat sheaf G_X (i.e., the sheaf associated the constant presheaf G^P with $G^P(U) = G$ for all $U \to X$) is representable by the group scheme

$$\underline{G}_X = \coprod_{g \in G} X = \coprod_{g \in G} X_{[g]}$$

with the obvious group law.

Proof Consider the category Sch/X of all X-schemes. One can easily see that for every X-scheme Y we have:

$$\underline{G}_X(Y) = Hom_X(Y, \underline{G}_X) = \{\varphi : Y \to G \mid \varphi \text{ locally constant}\}$$

(If $\varphi : Y \to G$ is locally constant, then $Y = \coprod_{g \in G} \varphi^{-1}(g)$). Furthermore $\underline{G}_X$ is a sheaf for all three considered topologies, by 10.10. Finally for the Zariski sheaf G_X^{Zar} associated to G we have

$$G_X^{Zar}(Y) = \{\varphi : Y \to G \mid \varphi \text{ locally constant}\}$$

The claim follows, since this gives a sheaf for all three considered topologies.

Remark 10.15 If X is locally noetherian and $\pi_0(Y)$ is the set of connected components of Y, we have

$$G_X(Y) = G^{\pi_0(Y)}$$

for all $Y \to X$, which are locally of finite type (and therefore again locally noetherian).

Proposition 10.16 Let X be a scheme of characteristic $p > 0$ ($\Leftrightarrow$ the morphism $X \to \mathrm{Spec}(\mathbb{Z})$ factorizes over $\mathrm{Spec}(\mathbb{F}_p)$ $\Leftrightarrow$ for every open $U \subseteq X$ we have $p\Gamma(U, \mathcal{O}_U) = 0$). Let

$$F : \mathbb{G}_{a,X} \to \mathbb{G}_{a,X}$$

be the Frobenius homomorphism: For $U \to X$ flat (or étale) let

$$F : \Gamma(U, \mathcal{O}_U) \to \Gamma(U, \mathcal{O}_U)$$

be the ring homomorphism (!)

$$a \mapsto a^p$$

(this is additive, since $pa = 0$). Then one has an exact sequence of flat (or étale) sheaves

$$0 \to \mathbb{Z}/p\mathbb{Z}_X \to \mathbb{G}_{a,X} \overset{F-1}{\to} \mathbb{G}_{a,X} \to 0\,,$$

(called **Artin-Schreier sequence**), where $\mathbb{Z}/p\mathbb{Z}_X$ is the constant sheaf, associated to the abelian group $\mathbb{Z}/p\mathbb{Z}$, which is represented by the group scheme

$$\underline{\mathbb{Z}/p\mathbb{Z}}_X = X \times_{\mathrm{Spec}\,(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_p[T]/\langle T^P - T\rangle)\,.$$

Proof: Since

$$T^p - T = \prod_{i\in\mathbb{F}_p} (T - i)$$

in $\mathbb{F}_p[T]$, we have an isomorphism

$$\mathbb{F}_p[T]/\langle T^p - T\rangle \cong \prod_{i=0}^{p-1} \mathbb{F}_p\,.$$

Therefore $Spec(\mathbb{F}_p[T]/\langle T^p - T\rangle)$ is canonically isomorphic to the constant group scheme $\mathbb{Z}/p\mathbb{Z}_{Spec(\mathbb{F}_p)}$. Furthermore, for every X-scheme Y we have

$$\begin{aligned} Hom_X(Y, \underline{\mathbb{Z}/p\mathbb{Z}}_X) &= Hom(Y, \underline{\mathbb{Z}/p\mathbb{Z}}_{\mathbb{F}_p}) \\ &= Hom(\mathbb{F}_p[T]/\langle T^p - T\rangle, \Gamma(Y, \mathcal{O}_Y)) = \{a \in \Gamma(Y, \mathcal{O}_Y) \mid a^p - a = 0\}\,. \end{aligned}$$

This shows the exactness of

$$0 \to \mathbb{Z}/p\mathbb{Z}_X \to \mathbb{G}_{a,X} \overset{F-1}{\to} \mathbb{G}_{a,X}\,.$$

Furthermore, $F - 1$ is an epimorphism of sheaves for the étale (and flat) topology, since for $U = \mathrm{Spec}(A) \subseteq Y$ affine open and $a \in A$

$$V = \mathrm{Spec}(A[T]/\langle T^p - T - a\rangle) \to U$$

is a covering, so that $a = (F-1)b = b^p - b$ if b is the image of T in $B = A[T]/\langle T^p - T - a\rangle$.

Lemma/Definition 10.17 Let X again be a scheme of characteristic $p > 0$ and let $\alpha_{p,X} \subseteq \mathbb{G}_{a,X}$ be the subsheaf defined by

$$\alpha_{p,X}(U) = \{a \in \Gamma(U, \mathcal{O}_U) \mid a^p = 0\}\,.$$

Then $\alpha_{p,X}$ is represented by the group scheme $X \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_p[T]/\langle T^p\rangle)$. The sequence

$$0 \to \alpha_{p,X} \to \mathbb{G}_{a,X} \xrightarrow{F} \mathbb{G}_{a,X}$$

is exact for the Zariski or étale topology. The sequence

$$0 \to \alpha_{p,X} \to \mathbb{G}_{a,X} \xrightarrow{F} \mathbb{G}_{a,X} \to 0$$

is exact for the flat topology, but not in general for the étale topology.

Proof: The first claims are obviously true. The morphism F is an epimorphism in the flat topology, since for every $\mathbb{F}_p$-algebra A the algebra $B = A[T]/\langle T^p - a\rangle$ is faithfully flat over A. For a separably closed field L the Frobenius morphism

$$\begin{array}{ccc} L & \xrightarrow{F} & L \\ a & \mapsto & a^p \end{array}$$

is not surjective in general, except if L is a perfect field.

Lemma/Definition 10.18 Let X be a scheme and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then

$$U \mapsto \Gamma(U, \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_U)$$

(for X-schemes $\pi_U : U \to X$) is a sheaf on $(Sch/X)_{\mathrm{fl}}$, the site of all X-schemes with the flat topology. Here $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ stands for $\pi_U^{-1}\mathcal{M} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_U = \pi_U^*\mathcal{M}$, the quasi-coherent pull-back. In particular, this gives a sheaf on the small sites $X_{\text{ét}}$ and X_{fl}, called $\mathcal{M}_{\text{ét}}$ and $\mathcal{M}_{\mathrm{fl}}$, respectively.

Proof We use the criterion of Theorem 10.8. Condition 10.8 (a) is obviously true, since, by construction, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ is a Zariski sheaf on U. For 10.8 (b) let $U' = \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be affine and faithfully flat. Then $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ corresponds to an A-module M, $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_U$ corresponds to a B-module $B \otimes_A M$, and we have to show the exactness of

$$M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M\,.$$

But by the theory of flat descent this holds, since $A \to B$ is faithfully flat (see Theorem Lemma 14.6 below).

Remark 10.19 The case $\mathbb{G}_{a,X} = \mathcal{O}_{X,\text{ét}}$ is a special case.

11 The decomposition theorem

Let X be a scheme, let $i : Y \hookrightarrow X$ be a closed immersion and let $j : U \hookrightarrow X$ be the open immersion of the open complement $U = X - Y$.

If F is an étale sheaf on X, then $F_1 = i^*F$ is an étale sheaf on Y and $F_2 = j^*F$ is an étale sheaf on U. Let $F \overset{ad}{\to} j_*j^*F$ be the adjunction map (which, under the isomorphism $Hom_U(j^*F, j^*F) \cong Hom_X(F, j_*j^*F)$, corresponds to id_{j^*F}).
Then $i^*(ad)$ gives a morphism

$$\phi_F : F_1 = i^*F \to i^*j_*j^*F = i^*j_*F_2 \,.$$

Therefore we can associate the triple (F_1, F_2, ϕ_F) to every sheaf F on X.

Let $T(X, Y)$ be the the category of triples (F_1, F_2, ϕ), where F_1 is an étale sheaf on Y, F_2 an étale sheaf on U and $\phi : F_1 \to i^*j_*F_2$ is a morphism of étale sheaves on Y. Morphisms

$$(F_1, F_2, \phi) \to (F_1', F_2', \phi')$$

are pairs (ψ_1, ψ_2), where $\psi_1 : F_1 \to F_1'$ and $\psi_2 : F_2 \to F_2'$ are morphisms of sheaves on Y resp. U, such that the diagram

(11.0)
$$\begin{tikzcd}
F_1 \arrow[r, "\phi"] \arrow[d, "\psi_1"'] & i^*j_*F_2 \arrow[d, "i^*j_*(\psi_2)"] \\
F_1' \arrow[r, "\phi'"] & i^*j_*F_2'
\end{tikzcd}$$

commutes.

Theorem 11.1 Let the functor

$$t : Sh(X_{\text{ét}}) \to T(X, Y)$$

be defined as follows:
(i) To a sheaf $F \in Sh_{\text{ét}}(X)$ assign the triple $(i^*F, j^*F, i^*(ad) : i^*F \to i^*j_*j^*F)$.
(ii) To a morphism $\varphi : F \to F'$ in $Sh(X_{\text{ét}})$ assign the morphism in $T(X, Y)$

$$\psi(\varphi) = (\psi_1 = i^*(\varphi) : i^*F \to i^*F', \psi_2 = j^*\varphi : j^*F \to j^*F')$$

Then t is an equivalence of categories.

Proof First of all, t is well-defined: by the preliminary remark, the triple in (i) is an object in $T(X, Y)$, and $\psi(\varphi)$ is a morphism in $T(X, Y)$, since the diagram

(11.1.1)
$$\begin{tikzcd}
i^*F \arrow[r, "i^*(ad_F)"] \arrow[d, "i^*(\varphi)"'] & i^*j_*j^*F \arrow[d, "i^*j_*j^*(\varphi)"] \\
i^*F' \arrow[r, "i^*(ad_{F'})"] & i^*j_*j^*F'
\end{tikzcd}$$

commutes (Functoriality of the adjointness and then applying the functor i^*).

Now we define a pseudoinverse s for t. For a triple (F_1, F_2, ϕ), let $s(F_1, F_2, \phi) \in Sh(X_{\text{ét}})$ be the fiber product of i_*F_1 and j_*F_2 over $i_*i^*j_*F_2$, so that the diagram

(11.1.2)
$$\begin{tikzcd} s(F_1, F_2, \phi) \arrow[r] \arrow[d] & j_*F_2 \arrow[d, "ad_i"] \\ i_*F_1 \arrow[r, "i_*(\phi)"] & i_*i^*j_*F_2 \end{tikzcd}$$

is cartesian. This assignment is functorial, since forming the fiber product is functorial: Every morphism

$$(\psi_1, \psi_2) : (F_1, F_2, \phi) \to (F_1', F_2', \phi')$$

induces a morphism

$$s(F_1, F_2, \phi) \to s(F_1', F_2', \phi)\,,$$

because of the commutativity (11.0) and the functoriality of ad_i. This gives the wanted functor $s : T(X, Y) \to Sh(X_{\text{ét}})$.

Now we construct an isomorphism of functors $id \xrightarrow{\sim} st$ on $Sh(X_{\text{ét}})$.

For every sheaf F in $Sh(X_{\text{ét}})$, the diagram

(11.1.3)
$$\begin{tikzcd} F \arrow[r, "ad_j"] \arrow[d, "ad_i(F)"'] & j_*j^*F \arrow[d, "ad_i(j_*j^*F)"] \\ i_*i^*F \arrow[r, "i_*i^*(ad_j)"] & i_*i^*j_*j^*F \end{tikzcd}$$

commutes, by the functoriality of the adjunction morphism ad_i (in sheaves). By the universal property of the fiber product (11.1.2) for the triple

$$(F_1, F_2, \phi) = (i^*F, j^*F, i^*(ad_j)) = t(F)\,,$$

(11.1.3) induces a canonical morphism

$$\alpha : F \to s(i^*F, j^*F, i^*(ad_j)) = st(F)\,.$$

We show that (11.1.3) is cartesian, too; then α is an isomorphism.

Here it suffices to show that the diagram is cartesian if one passes to the (geometric) stalks, since the stalks form a conservative family.

But if $\overline{x}$ is a geometric point over U, we obtain a diagram

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d] & F_{\overline{x}} \arrow[d] \\ 0 \arrow[r] & 0\,, \end{tikzcd}$$

which is cartesian. If $\overline{x}$ is over Y, we obtain the diagram

$$\begin{tikzcd} F_{\overline{x}} \arrow[r] \arrow[d, "id"'] & (j_*j^*F)_{\overline{x}} \arrow[d, "id"] \\ F_{\overline{x}} \arrow[r] & (j_*j^*F)_{\overline{x}} \end{tikzcd}$$

which is cartesian as well.

Furthermore we have an isomorphism of functors $ts \xrightarrow{\sim} id$. If

$$(F_1, F_2, \phi : F_1 \to i^* j_* F_2)$$

is an object in $T(X,Y)$, then $s(F_1, F_2, \phi)$ is defined by the cartesian diagram

$$\begin{tikzcd} s(F_1,F_2,\phi) \arrow[r] \arrow[d] & j_*F \arrow[d, "ad_i"] \\ i_*F_1 \arrow[r] & i_*i^*j_*F_2\,. \end{tikzcd} \tag{11.1.4}$$

If we apply the functor i^*, we obtain again a cartesian diagram

$$\begin{tikzcd} i^*s(F_1,F_2,\phi) \arrow[r] \arrow[d, "\beta_1"'] & i^*j_*F_2 \arrow[d, "id"] \\ F_1 \arrow[r] & i^*j_*F_2\,, \end{tikzcd}$$

since i^* is an exact functor (see 7.8) and $i^*(ad_i) = id$ by definition of the adjunction map ad_i, as well as canonically und functorial $i^* i_* F_1 \xrightarrow{\sim} F_1$ for every étale sheaf on Y by the following Lemma. Thus β_1 is an isomorphism.

Correspondingly we apply j^* to (11.1.4) and we obtain the cartesian diagram

$$\begin{tikzcd} j^*s(F_1,F_2,\phi) \arrow[r, "\beta_2"] \arrow[d] & j^*j_*F_2 \arrow[d] \\ 0 \arrow[r] & 0\,, \end{tikzcd}$$

hence an isomorphism

$$\beta_2 : j^* s(F_1, F_2, \phi) \underset{\sim}{\overset{\beta_2'}{\to}} j^* j_* F_2 \xrightarrow{\sim} F_2\,,$$

where the last isomorphism holds by the following lemma.

Finally, the following diagram is commutative:

$$\begin{tikzcd} i^*(ad_j): & i^*s(F_1,F_2,\phi) \arrow[r] \arrow[d, "\beta_1"', "\wr"] & i^*j_*j^*s(F_1,F_2,\phi) \arrow[d, "i^*j_*(\beta_2)", "\wr"'] \\ & F_1 \arrow[r, "\phi"] & i^*j_*F_2\,, \end{tikzcd}$$

and we obtain an isomorphism

$$ts(F_1, F_2, \phi) \xrightarrow{\sim} (F_1, F_2, \phi)\,.$$

Lemma 11.2 (a) For an étale sheaf F_1 on Y, the adjunction map $Ad_i : i^* i_* F_1 \to F_1$ is an isomorphism.

(b) For an étale sheaf F_2 on U, the adjunction map $Ad_j : j^* j_* F_2 \to F_2$ is an isomorphism.

Proof (a) By Lemma 9.18, for every $y \in Y$ and every geometric point $\overline{y}$ over y we have a canonical isomorphism

$$(i^* i_* \mathcal{F})_{\overline{x}} \cong (i_* \mathcal{F})_{f(\overline{x})} \cong \mathcal{F}_{\overline{x}} \,.$$

(b) For an étale sheaf $\mathcal{F}$ on U and an étale morphism $X' \to X$ we have by definition

$$j^P(j_*\mathcal{F})(U') = \varinjlim_{(X',\psi) \in I_{U'}^{op}} (j_*\mathcal{F})(X') \,,$$

where $I_{U'}$ is the category of all pairs (X', ψ) with X' in $X_{\text{ét}}$ and a morphism $\psi : U' \to U \times_X X'$, where morphisms are commutative diagrams

$$\begin{tikzcd}
 & U \times_X X'' \arrow[dd, "(id \times f)"] \\
U' \arrow[ur] \arrow[dr] & \\
 & U \times_X X'
\end{tikzcd}$$

with $f : X'' \to X'$ (see the proof of Proposition 3.2), and where $I_{U'}^{op}$ is the dual category to $I_{U'}$.

But in our situation, $I_{U'}$ has the initial object $U' \to U \times_X U'$ (with obvious morphism), since for an object $\psi : U' \to U \times_X X'$ we have a morphism $\alpha : U' \to X'$ and hence a canonical morphism

$$\begin{tikzcd}
 & U \times_X U' \arrow[dd, "(id \times \alpha)"] \\
U' \arrow[ur] \arrow[dr, "\psi"'] & \\
 & U \times_X X' \,.
\end{tikzcd}$$

(Note that $U' \to U \to X$ is an étale morphism). Thus follows

$$j^P(j_*\mathcal{F})(U') = \mathcal{F}(U \times_X U') = \mathcal{F}(U') \,.$$

Since $\mathcal{F}$ is already a sheaf, we have $j^* j_* \mathcal{F} \cong \mathcal{F}$.

Since s and t induce inverse maps on morphisms (this can again be checked on the stalks, since these form a conservative family), we get that t is an equivalence of categories, with quasi-inverse s.

Remark 11.3 In particular, Theorem 11.1 implies that $T(X,Y) \cong Sh(X_{\text{ét}})$ is an abelian category. One can easily see that a sequence in $T(X,Y)$

$$0 \to (F_1', F_2', \phi') \to (F_1, F_2, \phi) \to (F_1'', F_2'', \phi'') \to 0$$

is exact if and only if the sequences

$$0 \to F_1' \to F_1 \to F_1'' \to 0$$

and

$$0 \to F_2' \to F_2 \to F_2'' \to 0$$

are exact, since a sequence

$$0 \to F' \to F \to F'' \to 0$$

is exact if and only if

$$0 \to i^*F' \to i^*F \to i^*F'' \to 0$$

and

$$0 \to j^*F' \to j^*F \to j^*F'' \to 0$$

are exact (consider the stalks).

Definition 11.4 Identifying $S(X_{\text{ét}})$ with $T(X,Y)$, we can define the following six functors

$$\begin{array}{ccccc} & \overset{i^*}{\leftarrow} & & \overset{j_!}{\leftarrow} & \\ S(Y_{\text{ét}}) & \overset{i_*}{\rightarrow} & S(X_{\text{ét}}) & \overset{j^*}{\rightarrow} & S(U_{\text{ét}}) \\ & \overset{i^!}{\leftarrow} & & \overset{j_*}{\leftarrow} & \end{array}$$

by

$$\begin{array}{ccccccc} F_1 & \overset{i^*}{\leftarrow\!\shortmid} & (F_1,F_2,\phi) & , & (0,F_2,0) & \overset{j_!}{\leftarrow\!\shortmid} & F_2 \\ F_1 & \overset{i_*}{\mapsto} & (F_1,0,0) & , & (F_1,F_2,\phi) & \overset{j^*}{\mapsto} & F_2 \\ \ker\phi & \overset{i^!}{\leftarrow\!\shortmid} & (F_1,F_2,\phi) & & (i^*j_*F_2,F_2,id) & \overset{j_*}{\leftarrow\!\shortmid} & F_2 \end{array}$$

One calls $j_!F$ the "extension by zero" of F and $i^!F \subseteq F$ the "subsheaf of the sections with support in Y".

It is obvious that these assignments are functorial.

Theorem 11.5 (a) Under the identification between $S(X_{\text{ét}})$ and $T(X,Y)$, the functors i^*, i_*, j^* and j_* correspond to the functors with the same name between $S(X_{\text{ét}})$ and $(Y_{\text{ét}})$, and $S(X_{\text{ét}})$ and $S(U_{\text{ét}})$.

(b) Every functor in 11.3 is left adjoint to the functors below it.

(c) The functors i^*, i_*, j^* and $j_!$ are exact; j_* and $i^!$ are left exact.

(d) The compositions $i^*j_!, i^!j_!$ and j^*i_* are zero.

(e) The functors i_* and j_* are fully faithful, and i_* induces an equivalence of categories

$$Sh(Y_{\text{ét}}) \underset{\sim}{\overset{i_*}{\to}} \{F \in Sh(X_{\text{ét}}) \mid F \text{ has support in } Y\}$$

(f) The functors $j_*, j^*, i^!$ and i_* respect injectives.

Proof: (a) is clear from the identification between $S(X_{\text{ét}})$ and $T(X,Y)$.

(b) for (i^*, i_*) and (j^*, j_*) follows from (a), i.e., from the fact that these form adjoint pairs. For the two remaining cases we have canonical isomorphisms

$$Hom_X((0,F_2,0),(G_1,G_2,\phi)) \cong Hom_U(F_2,G_2)$$

and
$$Hom_X((F_2,0,0),(G_1,G_2,\phi)) \cong Hom_Y(F_2,\ker(G_1 \xrightarrow{\phi} i^*j_*G_2))\,.$$

(c) follows immediately from Remark 11.2, and

(d) follows from the description of the functors

In (e), the fully faithfulness of i_* again follows from the description. For j_*, a morphism
$$(i^*j_*F_2, F_2, id) \to (i^*j_*F'_2, F'_2, id)$$
is again determined only by $F_2 \to F'_2$, by the commutative diagram (11.0).

With this, the second claim in (d) follows, since (F_1, F_2, ϕ) has support in Y if and only if $F_2 = 0$ and hence $\phi = 0$.

(f) follows from the fact that these functors have exact left adjoints.

Corollary 11.6 (a) For an open immersion $j : U \hookrightarrow X$ and an étale sheaf $\mathcal{F}$ on U one has $j^*j_*\mathcal{F} = \mathcal{F}$.

(b) For a closed immersion $i : Y \hookrightarrow X$ and an étale sheaf $\mathcal{G}$ on Y one has $i^*i_*\mathcal{F} = \mathcal{F}$.

Proof: This follows from Theorem 11.4 (a) and the description of the functors in Definition 11.3. In fact, we get
$$j^*j_*\mathcal{F} = j^*(i^*j_*\mathcal{F}, \mathcal{F}, id) = \mathcal{F}\,,$$
and
$$i^*i_*\mathcal{G} = i^*(\mathcal{G}, 0, 0) = \mathcal{G}\,.$$

Corollary 11.7 For every étale sheaf F on X, one has exact sequences

(a)
$$0 \to j_!j^*F \to F \to i_*i^*F \to 0$$

(b)
$$0 \to i_*i^!F \to F \to j_*j^*F$$

Proof Again this follows from the description in triples. In fact, (a) corresponds to the exact sequence
$$0 \to (0, j^*F, 0) \to (i^*F, j^*F, \phi) \to (i^*F, 0, 0) \to 0\,,$$
and (b) corresponds to the exact sequence
$$0 \to (\ker\phi, 0, 0) \to (i^*F, j^*F, \phi) \to (i^*j_*j^*F, j^*F, \phi)\,.$$

12 Čech cohomology

The following generalizes the zero-th Čech cohomology (see definition 3.7) and the topological Čech cohomology. We consider presheaves with values in abelian groups.

Lemma/Definition 12.1 Let $\mathcal{S} = (\mathcal{X}, \mathcal{T})$ be a site.

(a) For a presheaf P on $\mathcal{X}$ and a covering $\mathfrak{U} = (U_i \to U)_{i \in I}$ in $\mathcal{T}$ and $n \geq 0$, the group

$$C^n(\mathfrak{U}, P) := \prod_{(i_0,\dots,i_n) \in I^{n+1}} P(U_{i_0} \times_U \dots \times_U U_{i_n})$$

is called the group of the n-**cochains for the covering** $\mathfrak{U}$ **with values in** P. Define the differential

$$d^n : C^n(\mathfrak{U}, P) \to C^{n+1}(\mathfrak{U}, P)$$

by

$$(d^n s)_{i_0,\dots,i_{n+1}} = \sum_{\nu=0}^{n+1} (-1)^\nu s_{i_0,\dots,\hat{i}_\nu,\dots,i_{n+1}}|_{U_{i_0} \times_U \dots \times_U U_{i_{n+1}}}$$

where the restriction with respect to the morphism

$$U_{i_0} \times_U \dots \times_U U_{i_{n+1}} \to U_{i_0} \times_U \dots \times_U \hat{U}_{i_\nu} \times_U \dots \times_U U_{i_n+1}$$

is taken and $\hat{a}$ denotes the omission of a. Then $d^{n+1}d^n = 0$ for all n and we obtain a complex $C^\cdot(\mathfrak{U}, P)$, called the Čech complex for the covering $\mathfrak{U}$ with values in P.

(b) The n-th cohomology

$$\check{H}^n(\mathfrak{U}, P) := H^n(C^\cdot(\mathfrak{U}, P))$$

is called the n-th Čech cohomology of P for covering $\mathfrak{U}$.

Proof that $d^{n+1}d^n = 0$: left to the readers! (standard).

Remark 12.2 For $n = 0$, we obviously obtain the zero-th Čech-cohomology from Definition 3.7 (a).

Lemma 12.3 Let $(V_j \to U)_{j \in J}$ and $(U_i \to U)_{i \in I}$ be coverings and let

$$f = (\varepsilon, f_j) : (V_j \to U) \to (U_i \to U)$$

be a refinement map. This induces maps for all n

$$\check{H}^n((U_i \to U), P) \to \check{H}^n((V_j \to U), P)\,.$$

Proof We have a map $\varepsilon : J \to I$ and morphisms $f_j : V_j \to U_{\varepsilon(j)}$. With this we define the map

$$f^n : C^n((U_i \to U), P) \to C^n((V_j - U), P)$$

as follows: If $s = (s_{i_0,\dots,i_n}) \in C^n((U_i \to U), P)$, then define

$$(f^n s)_{j_0,\dots,j_n} = res_{f_{j_0} \times \dots \times f_{j_n}}(s_{\varepsilon(j_0),\dots,\varepsilon(j_n)})\,,$$

where the restriction is taken with respect to

$$f_{j_0} \times \ldots \times f_{j_n} : V_{j_0} \times_U \ldots \times_U V_{j_n} \to U_{\varepsilon(j_0)} \times_U \ldots \times_U U_{\varepsilon(j_n)} .$$

These maps commute with the differentials d^n, hence give a morphism of complexes

$$f^* : C^{\cdot}((U_i \to U), P) \to C^{\cdot}((V_j \to U), P) ,$$

which induces the desired map in the cohomology.

Remark 12.4 On $\check{H}^0(-, P)$, this map coincides with the map defined in 3.7!

Lemma 12.5 If

$$f, g : (V_j \to U) \to (U_i \to U)$$

are two refinement maps, then we have

$$f^* = g^* : \check{H}^n((U_i \to U), P) \to \check{H}^n((V_j \to U), P)$$

for all $n \geq 0$.

Proof (compare Lemma 3.11) Let $f = (\varepsilon, f_j)$ and $g = (\eta, g_j)$. Define

$$k^n : C^n((U_i \to U), P) \to C^{n-1}(V_j \to U), P)$$

by

$$(k^n s)_{j_0,\ldots,j_{n-1}} = \sum_{i=0}^{p-1} (-1)^r res_{f_{j_0} \times \ldots \times (f_{j_r}, g_{j_r}) \times \ldots \times g_{j_{n-1}}} s_{\varepsilon(j_0),\ldots,\varepsilon(j_r),\eta(j_r),\ldots,\eta(j_{n-1})}$$

for $(f_{j_r}, g_{j_r}) : V_{j_r} \to U_{\varepsilon(j_r)} \times U_{\eta(j_r)}$. Then we have

$$d^{n-1}k^n + k^{n+1}d^n = g^n - f^n ,$$

i.e., (k^n) provides a chain homotopy between (f^n) and (g^n), and the claim follows.

Definition 12.6 The n-th Čech cohomology of U with values in P is defined as

$$\check{H}^n(U, P) := \check{H}^n(U, \mathcal{T}; P) = \lim_{\to} \check{H}^n(\mathfrak{U}, P)$$

where the limit runs over all coverings of U (in $\mathcal{T}$).

Remark 12.7 Because of 12.5, this is a limit over the inductively ordered *set* $\mathcal{T}(U)_0$ of all coverings $\mathfrak{U}$ of U, where $\mathfrak{U}' \geq \mathfrak{U}$, if there is a refinement map $f : \mathfrak{U}' \to \mathfrak{U}$ (see the proof of 3.11).

Lemma 12.8 Let

(12.8.1) $$0 \to P_1 \to P_2 \to P_3 \to 0$$

be an exact sequence of abelian presheaves (!) on $\mathcal{X}$.

(a) For every covering $\mathfrak{U}$ in $\mathcal{T}$ there is a long exact cohomology sequence

$$0 \to \check{H}^0(\mathfrak{U}, P_1) \to \check{H}^0(\mathfrak{U}, P_2) \to \check{H}^0(\mathfrak{U}, P_3) \xrightarrow{\delta} H^1(\mathfrak{U}, P_1) \to \dots ,$$

which is functorial with respect to refinement maps and morphisms of exact sequences (12.8.1).

(b) For every U in $\mathcal{X}$ there is a long exact cohomology sequence

$$0 \to \check{H}^0(U, P_1) \to \check{H}^0(U, P_2) \to \check{H}^0(U, P_3) \xrightarrow{\delta} \check{H}^1(U, P_1) \to \check{H}^1(U, P_2) \to \dots ,$$

which is functorial for restriction maps and for morphisms of exact sequences (12.8.1).

Proof (a): We have an exact sequence of complexes

$$0 \to C^\cdot(\mathfrak{U}, P_1) \to C^\cdot(\mathfrak{U}, P_2) \to C^\cdot(\mathfrak{U}, P_3) \to 0 , \tag{12.8.2}$$

since for $\mathfrak{U} = (U_i \to U)_{i\in I}$ and every $(i_0, \dots, i_n) \in I^{n+1}$ the sequence

$$0 \to P(U_{i_0,\dots,i_n}) \to P_2(U_{i_0,\dots,i_n}) \to P_3(U_{i_0,\dots,i_n}) \to 0 \tag{12.8.2}$$

is exact, where $U_{I_0,\dots,i_n} := U_{i_0} \times_U \dots \times_U U_{i_n}$. The sequence in (a) is the long exact cohomology sequence for (12.8.2). The functorialities follow, since (12.8.2) is functorial in $\mathfrak{U}$ and in (12.8.1).

(b) follows from (a) by passing to the inductive limit over all coverings of U (see Remark 12.7), since forming an inductive limit is an exact functor.

Remark 12.9 For an exact sequence of $\mathcal{T}$-sheaves

$$0 \to F_1 \to F_2 \to F_3 \to 0 , \tag{12.9.1}$$

one does not obtain a long exact sequence of Čech cohomology groups in general, since (12.9.1) is generally not exact as a sequence of presheaves.

Example 12.10 Let $A \to B$ a faithfully flat ring homomorphism, which is locally of finite type (resp. locally of finite presentation). Then $(V = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = U)$ is a covering in the flat topology. The associated Čech complex for the presheaf $\mathbb{G}_{a,X}$ is the complex

$$0 \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \dots$$

By descent theory (see [Mi] I. 2.17, and see also 14.6 below) we have

$$\check{H}^n((V \to U), \mathbb{G}_a) = \begin{cases} A & , \quad n = 0, \\ 0 & , \quad n > 0 . \end{cases}$$

One can also obtain the Čech cohomology as a derived functor:

Theorem 12.11 (a) For a covering $\mathfrak{U} = (U_i \to U)$, $\check{H}^n(\mathfrak{U}, -)$ is the n-th right derivative of the left exact functor

$$\begin{array}{rcl} \check{H}^0(\mathfrak{U}, -) : Pr(\mathcal{X}) & \to & \underline{Ab} \\ P & \mapsto & \check{H}^0(\mathfrak{U}, P) . \end{array}$$

(b) For $U \in ob(\mathcal{X})$, $\check{H}^n(U,-)$ is the n-th right derivative of the left exact functor

$$\begin{array}{rcl}\check{H}^0(U,-) : Pr(\mathcal{X}) & \to & \underline{Ab} \\ P & \mapsto & \check{H}^0(U,P)\,.\end{array}$$

Proof It follows from 12.8 that the functors $(\check{H}^n(\mathfrak{U},-))_{n\geq 0}$ resp. $(\check{H}^n(U,-))_{n\geq 0}$ form exact δ-functors on $Pr(\mathcal{X})$. Thus it suffices to show that $\check{H}^n(\mathfrak{U},-)$ resp. $\check{H}^n(U,-)$ is effaceable for $n > 0$: then these give universal δ-functors, this is also known for the right derivatives, and two universal δ-functors are obviously isomorphic. Since $Pr(\mathcal{X})$ has enough injectives, it suffices to show:

Lemma 12.12 For $n > 0$ we have $\check{H}^n(\mathfrak{U}, I) = 0 = \check{H}^n(U, I)$, if I is an injective presheaf.

Proof The second claim follows from the first. Furthermore, we have to show that for every covering $\mathfrak{U} = (U_i \to U)_{i\in I}$ in $\mathcal{T}$ the sequence

$$\prod_i I(U_i) \to \prod_{i_0,i_1} I(U_{i_0i_1}) \to \prod_{i_0,i_1,i_2} I(U_{i_0i_1i_2}) \to \ldots$$

is exact, where $U_{i_0,\ldots,i_n} = U_{i_j} \times_U U_{i_1} \times_U \ldots \times_U U_{i_n}$. This sequence can be identified with a sequence

(12.12.1) $$\prod_i Hom(\mathbb{Z}^P_{U_i}, I) \to \prod_{i_0,i_1} Hom(\mathbb{Z}^P_{U_{i_0,i_1}}, I) \to \ldots \quad ,$$

(see Lemma/Definition 5.8) which comes from an obvious complex of presheaves

(12.12.2) $$\bigoplus_i \mathbb{Z}^P_{U_i} \leftarrow \bigoplus_{i_0,i_1} \mathbb{Z}^P_{U_{i_0i_1}} \leftarrow \bigoplus_{i_0,i_1,i_2} \mathbb{Z}_{U_{i_0i_1i_2}} \leftarrow \ldots .$$

by applying $Hom(-, I)$. The last functor is exact since I is injective. Therefore it suffices to show the exactness of (12.12.2), hence the exactness of

(12.12.3) $$\bigoplus_i \mathbb{Z}^P_{U_i}(V) \leftarrow \bigoplus_{i_0,i_1} \mathbb{Z}_{U_{i_0i_1}}(V) \leftarrow \ldots$$

for every V in $\mathcal{X}$.

Now, we have $\mathbb{Z}^P_W(V) = \bigoplus_{Hom(V,W)} \mathbb{Z}$ for W in $\mathcal{X}$, and we have a canonical morphism $U_{i_0\ldots i_n} \to U$ for every $W = U_{i_0i_1\ldots i_n} = U_{i_j} \times_U U_{i_1} \times_U \ldots \times_U U_{i_n}$. This implies

$$Hom(V, W) = \coprod_{\phi\in Hom(V,U)} Hom_\phi(V, W)\,,$$

where $Hom_\phi(V, W)$ is the set of the morphisms $\varphi : V \to W$ for which

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\phi"'] & & W \arrow[dl] \\ & U & \end{tikzcd}$$

is commutative. Furthermore, by the universal property of the fiber product, we have

$$Hom_\phi(V, U_{i_0} \times_U \ldots \times_U U_{i_n}) = Hom_\phi(V, U_{i_0}) \times \ldots \times Hom_\phi(V, U_{i_n})\,.$$

As a consequence, if we define

$$S(\phi) = \coprod_{i \in I} Hom_\phi(V, U_i)\,,$$

then the complex (12.12.3) can described as follows :

$$\bigoplus_{\phi \in Hom(V,U)} \Big(\bigoplus_{S(\phi)} \mathbb{Z} \leftarrow \bigoplus_{S(\phi)\times S(\phi)} \mathbb{Z} \leftarrow \bigoplus_{S(\phi)^3} \mathbb{Z} \leftarrow \ldots \Big)$$

with the obvious differential in the bracket

$$1_{j_0,\ldots,j_p} \mapsto \sum_{\nu=0}^{p} (-1)^\nu 1_{j_0,\ldots,\hat{j}_\nu,\ldots j_p}\,.$$

Now the complex in the bracket is exact: A contracting homotopy is $(h^p, p \geq 0)$, with

$$\begin{array}{rccc} h^p : & \bigoplus_{S(\phi)^{p+1}} \mathbb{Z} & \to & \bigoplus_{S(\phi)^{p+2}} \mathbb{Z} \\ & 1_{i_0,\ldots,i_p} & \mapsto & 1_{e,i_0,\ldots,i_p}\,, \end{array}$$

where $e \in S(\phi)$ is a fixed element (check!).

Theorem 12.13 Let $U \in ob(\mathcal{X})$ and $\mathfrak{U} = (U_i \to U)$ be a covering in $\mathcal{T}$ and let F be a sheaf (with respect to $\mathcal{T}$). There are spectral sequences

$$\begin{array}{rcl} E_2^{p,q} = \check{H}^p(\mathfrak{U}, \underline{H}^q(F)) & \Rightarrow & H^{p+q}(U,F) \\ E_2^{p,q} = \check{H}^p(U, \underline{H}^q(F)) & \Rightarrow & H^{p+q}(U,F)\,. \end{array}$$

Here, cohomology and Čech-cohomology are taken with respect to $\mathcal{T}$, and $\underline{H}^q(F)$ is the presheaf

$$V \mapsto H^q(V,F)\,.$$

Proof First spectral sequence: We apply Grothendieck's Theorem (Theorem 6.8). Obviously, we have

$$\check{H}^0(\mathfrak{U}, \underline{H}^0(F)) = H^0(U,F)\,,$$

since F is a sheaf, hence $H^0(U,-)$ is the composition of $\underline{H}^0(-)$ and $\check{H}^0(\mathfrak{U},-)$. Furthermore, for an injective sheaf I, the presheaf $\underline{H}^0(I)$ is acyclic for $\check{H}^0(\mathfrak{U},-)$, i.e.,

$$R^n\check{H}^0(\mathfrak{U}, \underline{H}^0(I)) \overset{12.11}{=} \check{H}^n(\mathfrak{U}, \underline{H}^0(I)) = \check{H}^n(\mathfrak{U}, I) = 0 \quad \text{for} \quad n > 0\,.$$

In fact, the embedding $i : Sh(\mathcal{X},\mathcal{T}) \to Pr(\mathcal{X})$ respects injectives, since i has the exact left adjoint a. Therefore, I is also injective as a presheaf, and the last vanishing follows from Lemma 12.12.

The second spectral sequence follows in an analogous way, or by passing to the limit over all coverings of U.

Corollary 12.14 There is a spectral sequence

$$E_2^{p,q} = \underline{\check{H}}^p(\underline{H}^q(F)) \to \underline{H}^{p+q}(F)\,.$$

Proof Consider the second spectral sequence in 12.13 for all U in $\mathcal{X}$ and note that this is functorial (contravariant) in U.

Proposition 12.15 We have

$$\check{H}^0(U, \underline{H}^q(F)) = 0 \quad \text{for } q > 0\,,$$

and therefore also $\underline{\check{H}}^0(\underline{H}^q(F)) = 0$ for $q > 0$.

Proof Let $F \hookrightarrow I^{\cdot}$ be an injective resolution in $Sh(\mathcal{X}, \mathcal{T})$. Then $\underline{H}^q(F)$ is the q-th cohomology presheaf of the complex $i(I^{\cdot})$. Since a is exact, a commutes with taking the cohomology, hence $a\underline{H}^q(F) = \mathcal{H}^q(aiI^{\cdot}) = \mathcal{H}^q(I^{\cdot}) = 0$ for $q > 0$ (Here, $\mathcal{H}^q$ denotes the q-th cohomology sheaf). But $\underline{\check{H}}^0(\underline{H}^q(F))$ is a sub-presheaf of $a\underline{H}^q(F) = \underline{\check{H}}^0\underline{\check{H}}^0(H^q(F))$ (since $\underline{\check{H}}^0(P)$ is separated for every presheaf P, see Lemma 3.10 (c) and note that by definition $\tilde{P} = \underline{\check{H}}^0(P)$). Therefore, $\underline{\check{H}}^0(\underline{H}^q(F)) = 0$ for all $q > 0$, hence $\check{H}^0(U, \underline{H}^q(F)) = 0$ for all U.

Corollary 12.16 For every sheaf F on $(\mathcal{X}, \mathcal{T})$ and every U in $\mathcal{X}$ there are canonical isomorphisms

$$\begin{array}{rcl} \check{H}^0(U,F) & \cong & H^0(U,F) \\ \check{H}^1(U,F) & \cong & H^1(U,F) \end{array}$$

and an exact sequence

$$0 \to \check{H}^2(U,F) \to H^2(U,F) \to \check{H}^1(U, \underline{H}^1(F)) \to \check{H}^3(U,F) \to H^3(U,F)$$

Proof The spectral sequence

$$\check{H}^p(U\underline{H}^q(F)) \Rightarrow H^{p+q}(U,F)$$

has the following shape (on the q-axis the initial terms are zero, except in the origin)

$$\begin{array}{lllll} q & & & & \\ 0 & & & & \\ 0 & \bullet & & & \\ 0 & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \\ & & & & \quad p \end{array}$$

The first claim in 12.16 ist obvious, since F is a sheaf, and from the above shape we get

$$\begin{array}{rcl} H^1(U,F) \ \cong \ E_2^{1,0} & = & \check{H}^1(U, \underline{H}^0(F)) \\ & = & \check{H}^1(U,F)\,, \end{array}$$

since $E_2^{0,1} = 0$ and all differentials leaving and entering $E_2^{1,0}$ are zero. Deducing the last sequence is analogous to the proof of the usual sequence of the low terms (Lemma 6.7).

Corollary 12.17 Let X be a scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open covering of X such that

$$U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_n}$$

is affine for all n and all $i_0, \ldots, i_n \in I$. Then one has a canonical isomorphism

$$\check{H}^n(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^n(X, \mathcal{F})$$

for all $n \geq 0$.

Proof We use the spectral sequence

$$\check{H}^p(\mathfrak{U}, \underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}) .$$

For $q > 0$, $H^q(U_{i_0} \cap \ldots \cap U_{i_n}, \mathcal{F}) = 0$ by Serre's vanishing theorem. This implies the claim.

Definition 12.18 A sheaf F on a site $(\mathcal{X}, \mathcal{T})$ is called **flabby**, if $H^n(U, F) = 0$ for all $U \in ob(\mathcal{X})$ and all $n > 0$.

Example 12.19 Every injective sheaf is flabby.

Proposition 12.20 For a sheaf F the following conditions are equivalent:

(a) F is flabby.

(b) For every U in $\mathcal{X}$ and every covering $\mathfrak{U}$ of U (in a cofinal family), $\check{H}^n(\mathfrak{U}, F) = 0$ for $n > 0$.

(c) $\check{H}^n(U, F) = 0$ for all U in $\mathcal{X}$ and all $n > 0$.

Proof (a) $\Rightarrow$ (b): If F is flabby, then $\underline{H}^q(F) = 0$ is flabby for $q > 0$. The first spectral sequence of 12.13 therefore provides an isomorphism

$$\check{H}^n(\mathfrak{U}, F) \xrightarrow{\sim} H^n(U, F) = 0 \quad \text{for} \quad n > 0 .$$

(b) $\Rightarrow$ (c): Follows by passing to the inductive limit over all coverings of U.

(c) $\Rightarrow$ (a): By assumption, we have $\underline{\check{H}}^n(F) = 0$ for $n > 0$. By Corollary 12.16, we have $\underline{H}^1(F) = 0$. Now we use induction over n, via the spectral sequence of 12.14

$$\underline{\check{H}}^p(H^q(F)) \Rightarrow \underline{H}^{p+q}(F) .$$

By assumption, $\check{H}^2(\underline{H}^0(F)) = \underline{\check{H}}^2(F) = 0$, furthermore $\underline{\check{H}}^1(\underline{H}^1(F)) = 0$ and $\underline{\check{H}}^0(\underline{H}^2(F)) = 0$ by 12.15. From the spectral sequence we get $\underline{H}^2(F) = 0$. The same argument shows inductively

$$\underline{\check{H}}^i(\underline{H}^j(F)) = 0 \quad \text{for} \quad i + j \leq n$$

and hence $\underline{H}^n(F) = 0$.

Corollary 12.21 Let $f : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ be a morphism of sites. If F' is a flabby sheaf on $(\mathcal{X}', \mathcal{T}')$, them f_*F' is flabby, too.

Proof Let $f^0 : \mathcal{X} \to \mathcal{X}'$ be the underlying functor. If $\mathfrak{U} = (U_i \to U)$ is a covering in $\mathcal{T}$, then $\mathfrak{U}' = (f^0 U_i \to f^0 U)$ is a covering in $\mathcal{T}'$ and we have

$$(f_* F')(V) = F(f^0 V)$$

for all V in $\mathcal{X}$. Then we have

$$\check{H}^n(\mathfrak{U}, f_* F') = \check{H}^n(\mathfrak{U}', F') = 0 \quad \text{for} \quad n > 0\,.$$

Corollary 12.22 (Leray spectral sequence) (a) If $f : (\mathcal{X}'', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ is a morphism of sites, then for every sheaf F' on $(\mathcal{X}', \mathcal{T}')$ and every $U \in \mathcal{X}$ one has a spectral sequence

$$H^p(U, R^q f_* F') \Rightarrow H^{p+q}(f^0 U, F')\,.$$

(b) If $(\mathcal{X}'', \mathcal{T}'') \to (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$ are morphisms of sites, then for every sheaf F'' on $(\mathcal{X}'', \mathcal{T}'')$ there is a spectral sequence

$$R^p f_* R^q g_* F'' \Rightarrow R(gf)_* F''\,.$$

Proof This follows from Theorem 6.8 (Grothendieck's spectral sequence), since f_* maps flabby sheaves to flabby sheaves, therefore acyclic sheaves for $H^0(U, -)$, and g_* as well maps flabby sheaves to flabby sheaves, and therefore to acyclic sheaves for f_*. In fact, $R^n f_* F'$ is the associated sheaf to the presheaf $U \mapsto H^n(f^0 U, F')$, and the presheaf is already 0, if F' is flabby.

13 Comparison of sites

Proposition 13.1 (Change of the category) Let $(\mathcal{X}', \mathcal{T}')$ be a site, let $\mathcal{X} \subseteq \mathcal{X}'$ be a full subcategory, and let $\mathcal{T}$ be the restriction of $\mathcal{T}'$ to $\mathcal{X}$.

We assume:

(13.1.1) For every object U in $\mathcal{X}$ and every covering $(U_i \to U)$ in $\mathcal{T}'$, all U_i are already in $\mathcal{X}$.

For the morphism of sites
$$\alpha : (\mathcal{X}', \mathcal{T}') \to (\mathcal{X}, \mathcal{T}),$$
which is given by the embedding $\mathcal{X} \hookrightarrow \mathcal{X}'$, we then have:

(a) The functor $\alpha_* : Sh(\mathcal{X}', \mathcal{T}') \to Sh(\mathcal{X}, \mathcal{T})$ is exact and the adjunction map $F \to \alpha_*\alpha^*F$ is an isomorphism for all $F \in Sh(\mathcal{X}, \mathcal{T})$.

(b) The functor $\alpha^* : Sh(\mathcal{X}, \mathcal{T}) \to Sh(\mathcal{X}', \mathcal{T}')$ is fully faithful and left exact.

(c) The canonical homomorphisms
$$\begin{array}{rcl} H^n(U, \mathcal{T}; \alpha_*F') & \to & H^n(U, \mathcal{T}'; F') \\ H^n(U, \mathcal{T}; F) & \to & H^n(U, \mathcal{T}'; \alpha^*F) \end{array}$$
are isomorphisms for all $U \in \mathcal{X}$, all $F' \in Sh(\mathcal{X}', \mathcal{T}')$, all $F \in Sh(\mathcal{X}, \mathcal{T})$ and all $n \geq 0$.

Proof (a) α_* is simply the restriction; so that the exactness is obvious by (13.1.1). Furthermore, for every $U \in \mathcal{X}$ and $F \in Sh(\mathcal{X}, \mathcal{T})$ we have
$$(\alpha^P F)(U) = F(U),$$
since the category I_U, over which the limit is formed for $(\alpha^P F)(U)$, has the initial object (U, id_U). Since F restricted to $(\mathcal{X}, \mathcal{T})$ is a sheaf, we have $(\alpha^*F)(U) = (a\alpha^P F)(U) = F(U)$. Moreover,
$$(\alpha_*\alpha^*F)(U) = (\alpha^*F)(U) = F(U),$$
hence the second claim follows.

(b) This follows from the proof above.

(c) We have a spectral sequence

(13.1.2)
$$E_2^{p,q} = H^p(U, R^q\alpha_*F) \Rightarrow H^{p+q}(U, F),$$

either by Corollary 12.22, or by the Grothendieck-Leray spectral sequence 6.7. This exists, since $H^0(U, \alpha_*F) = H^0(U, F)$, and since α_* has a the left exact left adjoint α^*, so that by 7.10, α_* maps injectives to injectives (hence acyclic sheaves).

Since α_* is exact, we have $R^q\alpha_*F = 0$ for $q > 0$, and (13.1.2) provides an edge isomorphism
$$H^n(U, \alpha_*F) \xrightarrow{\sim} H^n(U, F).$$

The second claim follows from the fact that the composition
$$H^n(U, F) \to H^n(U, \alpha^*F) \xrightarrow{\sim} H^n(U, \alpha_*\alpha^*F) \xrightarrow{\sim} H^n(U, F)$$

is the identity (consider injective resolutions).

Example 13.2 One can apply this to the morphism of sites

$$\alpha : (Sch/X)_E \to X_E\,,$$

where X is a scheme, E is a admissible category of morphisms, $(Sch/X)_E$ is the category of all X-schemes with the E-coverings as topology (the large E-site) and X_E is the category of all X-schemes U, for which the structural morphism $U \to X$ is in E, with the E-coverings as topology (the small E-site). In particular, the large étale site $(Sch/X)_{\text{ét}}$ and the small étale site $X_{\text{ét}}$ give the "same cohomology" (via α_* and α^*, respectively).

Proposition 13.3 (Change of the topology) Let $\mathcal{X}$ be a category and let $\mathcal{T} \subset \mathcal{T}'$ be topologies on $\mathcal{X}$ (every covering for $\mathcal{T}$ is a covering for $\mathcal{T}'$). Let

$$\beta : (\mathcal{X}, \mathcal{T}') \to (\mathcal{X}, \mathcal{T})$$

be the morphism of sites, given by $id_{\mathcal{X}}$. Assume that for every covering $\mathfrak{U} = (U_i \to U)$ in $\mathcal{T}'$ there is a covering $(V_j \to U)$ in $\mathcal{T}$, which refines $\mathfrak{U}$. Then $\beta_* : Sh(\mathcal{X}, \mathcal{T}') \to Sh(\mathcal{X}, \mathcal{T})$ is exact and thus

$$H^n(\mathcal{X}_{\mathcal{T}}, \beta_* F') \tilde{\to} H^n(\mathcal{X}_{\mathcal{T}'}, F')$$

for every sheaf $F' \in Sh(\mathcal{X}, \mathcal{T}')$.

Proof: The functor β_* is the identity. We only have to show that every epimorphism $F \to F''$ for $\mathcal{T}'$ is an epimorphism for $\mathcal{T}$ as well. If we have $U \in ob(\mathcal{X})$ and $s \in F''(U)$, then there is a covering $(U_i \to U)$ in $\mathcal{T}'$, so that, for all i, $s|_{U_i}$ is in the image of $F(U_i) \to F''(U_i)$. If now $(V_j \to U) \in \mathcal{T}$ is a refinement of $(U_i \to U)$, and if $V_j \to U$ factorizes as $V_j \to U_i \to U$, we then obtain a commutative diagram

$$\begin{array}{ccc} F(V_j) & \longrightarrow & F''(V_j) \\ \uparrow & & \uparrow \\ F(U_i) & \longrightarrow & F''(U_i)\,, \end{array}$$

which shows that $s|_{V_j}$ is in the image of the top map. Thus β_* is exact. The second claim follows from the Leray-spectral sequence, since $R^p\beta_* F' = 0$ for $p > 0$, by the exactness of β_*.

Corollary 13.4 Let X be a scheme, and let $E \subset E'$ be two admissible classes of morphisms, so that the condition of refinement of 13.3 holds for the corresponding topologies $(E) \subset (E')$. For the morphism of small sites

$$\gamma : X_{E'} \to X_E\,,$$

the functor $\gamma_* : Sh(X_{E'}) \to Sh(X_E)$ is exact, hence for every sheaf F' on $X_{E'}$ we have

$$H^n(U, E; \gamma_* F') \tilde{\to} H^n(U, E', F')\,.$$

Proof Let $E - Sch/X$ be the category of the X-schemes U, whose structural morphism $U \to X$ is in E. Then γ factorizes as

$$\gamma : X_{E'} \xrightarrow{\beta} (E' - Sch/X)_E \xrightarrow{\alpha} (E - Sch/X)_E = X_E\,,$$

and the claim follows from 13.1 (exactness of α_*) and 13.3 (exactness of β_*).

Example 13.5 We have the following examples for Corollary 13.4

(a) (étale morphisms of finite type) $\subset$ (ét)

(b) (ét) $\subset$ (fl), if X is quasi-compact (Milne, Étale Cohomology', I 3.26).

(c) $(fpqc)$ $\subset$ (fl), where $(fpqc)$ is the class of the flat "quasi-compact" morphism. (Milne, I 2.25).

Theorem 13.6 (quasi-coherent $\mathcal{O}_X$-modules) Let X be a scheme and $\mathcal{M}$ a quasi-coherent $\mathcal{O}_X$-module. Let $\mathcal{M}_{\text{ét}}$ and M_{fl} be the corresponding sheaves in the étale topology and flat topology, respectively (see 10.18). Then we have

$$H^n(X_{\text{Zar}}, \mathcal{M}) \xrightarrow{\sim} H^n(X_{\text{ét}}, \mathcal{M}_{\text{ét}}) \xrightarrow{\sim} H^n(X_{\text{fl}}, \mathcal{M}_{fl})$$

for all n.

Proof We give the proof for the flat topology, the case of the étale topology is analogous. Let

$$f : X_{\text{fl}} \to X_{\text{Zar}}$$

be the morphism of sites (see 13.4). It is obvious that $f_*\mathcal{M}_{\text{fl}} = \mathcal{M}$, therefore it suffices to show that $R^n f_*\mathcal{M}_{\text{fl}} = 0$ for $n > 0$ (then the claim follows from the Leray-spectral sequence). Since $R^n f_*\mathcal{M}_{\text{fl}}$ is the Zariski-sheaf associated to the presheaf $U \mapsto H^n(U_{\text{fl}}, \mathcal{M}_{\text{fl}})$ (see Theorem 5.16), it suffices to show that $H^n(U_{\text{fl}}\mathcal{M}_{\text{fl}}) = 0$, if $U = \text{Spec}(A) \subseteq X$ is affine open. Furthermore, by Corollary 13.4, we can consider the small site U_E , where E is the class of flat affine morphisms of finite presentation. We want to show that $\mathcal{M}_{\text{fl}}$ is flabby, and by 13.20 it suffices to show that $\check{H}^n(\mathfrak{U}, \mathcal{M}_{\text{fl}}) = 0$ for all $n > 0$ and all coverings $\mathfrak{U} = (U_i \to U)_{i\in I}$ in E. By 10.6 and the quasi compactness of U, we can assume that I is finite (cofinite system of E-coverings!). Then $V = \coprod\limits_i U_i = \text{Spec}(B)$ is affine and $A \to B$ is faithfully flat, and the Čech complex is the obvious complex

$$B \otimes_A M \to B \otimes_A B \otimes_A M \to B \otimes_A B \otimes_A B \otimes_A M \to \dots ,$$

which is exact in degrees ≥ 1, as was noted in 12.10.

Remark 13.7 (comparison isomorphism over $\mathbb{C}$). Let X be a smooth variety over $\mathbb{C}$. Then the theorem of implicit functions implies that the set $X(\mathbb{C})$ is a complex manifold: It suffices to show this locally. But locally we have

$$X = \text{Spec}(\mathbb{C}[X_1, \dots, X_n]/\langle f_1, \dots, f_m\rangle)\, , m \leq n\, ,$$

where the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial X_j}(P)\right)$$

has rank m for all closed points P of X. Then we have

$$X(\mathbb{C}) \cong \{a = (a_1, \dots, a_n) \in \mathbb{C}^n \mid f_j(a) = 0 \quad \forall\, j\}\, ,$$

and the f_i define continuous maps

$$f_i : \mathbb{C}^n \to \mathbb{C}\,,$$

and the matrix above is the usual Jacobi matrix at P. The theorem on implicit function gives local homomorphisms

$$X(\mathbb{C}) \supseteq V \xrightarrow{\sim} U \subseteq \mathbb{C}^{n-m}\,,$$

and one obtains charts for $X(\mathbb{C})$ as a manifold.

By Artin and Grothendieck there are isomorphisms for all m and n

$$H^n(X(\mathbb{C}), \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} H^n(X_{\text{ét}}, \mathbb{Z}/m\mathbb{Z})\,,$$

for every smooth variety $X/\mathbb{C}$. Here the left hand side is the (topological) cohomology of sheaves of the constant sheaf, and the right hand side is the étale cohomology.

14 Descent theory and the multiplicative group

Lemma 14.1 Let X be a scheme. There is a canonical isomorphism

$$H^1(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \cong Pic(X)\,,$$

where $Pic(X)$ is the Picard group of X.

Proof In view of 12.16 it suffices to construct a canonical isomorphism

$$Pic(X) \xrightarrow{\sim} \check{H}^1(X, \mathcal{O}_X^\times)\,. \tag{14.1.1}$$

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then there is an open covering $\mathfrak{U} = (U_i)_{i\in I}$ of X, such that there are isomorphisms

$$\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{L}|_{U_i} \qquad \text{for all } i \in I\,.$$

On $U_i \cap U_j$, these induce isomorphisms

$$\varphi_{ij} = \varphi_j^{-1}\varphi_i : \mathcal{O}_{U_i|_{U_i\cap U_j}} \xrightarrow{\varphi_i} \mathcal{L}|_{U_i\cap U_j} \xrightarrow{\varphi_j^{-1}} \mathcal{O}_{U_j|_{U_i\cap U_j}}\,,$$

which correspond to elements $s_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$.
Here we have

$$\begin{array}{ccc} s_{ik|_{U_{ijk}}} \cdot s_{jk|_{U_{ijk}}} \cdot s_{ij|_{U_{ijk}}} & = & 1 \\ \shortparallel & & \\ \varphi_i^{-1}\varphi_k\varphi_k^{-1}\varphi_j\varphi_j^{-1}\varphi_i & & , \end{array} \tag{14.1.2}$$

i.e., we have a Čech-1-cocycle in

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) = H^1(\prod_i \mathcal{O}_X^\times(U_i) \to \prod_{i,j} \mathcal{O}_X^\times(U_i\cap U_j) \to \prod_{i,j,k} \mathcal{O}_X^\times(U_i \cap U_j \cap U_k))$$

Conversely, such a cocycle gives glueing isomorphisms

$$\varphi_{i,j} : \mathcal{O}(U_i \cap U_j) \underset{\sim}{\xrightarrow{\cdot s_{ij}}} \mathcal{O}(U_i \cap U_j)\,,$$

which glue together the free modules $\mathcal{O}_{U_i}$ on U_i to an invertible $\mathcal{O}_X$-module $\mathcal{L}$ (the cocycle condition (14.1.2) gives the cocycle condition/transitivity for φ_{ij}). This assignment is additive. Furthermore, $\mathcal{L}$ is trivial if and only if the 1-cocycle (s_{ij}) is trivial in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. This gives an isomorphism

$$\check{H}^1(\mathfrak{U}, \mathcal{O}_X^\times) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{isomorphism classes of invertible } \mathcal{O}_X\text{-modules } \mathcal{L}, \\ \text{which are trivialized on } (U_i) \end{array} \right\}$$

We obtain (14.1.1) by taking the inductive limit over all coverings.

Remark 14.2 The same argument holds again for all (locally) ringed spaces.

Lemma 14.3 Let A be a ring and let M be an A-module. If M is a flat A-module, then the following holds:

(1) If $\sum_{i=1}^{r} a_i m_i = 0$ with $a_i \in A$ and $m_i \in M$, then there are an $s \in \mathbb{N}$ and elements $b_{ij} \in A$ and $y_j \in M$ $(j = 1, \dots, s)$ with

$$\sum_i a_i b_{ij} = 0$$

for all j and $m_i = \sum_j b_{ij} y_j$ for all i.

Proof Consider the exact sequence

$$\begin{array}{ccccc} K & \to & A^r & \xrightarrow{f} & A \\ & & (b_1, \dots, b_r) & \mapsto & \sum_{i=1}^{r} b_i a_i \,, \end{array}$$

$K = \ker(f)$. Then

$$\begin{array}{ccccc} K \otimes_A M & \to & M^r & \xrightarrow{f_M} & M \\ & & (n_1, \dots, n_r) & \mapsto & \sum_{i=1}^{r} a_i n_i \end{array}$$

is exact. By assumption, we have $f_M(m_1, \dots, m_r) = 0$, therefore there is an element

$$\sum_{j=1}^{s} \beta_j \otimes y_i \in K \otimes_A M$$

$(\beta_j \in K, y_i \in M)$, which is mapped to $(m_1, \dots, m_r)$.

If we write $\beta_j = (b_{ij}, \dots, b_{rj})$ with $b_{ij} \in A$, the claim follows.

Remark 14.4 The converse holds as well: If (1) holds, then M is flat.

Lemma 14.5 Let M be a finite generated module over a local ring A. Then the following conditions are equivalent:

(a) M is flat.

(b) M is free.

Proof We only have to show (a) $\Rightarrow$ (b). Let $\mathfrak{m}$ be the maximal ideal of A and let $m_1, \dots, m_n \in M$ be in such way that their images $\overline{m}_1, \dots, \overline{m}_n$ in $M/\mathfrak{m}M$ form a basis of this $A/\mathfrak{m}$-vector space. Then, by the Nakayama-Lemma the morphism

$$\begin{array}{rccc} & A^n & \twoheadrightarrow & M \\ \text{basis element} & e_i & \mapsto & m_i \end{array}$$

is surjective. It suffices to show that $m_1, \dots, m_n \in M$ are linearly independent over A, if $\overline{m}_1, \dots, \overline{m}_n$ are linearly independent in $M/\mathfrak{m}M$. We use induction over n. Let $n = 1$ and $am_1 = 0$ for $a \in A$. By Lemma 14.3 there are $b_1, \dots, b_s \in A$ and $y_1, \dots, y_s \in M$ with $ab_j = 0$ for all j and $m_1 = \sum_j b_j \cdot y_j$. Since $\overline{m}_1 \neq 0$ there is a j with $b_j \notin \mathfrak{m}$, i.e., $b_j \in A^\times$ a unit. From $ab_j = 0$ we get $a = 0$.

Now let $n > 1$ and $\sum_{i=1}^{n} a_i m_i = 0$. By Lemma 14.3 there exist $y_1, \ldots, y_s \in M$ and $b_{ij} \in A$ $(i = 1, \ldots, s)$ with

$$m_i = \sum_{j=1}^{s} b_{ij} y_j \quad , \quad \sum_{i=1}^{n} a_i b_{ij} = 0 \, .$$

Since $\overline{m}_n \neq 0$ there is a j with $b_{nj} \notin \mathfrak{m}$, i.e., b_{nj} a unit. Then we have

$$a_n = \sum_{i=1}^{n-1} c_i a_i \quad , \quad \text{with } c_i = -b_{ij}/b_{in} \, ,$$

and therefore

$$0 = \sum_{i=1}^{n} a_i m_i = a_1(m_1 + c_1 m_n) + \ldots + a_{n-1}(m_{n-1} + c_{n-1} m_n)$$

Since the considered $\overline{m_1 + c_1 m_n}, \ldots, \overline{m_{n-1} + c_{n-1} m_n}$ are linear independent over $A/\mathfrak{m}$, the induction hypothesis implies $a_1 = \ldots = a_{n-1} = 0$ and hence also $a_n = \sum_{i=1}^{n-1} c_i a_i = 0$.

Now we consider the descent theory for faithfully flat ring homomorphisms.

Theorem 14.6 (Descent theory I) Let $A \to B$ be a faithfully flat ringhomomorphism. Then, for every A-module M, the sequence

$$\begin{array}{cccccc} 0 & \to & M & \overset{\gamma}{\to} & B \otimes_A M & \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} & B \otimes_A B \otimes_A M \\ & & & & & \mapsto & 1 \otimes b \otimes m \\ & & m & \mapsto & 1 \otimes m, \, b \otimes m & & \\ & & & & & \mapsto & b \otimes 1 \otimes m \end{array}$$

is exact. This means that M is the difference kernel of α_1 and α_2, i.e., that

$$M = \ker(\alpha_1 - \alpha_2) \, .$$

Proof: By Lemma 10.4, the sequence

$$0 \to M \overset{\gamma}{\to} B \otimes_A M \overset{\alpha_1 - \alpha_2}{\longrightarrow} B \otimes B \otimes M \tag{14.6.1}$$

is exact if and only if the sequence

$$0 \to B \otimes M \overset{1 \otimes \gamma}{\longrightarrow} B \otimes B \otimes M \overset{1 \otimes (\alpha_1 - \alpha_2)}{\to} B \otimes B \otimes B \otimes M \, ,$$

tensored by B, is exact. Let

$$\begin{array}{rccc} \mu : & B \otimes B & \to & B \\ & b_1 \otimes b_2 & \mapsto & b_1 b_2 \end{array}$$

be the multiplication map. Then

$$\begin{array}{rccc} \mu \otimes 1 : & B \otimes B \otimes M & \to & B \otimes M \\ & b_1 \otimes b_2 \otimes m & \to & b_1 b_2 \otimes m \end{array}$$

is a left inverse of $1 \otimes \gamma$ ($\mu \otimes 1 \circ 1 \otimes \gamma = id$), hence $1 \otimes \gamma$ is injective.

Now let $z = \sum_i x_1 \otimes y_i \otimes m_i$ be in the kernel of $1 \otimes (\alpha_1 - \alpha_2)$, so that

$$\sum_i x_i \otimes 1 \otimes y_i \otimes m_i = \sum_i x_i \otimes y_i \otimes 1 \otimes m_i\,.$$

By applying μ to the first two places we obtain

$$\sum_i x_i \otimes y_i \otimes m_i = \sum_i x_i y_i \otimes 1 \otimes m_i\,,$$

and hence $\sum_i x_i y_i \otimes 1 \otimes m_i$ is the image of $\sum_i x_i y_i \otimes u_i$ under $1 \otimes \gamma$.

Theorem 14.7 (Descent theory II) Let $A \to B$ be a faithfully flat ringhomomorphism.
(a) Let M be a A-module. For a B-module

$$M' = B \otimes_A M$$

one has a canonical isomorphism of $B \otimes B$-modules (all tensor products are over A)

$$(14.7.1) \qquad \begin{array}{rrcl} \phi: & M' \otimes B & \overset{\sim}{\to} & B \otimes M' \\ & (b \otimes m) \otimes b' & \mapsto & b \otimes (b' \otimes m)\,. \end{array}$$

By this one can retrieve M from M':

$$(14.7.2) \qquad M = \{m' \in M' \mid \phi(m' \otimes 1) = 1 \otimes m'\}\,,$$

because this amounts to the exactness of the sequence (14.6.1)

$$0 \to M \to B \otimes M \overset{\alpha_1 - \alpha_2}{\longrightarrow} B \otimes B \otimes M$$

where

$$(14.7.3) \qquad \begin{array}{rcl} \alpha_1(b \otimes m) & = & 1 \otimes b \otimes m \\ \alpha_2(b \otimes m) & = & b \otimes 1 \otimes m\,. \end{array}$$

In fact for $m' = b \otimes m \in B \otimes M = M'$ we have

$$\begin{array}{rcl} \phi(m' \otimes 1) - 1 \otimes m' & = & (b \otimes 1) \otimes m - 1 \otimes b \otimes m \\ & = & \alpha_2(b \otimes m) - \alpha_1(b \otimes m)\,. \end{array}$$

(b) For the induced morphisms (ϕ_i keeps the entry on the i-th position and is ϕ on the remaining positions)

$$\begin{array}{rrcl} \phi_1: & B \otimes M' \otimes B & \to & B \otimes B \otimes M' \\ & b \otimes (b_2 \otimes m) \otimes b_3 & \mapsto & b \otimes b_2 \otimes (b_3 \otimes m) \end{array}$$

$$\begin{array}{rrcl} \phi_2: & M' \otimes B \otimes B & \to & B \otimes B \otimes M' \\ & (b \otimes m) \otimes b_2 \otimes b_3 & \mapsto & b \otimes (b_2 \otimes m) \otimes b_3 \end{array}$$

and

$$\begin{array}{rrcl} \phi_3: & M' \otimes B \otimes B & \to & B \otimes M_{\otimes} B \\ & (b \otimes m) \otimes b_2 \otimes b_3 & \mapsto & b \otimes (b_2 \otimes m) \otimes b_3 \end{array}$$

we get the co-called cocycle conditions

$$\phi_2 = \phi_1\phi_3\,. \tag{14.7.4}$$

(c) Conversely, let M' be a B-module, let

$$\phi : M' \otimes B \xrightarrow{\sim} B \otimes M' \tag{14.7.5}$$

be an isomorphism of $B \otimes B$-modules, and let

$$\begin{array}{rrcl} \phi_1 : & B \otimes M' \otimes B & \to & B \otimes B \otimes M' \\ \phi_2 : & M' \otimes B \otimes B & \to & B \otimes B \otimes M' \\ \phi_3 : & M' \otimes B \otimes B & \to & B \otimes M' \otimes B \end{array}$$

be the induced isomorphisms, where ϕ_i keeps the entry on the i-th position and is defined on the other positions by ϕ. (For ϕ_2 we have explicitly

$$m' \otimes b_2 \otimes b_3 \mapsto \sum_i b_i \otimes b_2 \otimes m'_i\,,$$

if $\phi(m' \otimes b_3) = \sum\limits_i b_i \otimes m'_i$.)

If now the cocycle condition

$$\phi_2 = \phi_1\phi_3 \tag{14.7.6}$$

holds, then there is a canonical A-module M with $B \otimes_A M \cong M'$, namely the A-module

$$M = \{m' \in M' \mid \phi(m' \otimes 1) = 1 \otimes m'\}\,,$$

for which the canonical map

$$\begin{array}{rrcl} \gamma : & B \otimes_A M & \xrightarrow{\sim} & M' \\ & b \otimes m & \mapsto & bm \end{array} \tag{14.7.7}$$

is an isomorphism. In fact, let

$$\tau : M' \to B \otimes M$$

be defined by $\tau(m') = 1 \otimes m' - \phi(m' \otimes 1)$. By definition, we then have an exact sequence

$$0 \to M \to M' \xrightarrow{\tau} B \otimes M'\,.$$

If we tensorize with B on the right, we obtain the top row in the following diagram

$$\begin{tikzcd} 0 \arrow[r] & M \otimes B \arrow[r] \arrow[d, "\gamma"] & M' \otimes B \arrow[r] \arrow[d, "\phi", "\wr"'] & B \otimes M' \otimes B \arrow[d, "1\otimes\phi", "\wr"'] \\ 0 \arrow[r] & M' \arrow[r] & B \otimes M' \arrow[r] & B \otimes B \otimes M'\,, \end{tikzcd}$$

where the lower sequence is the exact sequence from 14.6, applied to the A-module M'. The map γ is defined by $\gamma(m \otimes b) = bm$ (and therefore corresponds to the map (14.7.7)). We show that γ is an isomorphism. Since the rows are exact (the top row is exact because of the flatness of B over A) and both vertical maps on the right are isomorphisms, the claim follows, if we show that the diagram is commutative.

The left hand square commutes, since, by definition of M, we have

$$\phi(m \otimes b) = (1 \otimes b)\phi(m \otimes 1) = (1 \otimes b)(1 \otimes m) = 1 \otimes bm$$

for $m \in M$ and $b \in B$.
For the right hand square we have the following for the "lower way": For $m' \in M'$ let

$$\phi(m' \otimes 1) = \sum_i b_i \otimes m'_i$$

with $b_i \in B$ and $m'_i \in M'$. Then we have

$$\phi(m' \otimes b) = (1 \otimes b)\phi(m' \otimes 1) = \sum_i b_i \otimes bm'_i\,,$$

and thus the image of this in $B \otimes B \otimes M'$ is equal to

$$\sum_i 1 \otimes b_i \otimes bm'_i - \sum_i b_i \otimes 1 \otimes bm'_i\,.$$

On the "upper way", $m' \otimes b$ is mapped to

$$\begin{array}{rcl} 1 \otimes m' \otimes b & - & \phi(m \otimes 1) \otimes b \\ = 1 \otimes m' \otimes b & - & \sum\limits_i b_i \otimes m'_i \otimes b \end{array}$$

and this, by $1 \otimes \phi$, is mapped to

$$\begin{array}{rcl} 1 \otimes \phi(m' \otimes b) & - & \sum\limits_i b_i \otimes \phi(m'_i \otimes b) \\ = \sum\limits_i 1 \otimes b_i \otimes bm'_i & - & \sum\limits_i b_i \otimes \phi(m'_i \otimes b)\,. \end{array}$$

Hence we have to show that

$$\sum_i b_i \otimes 1 \otimes bm'_i = \sum_i b_i \otimes \phi(m'_i \otimes b)\,.$$

But this means that we have

$$\begin{array}{ccc} \phi_2(m \otimes 1 \otimes b) & = & \phi_1(\phi_3(m \otimes 1 \otimes b)) \\ \| & & \| \\ \sum\limits_i b_i \otimes 1 \otimes bm'_i & & \phi_1(\sum\limits_i b_i \otimes 1 \otimes m'_i) \\ & & \| \\ & & \sum\limits_i b_i \otimes \phi(1 \otimes m'_i)\,, \end{array}$$

which holds because of the assumption that $\phi_2 = \phi_1\phi_3$.

Theorem 14.8 For every scheme X, the canonical morphisms

$$H^1(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \xrightarrow{\sim} H^1(X_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\sim} H^1(X_{\mathrm{fl}}, \mathbb{G}_m)$$

are isomorphisms.

Proof for the flat topology (the étale case is analogous). We use the Leray-spectral sequence for

$$\alpha : X_{\mathrm{fl}} \to X_{\mathrm{Zar}} \,.$$

By the sequence of the lower terms

$$0 \to H^1(X_{\mathrm{Zar}}, \alpha_* \mathbb{G}_m) \to H^1(X_{\mathrm{fl}}, \mathbb{G}_m) \to H^0(X_{\mathrm{Zar}}, R^1\alpha_* \mathbb{G}_m)$$

where $\alpha_*\mathbb{G}_m = \mathcal{O}_X^\times$, it suffices to show that $R^1\alpha_*\mathbb{G}_m = 0$. This means that for all for all $x \in X$ the stalk $(R^1\alpha_*\mathbb{G}_m)_x = 0$. But this stalk is

$$\varinjlim_{x\in U} H^1(U_{\mathrm{fl}}, \mathbb{G}_m) \cong \varinjlim_{x\in U} \check{H}^1(U_{\mathrm{fl}}, \mathbb{G}_m)\,,$$

where U goes through all open neighborhoods of x.

Since inductive limits commute, it suffices to show that for every flat covering $(U_i \to U)_{i\in I}$ with $U \subseteq X$ open the limit

$$\varinjlim_{V\subseteq U \text{ open}} \check{H}^1((U_i \times_U V \to V, \mathbb{G}_m) = 0\,.$$

We can assume that $U = \mathrm{Spec}(A)$ is affine and further that I is finite and every U_i is affine, and hence, that we have a faithfully flat morphism

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)\,.$$

Furthermore we can pass to the limit and assume that $A = \mathcal{O}_{X,x}$ is a local ring. We consider a class in

$$H^1(B^\times \to (B\otimes_A B)^\times \to (B\otimes_A B\otimes_A B)^\times)\,, \tag{14.8.1}$$

represented by the 1-cocycle $\alpha \in (B\otimes_A B)^\times$. We obtain an isomorphism

$$\phi : B\otimes_A B \xrightarrow{\sim} B\otimes_A B\,, \tag{14.8.2}$$

which fulfills the cocycle condition the property (14.7.6). Therefore there is an A-module M with $B\otimes_A M \cong B$. Since the B-module B is finitely generated and flat, this also holds for M. Since A is local, M is a free A-module of degree 1, i.e., $M \cong A$.

It follows from the descent theory that the isomorphism ϕ in (14.8.2) is the one which is constructed by (14.7.1) from M. Since $M \cong A$, we get that the associated 1-cocycle is trivial, i.e., comes from $B^\times$.

q.e.d.

15 Schemes of dimension 1

Proposition 15.1 Let X be a regular integral noetherian scheme and let $j : \operatorname{Spec} K \hookrightarrow X$ be the inclusion of the generic point. Then there is an exact sequence of étale sheaves

$$0 \to \mathbb{G}_{m/X} \to j_*(\mathbb{G}_{m/k}) \to \underset{x \in X^1}{\oplus} (i_x)_* \mathbb{Z} \to 0\,, \tag{15.1.1}$$

where X^1 is the set of the points of codimension 1 of X and i_x: $\operatorname{Spec}(k(x)) \hookrightarrow X$ is the canonical morphism.

We need:

Lemma 15.2 Let $f : U \to X$ be étale. Then the following holds.

(a) For $y \in U$ and $x = f(y)$, we have $\dim \mathcal{O}_{U,y} = \dim \mathcal{O}_{X,x}$.

(b) U is regular if an only if X is regular.

Proof (a): Without restriction, $X = \operatorname{Spec} A$, A is local and $U = \operatorname{Spec} B$ is affine. Let $\mathfrak{m} \subseteq A$ (maximal) and $\mathfrak{n} \subseteq B$ be the prime ideals which correspond to x and y. Then $\operatorname{Spec} B_{\mathfrak{n}} \to \operatorname{Spec} A$ is faithfully flat (10.5), therefore surjective, hence $\dim B_{\mathfrak{n}} \geq \dim A$. Conversely, by Zariski's main theorem 9.3, B/A is finite without restriction. Then $\varphi : A \to B$ induces an integral ring extension $A/\ker\varphi \hookrightarrow B$, and, by Cohen-Seidenberg, $\dim A \geq \dim(A/\ker\varphi)) = \dim B \geq \dim B_{\mathfrak{n}}$.

(b) Let $\mathfrak{m} \subset A$ and $\mathfrak{n} \subset B$ be as above. Then by (a) $d = \dim A = \dim B_{\mathfrak{n}}$. On the other hand we have $\mathfrak{m}B_{\mathfrak{n}} = \mathfrak{n}B_{\mathfrak{n}}$, and hence isomorphisms

$$\mathfrak{m}/\mathfrak{m}^2 \otimes_{k(\mathfrak{m})} k(\mathfrak{n}) \cong \mathfrak{m}/\mathfrak{m}^2 \otimes_{A(\mathfrak{m})} B_{\mathfrak{n}}/\mathfrak{m}B_{\mathfrak{n}} = \mathfrak{m}/\mathfrak{m}^2 \otimes_A B_{\mathfrak{n}} \cong \mathfrak{n}/\mathfrak{n}^2\,,$$

i.e., $\dim_{k(\mathfrak{m})} \mathfrak{m}/\mathfrak{m}^2 = \dim_{k(\mathfrak{n})} \mathfrak{n}/\mathfrak{n}^2$.

Proof of Proposition 15.1: For $U \to X$ étale we have morphisms

$$\mathbb{G}_m(U) \overset{\alpha}{\to} \mathbb{G}_m(U \times_X \operatorname{Spec} K) \overset{\beta}{\to} \underset{x \in X^{(1)}}{\oplus} \mathbb{Z}(U \times_X \operatorname{Spec}(K))\,,$$

where α is the restriction and β is defined as follows: We have $U \times_X \operatorname{Spec} K = \coprod_{\eta \in U^0} \operatorname{Spec} k(\eta)$ by 15.2 (a) and

$$U \times_X \operatorname{Spec} k(x) = \coprod_{\substack{y \in U_1 \\ f(y) = x}} \operatorname{Spec}(k(y))$$

The component $\beta_y : \mathbb{G}_m(k(\eta)) = k(\eta)^\times \to \mathbb{Z}$ of β at y is 0 if $y \notin \overline{\{\eta\}}$ ($\Leftrightarrow \eta \notin \operatorname{Spec} \mathcal{O}_{U,y}$) and the discrete valuation associated to y on $k(\eta)^\times$, if η is the generic point of $\operatorname{Spec} \mathcal{O}_{U,y}$, hence $k(y) = \mathit{Quot}\,\mathcal{O}_{U,y}$. It follows immediately that $\beta\alpha = 0$.

By forming the associated sheaf to $\underset{x \in X^{(1)}}{\oplus} (i_x)_* \mathbb{Z}$ we obtain the wanted sequence. For the exactness it suffices to prove the exactness if U is replaced by a local ring $\mathcal{O}_{U,y}$ for $y \in U^1$. But then the sequence is

$$0 \to \mathcal{O}_{U,y}^\times \to (\mathit{Quot}(\mathcal{O}_{U,y}))^\times \overset{v_y}{\to} \mathbb{Z} \to 0$$

and hence is exact.

Now we consider the long exact cohomology sequence associated to

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to \bigoplus_{x\in X^1} (i_x)_*\mathbb{Z} \to 0\,.$$

We need the following.

Lemma 15.3 Let X be a quasi compact, quasi separated scheme. Then, for every inductive system $(F_i)_{i\in I}$ of abelian étale sheaves on X, we have:

$$\varinjlim_{i\in I} H^n_{\text{ét}}(X, F_i) \xrightarrow{\sim} H^n_{\text{ét}}(X, \varinjlim_{i\in I} F_i)\,.$$

In particular, étale cohomology commutes with direct sums.

Proof: See Tamme II, Introduction to étale Cohomology, 1.5.3.

$S(X_{\text{ét}})$ is equivalent to $S(X_{\text{ét, f.p.}})$ for the noetherian site of all étale X-schemes of finite presentation. Thus we have

$$H^n_{\text{ét}}(X, \bigoplus_{x\in X^1} (i_x)_*\mathbb{Z}) = \bigoplus_{x\in X^1} H^n_{\text{ét}}(X, (i_x)_*\mathbb{Z})$$

Lemma 15.4 Let $(X_i)_{i\in I}$ be a projective system of quasi compact and quasi separated schemes, with affine transition morphisms. Let $i_0 \in J$ and F be an étale sheaf on X_{i_0}. Then the natural map

$$\varinjlim_{i\in I} H^n_{\text{ét}}(X_i, F|_{X_i}) \xrightarrow{\sim} H^n_{\text{ét}}(\varprojlim_i X_i, F|_{\varprojlim_i X})$$

is an isomorphism, where $F|_{X_i}$ and $F|_{X=\varprojlim_i X_i}$ denote the pull-backs of F, respectively.

Proof: See Milne, Étale Cohomology, Lemma 1.16.

Corollary 15.5 Let $f : Y \to X$ be a quasi compact, quasi separated morphism of schemes, let F be an étale sheaf on Y and let $\overline{x}$ be a geometric point of X. Then we have

$$(R^n f_*F)_{\overline{x}} \cong H^n_{\text{ét}}(Y \times_X \operatorname{Spec}(\mathcal{O}^h_{X,\overline{x}}), F|...)\,.$$

Proof Let P be the presheaf $U \rightsquigarrow H^n_{\text{ét}}(Y \times_X U, F)$ on X. Then we have $R^n f_*F = aP$ (the associated sheaf), hence

$$\begin{aligned}(R^n f_*F)_{\overline{x}} = P_{\overline{x}} \quad &= \varinjlim_{U \text{ étale neighborhood of } \overline{x} \text{ in } X} H^n_{\text{ét}}(Y \times_X U, F|Y \times_X U) \\ &\overset{15.4}{=} H^n_{\text{ét}}(Y \times_X \operatorname{Spec}\mathcal{O}^{sh}_{X,\overline{x}}, F|...)\,.\end{aligned}$$

The claim above now follows from the fact that $\mathcal{O}^{sh}_{X,\overline{x}} \otimes_{\mathcal{O}_{X,\overline{x}}} K = K_{\overline{x}}$.

Note that we have

$$H^i_{\text{ét}}(\mathrm{Spec}K_{\overline{x}}, \mathbb{G}_m) = H^i(K_{\overline{x}}, (K^{sep}_{\overline{x}})^\times)$$

We need some facts from the Galois cohomology.

Lemma 15.6 (Hilbert 90) $H^1(K, (K^{sep})^\times) = 0$ for every field K.

Lemma 15.7 For every field L, we have $H^2(L, (K^{sep})^*) = Br(L)$, the Brauer group of L.

By Corollary 15.5, for every geometric point $\overline{x}$ of X we have

$$(R^1 j_* \mathbb{G}_{m,K})_{\overline{x}} = H^1(K_{\overline{x}}, K^\times_{\overline{x}}) = 0\,,$$

where $K_{\overline{x}} = Quot(\mathcal{O}_{X,\overline{x}})$, since we have $\mathrm{Spec}(K) \times_X \mathrm{Spec}(\mathcal{O}_{X,\overline{x}}) = \mathrm{Spec}(K_{\overline{x}})$. Therefore

$$R^1 j_* \mathbb{G}_{m,K} = 0\,.$$

From the Leray-spectral sequence for j_* we thus get

$$H^1(X, j_*\mathbb{G}_{m,K}) = H^1(K, (K^{sep})^\times) = 0$$

and an exact sequence

$$0 \to H^2(X, j_*\mathbb{G}_{m,K}) \to Br(K) \to H^0(X, R^2 j_*\mathbb{G}_{m,X}) \to H^3(X, j_*\mathbb{G}_{m,X})\,.$$

On the other hand, from the sequence (15.1.1) we get an exact sequence

$$0 \to \Gamma(X, O_X^\times) \to K^\times \to \bigoplus_{x \in X^1} \mathbb{Z} \to H^1(X, \mathbb{G}_m) \to 0\,,$$

therefore the known isomorphism (for regular X)

$$H^1(X, \mathbb{G}_m) \xrightarrow{\sim} Pic(X)\,.$$

Now let $\dim X = 1$ and let $k(x)$ be perfect for all $x \in X^1$.

Lemma 15.8 $R^i j_* \mathbb{G}_{m,K} = 0$ for $i \geq 1$.

Proof For all geometric points $\overline{x}$ of X, $K_{\overline{x}} = Quot(\mathcal{O}_{X,\overline{x}})$, and $\mathcal{O}_{X,\overline{x}}$ is a discrete valuation ring with algebraic closed residue field, or a separably closed field. In the first case $K_{\overline{x}}$ has the cohomological dimension 1, in the second case the cohomological dimension 0. Thus

$$(R^i j_* \mathbb{G}_{m,K})_{\overline{x}} = H^i(K_{\overline{x}},\, (K^{sep}_{\overline{x}})^\times) = 0$$

for $i \geq 1$.

In this case we have isomorphisms

$$H^i(X, j_*\mathbb{G}_{m,K}) \cong H^i(K, (K^{sep})^\times)$$

for all i.

The sequence (15.1.1) gives an exact sequence

$$0 \to H^2(X, \mathbb{G}_m) \to Br(K) \to \underset{x \in X^1}{\oplus} H^2(k(x), \mathbb{Z}) \to H^3(X, \mathbb{G}_m) \to \dots$$

since $(i_x)_*$ is exact, and with the Leray spectral sequence for $(i_x)_*$ we get

$$H^i(X, (i_x)_*\mathbb{Z}) \cong H^i(\mathrm{Spec}(k(x), \mathbb{Z})$$

for all i.

Finally, let X be a smooth projective curve over an algebraically closed field k, and as before let $K = K(X)$ be the function field of X.

Theorem 15.9 (Theorem of Tsen) K has the cohomological dimension 1.

More precisely, Tsen showed that K is a so-called C_1-field, and hence $cd(K) \leq 1$. Hence $cd(k(x)) = 0$ for $x \in X^1$, $H^i(X, \mathbb{G}_m) \overset{\sim}{\to} H^i(X, j_*, \mathbb{G}_{m,K}) = 0$ for $i \geq 2$.

Let n be invertible in k. From the Kummer-sequence

$$0 \to \mu_n \to \mathbb{G}_m \overset{n}{\to} \mathbb{G}_m \to 0\,,$$

the next lemma follows by passing to cohomology

Lemma 15.10 (i) $H^0(X, \mu_n) = \mu_n$ (since $H^0(X, \mathbb{G}_m) = k^\times$)
(ii) $H^1(X, \mu_n) = Pic(X)[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$
(iii) $H^2(X, \mu_n) \cong Pic(X)/n\, Pic(X) = \mathbb{Z}/n\mathbb{Z}$, where g is the genus of X, thus $g = \dim_k H^0(X, \Omega^1_{X/k})$.

Proof The first isomorphisms are obvious from the long exact cohomology sequences. The further isomorphisms in (ii) and (iii) follow from the fact that for $Pic^0(X) = \ker(Pic(X) \overset{\deg}{\twoheadrightarrow} \mathbb{Z})$ we have

$$Pic^0(X) \cong Jac(X)(k)\,,$$

where $Jac(X)$ is the Jacobian variety of X. This is an abelian variety of dimension g, where g is the genus of X.

Corollary 15.11 The cohomology groups $H^i(X, \mathbb{Z}/n)$ are finite.

Proof Over an algebraically closed field we can identify $\mathbb{Z}/n$ with μ_n.