Perverse sheaves and the Weil conjectures

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1 Triangulated categories

Definition 1.1 An **additive category** $\mathcal{C}$ is a category whose Hom-sets are abelian groups, for which the composition of morphisms is bilinear, and for which there exist finite coproducts and products. (In particular, there is a zero object 0, which is final and cofinal). Then the canonical morphism

$$X \coprod Y \to X \prod Y$$

is an isomorphism from the coproduct to the product.

Definition 1.2 A **translation functor** T on a category $\mathcal{C}$ is an automorphism

$$T : \mathcal{C} \to \mathcal{C} .$$

For an object X in $\mathcal{C}$, we also write $X[n]$ for $T^n X$, and for a morphism f in $\mathcal{C}$ we also write $f[n]$ for $T^n f$. A **triangle** in $\mathcal{C}$ (with respect to T) is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

with morphisms u, v and w. Write also

$$\begin{tikzcd} & Z \arrow[dl, "w"', "+1"] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

where $+1$ indicates that w is a morphism $Z \to X[1]$. A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1] \end{tikzcd}$$

Definition 1.3 Let $\mathcal{D}$ be an additive category and let T be an additive automorphism of $\mathcal{D}$. For $i \in \mathbb{Z}$ the morphisms in

$$\mathrm{Hom}^i(X, Y) := \mathrm{Hom}(X, T^i(Y)) = \mathrm{Hom}(X, Y[i])$$

are called morphisms of degree i. Define the composition

$$\mathrm{Hom}^i(X, Y) \times H^j(Y, Z) \to \mathrm{Hom}^{i+j}(X, Z)$$

by

$$(f, g) \mapsto T^i(g) \circ f .$$

We obtain a $\mathbb{Z}$-graded category with homomorphism groups $\prod_{n \in \mathbb{Z}} \mathrm{Hom}^n(X, Y))$.

$\mathcal{D}$ is called a **triangulated category**, if there is a class of triangles – called distinguished triangles – for which the following holds:

(TR1) (a) Every triangle which is isomorphic to a distinguished triangle is distinguished.

(b) Every morphism $X \xrightarrow{f} Y$ can be extended to an distinguished triangle

$$X \xrightarrow{f} Y \to Z \to T(X)\,.$$

(c) The triangle $X \xrightarrow{id} X \to 0 \to T(X)$ is distinguished.

(TR2) The triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

is distinguished if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$ is distinguished.

(TR3) Every diagram with morphisms u, v, where the first square commutes and the rows are distinguished triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dotted, "w"] & T(X) \arrow[d, "T(u)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

can be extended to a morphism of triangles by a morphism w as indicated.

(TR4) (Octahedral axiom) Let

$$X \xrightarrow{u} Y \xrightarrow{i} Z' \longrightarrow TX$$

$$Y \xrightarrow{v} Z \longrightarrow X' \xrightarrow{j} TY$$

$$X \xrightarrow{w} Z \longrightarrow Y' \longrightarrow TX$$

be three distinguished triangles with $w = v \circ u$. Then the following equivalent properties hold.

(i) There are morphisms $f : Z' \to Y'$, $g : Y' \to X'$, such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "id"'] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r] \arrow[d, "f"] & TX \arrow[d, "id"] \\
X \arrow[r, "w"] \arrow[d, "u"'] & Z \arrow[r] \arrow[d, "id"] & Y' \arrow[r] \arrow[d, "g"] & TX \arrow[d, "T_u"] \\
Y \arrow[r, "v"] & Z \arrow[r] & X' \arrow[r, "j"] \arrow[d] & TY \arrow[dl, "T(i)"] \\
 & & TZ' &
\end{tikzcd}$$

commutes, and the third column is a distinguished triangle.

(ii) The diagram

$$
\begin{tikzcd}
 & & Z' \arrow[dddd, "f"] \arrow[ddr] & \\
 & Y \arrow[ur, "i"] \arrow[dd, "v"] & & \\
X \arrow[ur, "u"] \arrow[dr, "w"'] & & & TX \\
 & Z \arrow[dr] \arrow[dd] & & \\
 & & Y' \arrow[uur] \arrow[d, "g"] & \\
 & X' \arrow[r, equal] \arrow[d, "j"] & X' \arrow[d] & \\
 & T(Y) \arrow[r, "T(i)"] & T(Z') &
\end{tikzcd}
$$

is commutative, and the column with the morphisms f and g is a distinguished triangle.

(iii) The diagram

$$
\begin{tikzcd}
 & & & Z' \arrow[r] \arrow[dd, "f"] & TX \arrow[dd, equal] \\
 & Y \arrow[urr, "i"] \arrow[dr, "v"] & & & \\
X \arrow[ur, "u"] \arrow[rr, "w"] & & Z \arrow[r] \arrow[dr] & Y' \arrow[r] \arrow[d, "g"] & TX \arrow[d, "Tu"] \\
 & & & X' \arrow[r, "j"] \arrow[d] & TY \arrow[dl, "T(i)"] \\
 & & & TZ' &
\end{tikzcd}
$$

is commutative, and the column with f and g is a distinguished triangle.

The equivalence of the conditions follows easily.

The octahedral axiom is an analogue of the second isomorphism theorem in abelian categories. This is most obvious in the versions of (ii) and (iii): Z' is the analogue of Y/X and in the category of complexes is given by $Cone(u)$, Y' is the analogue of Z/X and is given by $Cone(w)$, and X' is the analogue of Z/Y and is given by $Cone(v)$, and we obtain

$$Y'/Z' = \frac{Z/X}{Y/X} \cong \frac{Z}{Y} = X' .$$

Additionally we get the information that

$$X' \to T(Z')$$

is given by the composition $T(i)j$.

Lemma 1.4 The triangle $X \xrightarrow{-id} X \longrightarrow 0 \longrightarrow TX$ is distinguished.

Proof: We have an isomorphism of triangles

$$\begin{tikzcd}
X \arrow[r, "-id_X"] & X \arrow[r] & 0 \arrow[r] & TX \\
X \arrow[r, "id_X"] \arrow[u, "id_X"] & X \arrow[r] \arrow[u, "-id_X"] & 0 \arrow[r] \arrow[u] & TX \arrow[u, "T(id_X)"]
\end{tikzcd}$$

Definition 1.5 An additive functor $F : \mathcal{D} \to \mathcal{D}'$ between triangulated categories is called exact, or functor of triangulated categories, if it commutes with the translation functors T and T' of $\mathcal{D}$ and $\mathcal{D}'$:

$$FTX = T'FX\,,$$

and maps distinguished triangles in $\mathcal{D}$ to distinguished triangles in $\mathcal{D}'$.

For working with triangulated categories, it is important to know that the above properties (TR1) to (TR4) hold. But, as observed by Peter May, the above system of conditions is redundant, and this is helpful for constructing triangulated categories.

Lemma 1.6 (TR3) follows from (TR4).

Proof Consider a diagram

(1)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] & TX \arrow[d, "T(i)"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & TX'
\end{tikzcd}$$

where the first square is commutative. We are looking for a morphism $k : Z \to Z'$ making the diagram commutative.

Applying (TR1), we have a distinguished triangle

$$X \xrightarrow{jf} Y' \xrightarrow{p} V \xrightarrow{q} TX\,.$$

By (TR4) we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, "s"] & TX \arrow[d, "id"] \\
X \arrow[r, "jf"] \arrow[d, "f"] & Y' \arrow[r, "p"] \arrow[d, "id"] & V \arrow[r, "q"] \arrow[d, "t"] & TX \arrow[d, "T(f)"] \\
Y \arrow[r, "j"] & Y' \arrow[r, "j'"] & Y'' \arrow[r, "j''"] \arrow[d, "u"] & TY \arrow[dl, "T(g)"] \\
 & & TZ &
\end{tikzcd}$$

Similarly, for $f'i(=jf)$ we get the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "id"] & X' \arrow[r, "i'"] \arrow[d, "f'"] & X'' \arrow[r, "i''"] \arrow[d, "s'"] & TX \arrow[d, "id"] \\
X \arrow[r, "f'i"] \arrow[d, "i"] & Y' \arrow[r, "p"] \arrow[d, "id"] & V \arrow[r, "q"] \arrow[d, "t'"] & TX \arrow[d, "T(i)"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] \arrow[d, "u'"] & TX' \arrow[dl, "T(i')"] \\
 & & TX'' &
\end{tikzcd}$$

so that

$$\begin{array}{ccccccc} pj = sg & , & tp = j' & , & qs = h & , & j''t = T(f)q \\ pf' = s'i' & , & t'p = g' & , & qs' = i'' & , & h't' = T(i)q \end{array} .$$

Now we let $k = t's : Z \to Z'$. Then we have
$kg = t'sg = t'pj = g'j$, and
$h'k = h't's = T(i)qs = T(i)h$,
so that k makes the diagram (1) commutative.

Lemma 1.7 Assume the following.

(TR2)(a) If the triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is distinguished, then

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

If (TR3) holds, then the converse of (TR2) (a) holds.

Proof Assume $(g, h, -f[1])$ is distinguished. By (TR2) (a) the triangle

$$X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} Z[1] \xrightarrow{-h[1]} X[2]$$

is distinguished.

Now choose a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g'} Z' \xrightarrow{h'} X[1] .$$

Then, by (TR3) and exercise 1.16, we have an isomorphism of triangles

$$\begin{tikzcd}
X[1] \arrow[r, "-f[1]"] \arrow[d, "id"] & Y[1] \arrow[r, "-g'[1]"] \arrow[d, "id"] & Z'[1] \arrow[r, "-h'[1]"] \arrow[d, "\alpha"] & X[2] \arrow[d, "id"] \\
X[1] \arrow[r, "-f[1]"] & Y[1] \arrow[r, "-g[1]"] & Z[1] \arrow[r, "-h[1]"] & X[2]
\end{tikzcd} .$$

and hence an isomorphism of triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "g'"] \arrow[d, "id"] & Z \arrow[r, "h'"] \arrow[d, "\alpha"] & X[1] \arrow[d, "id"] \\
Y \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & X[1],
\end{tikzcd}$$

so that the bottom triangle is distinguished by (TR1) (a).

We add some results of interest for working with triangulated categories.

Lemma 1.8 For a distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

in a triangulated category $\mathcal{D}$ all compositions of consecutive morphisms are zero.

Proof: By (TR2) it suffices to show $vu = 0$. Now consider the following diagram

$$\begin{tikzcd}
X \arrow[r, "id_X"] \arrow[d, "id"] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dotted, "g"] & X[1] \arrow[d, "id"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

Here by (TR1) and assumption both rows are distinguished triangles, and the first square commutes. Hence by (TR3) the indicated dotted arrow exists making the diagram commutative. This implies $vu = g0 = 0$.

Lemma 1.9 If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle, then for any object A in $\mathcal{D}$ the sequence

$$\mathrm{Hom}_{\mathcal{D}}(A, X) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{D}}(A, Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathcal{D}}(A, Z) \xrightarrow{w_*} \mathrm{Hom}_{\mathcal{D}}(A, X[1])$$

is exact.

Proof We already know that the composition of two consecutive morphisms is zero and again it suffices to show the exactness at $\mathrm{Hom}_{\mathcal{D}}(A, Y)$. Consider $f : A \to Y$ with $vf = 0$. Then we have a morphism of triangles

$$\begin{tikzcd}
A \arrow[r, "id_A"] \arrow[d, dotted, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \arrow[d, dotted, "g[1]"] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r] & A[1]
\end{tikzcd}$$

by (TR1), (TR2) and (TR3). Hence $u_*(g) = f$.

Lemma 1.10 Let

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f(1)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

be a morphism of distinguished triangles. If f and g are isomorphisms, then h is an isomorphism as well.

Proof: By the previous lemma we get a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}(A,X) \arrow[r, "u_*"] \arrow[d, "f_x"] & \mathrm{Hom}(A,Y) \arrow[r, "v_*"] \arrow[d, "g_*"] & \mathrm{Hom}(A,Z) \arrow[r, "w_*"] \arrow[d, "h_*"] & \mathrm{Hom}(A,X[1]) \arrow[r, "u[1]_*"] \arrow[d, "f[1]_*"] & \mathrm{Hom}(A,Y[1]) \arrow[d, "g[1]_*"] \\
\mathrm{Hom}(A,X') \arrow[r, "u'_*"] & \mathrm{Hom}(A,Y') \arrow[r, "v'_*"] & \mathrm{Hom}(A,Z') \arrow[r, "w'_*"] & \mathrm{Hom}(A,X'[1]) \arrow[r, "u'[1]_*"] & \mathrm{Hom}(A,Y'[1])
\end{tikzcd}$$

in which $f_*, g_*, f[1]_*$ and $g[1]_*$ are isomorphisms. Hence, by the five lemma, h_* is an isomorphism, and hence h is an isomorphism by the Yoneda lemma.

Now we discuss an important example of a triangulated category.

Let $\mathcal{A}$ be an additive category (not necessarily abelian!). Let $C(\mathcal{A})$ be the category of the (unbounded) complexes in $\mathcal{A}$: objects are diagrams

$$\ldots \to A^n \stackrel{d^n}{\to} A^{n+1} \stackrel{d^{n+1}}{\to} A^{n+2} \to \ldots$$

with

$$A^n \in Ob(\mathcal{A}), d^n \in Mor(\mathcal{A}), d^{n+1}d^n = 0 \qquad (n \in \mathbb{Z})$$

morphisms are commutative diagrams

$$\begin{tikzcd}
\ldots \arrow[r] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \ldots \arrow[d] \\
\ldots \arrow[r] & B^n \arrow[r] & B^{n+1} \arrow[r] & \ldots
\end{tikzcd}$$

A complex $A^{\cdot}$ is called bounded below, or bounded above, or bounded, respectively, if $A^n = 0$ for $n << 0$, or for $n >> 0$, or for $|n| >> 0$, respectively. We obtain corresponding categories $C^+(\mathcal{A}), C^-(\mathcal{A}), C^b(\mathcal{A})$.

Definition 1.11 For a complex $A^{\cdot}$ and $n \in \mathbb{Z}$ let $A^{\cdot}[n]$ be the n-th shifted complex:

$$\begin{array}{rcl} (A^{\cdot}[n])^i & = & A^{i+n} \\ d^i_{A^{\cdot}[n]} & = & (-1)^n d^{i+n} \end{array}$$

Definition 1.12 Let $f : A^{\cdot} \to B^{\cdot}$ be a morphism of complexes. The Cone $Cone(f)$ of f is the following complex:

$$Cone(f)^n = B^n \oplus A^{n+1}$$

$$d^n_{Cone(f)} = \begin{pmatrix} d^n_B & f^{n+1} \\ & -d^{n+1}_A \end{pmatrix}$$

In "elements" $(b,a) \mapsto (d^n b + f^{n+1}a, -d^{n+1}a)$. Note that $d(db + fa, -da) = (d^2 b + dfa - fda, d^2 b) = 0$.

We obtain a sequence of complexes

$$0 \to B^{\cdot} \to Cone(f) \to A^{\cdot}[1] \to 0$$

Definition 1.13 A (naive) double complex in $\mathcal{A}$ is a diagram in $\mathcal{A}$

$$\begin{tikzcd} & \vdots & \vdots & \\ {} \arrow[r] & A^{m,n+1} \arrow[r] \arrow[u] & A^{m+1,n+1} \arrow[r] \arrow[u] & {} \\ \cdots \arrow[r] & A^{m,n} \arrow[r, "d_I"] \arrow[u, "d_{\mathrm{II}}"] & A^{m+1,n} \arrow[r] \arrow[u] & \cdots \\ & \vdots \arrow[u] & \vdots \arrow[u] & \end{tikzcd}$$

with $d_{\mathrm{I}}^2 = 0 = d_{\mathrm{II}}^2$ and $d_{\mathrm{I}}d_{\mathrm{II}} = d_{\mathrm{II}}d_{\mathrm{I}}$, therefore a complex of complexes. If the considered sums exist (i.e. $A^{i,j} \in C^b(C(\mathcal{A}))$ or $C(C^b(\mathcal{A}) =)$ or $C^+(C^+(\mathcal{A}))$ or in $\mathcal{A}$ there exist arbitrary sums), then the associated simple complex $sA^{\cdot,\cdot}$ is defined as

$$\begin{aligned} (sA^{\cdot,\cdot})^n &= \bigoplus_{p+q=n} A^{p,q} \\ d \text{ auf } A^{p,q} &= d_{\mathrm{I}} + (-1)^p d_{\mathrm{II}}\,. \end{aligned}$$

Other characterization: With d_{I} and $(-1)^p d_{\mathrm{II}}$ on $A^{p,q}$, $sA^{\cdot,\cdot}$ is a non naive double complex ($d_{\mathrm{I}}d_{\mathrm{II}} + d_{\mathrm{II}}d_{\mathrm{I}} = 0$) for which $(d_{\mathrm{I}} + d_{\mathrm{II}})^2 = 0$.

Remark 1.14 With these definitions, $Cone(f)$ is the simple complex associated to the double complex

$$\begin{array}{rccc} & A^{\cdot} & \xrightarrow{f} & B^{\cdot} \\ \text{first degree:} & -1 & & 0 \end{array}$$

Definition 1.15 Two morphisms $f, g : A^{\cdot} \to B^{\cdot}$ of complexes are called **homotopic** ($f \sim g$), if there is a homotopy between f and g, i.e., a family $(h^n)_{n\in\mathbb{Z}}$ of homomorphisms $h^n : A^n \to B^{n-1}$ in $\mathcal{A}$ with

$$f - g = d^{n-1}h^n + h^{n+1}d^n$$

$$\begin{tikzcd} & A^n \arrow[r, "d"] \arrow[dl, "h^n"'] \arrow[d, "f-g"] & A^{n+1} \arrow[dl, "h^{n+1}"] \\ B^{n-1} \arrow[r, "d"'] & B^n & \end{tikzcd}$$

The class $\{f \sim 0\}$ of the zero homotopic morphisms form a two-sided ideal: If $g \circ f$ exists and $g \sim 0$ or $f \sim 0$, then $g \circ f \sim 0$; if $f \sim 0$ and $g \sim 0$, then $f \oplus g \sim 0$ in $C(\mathcal{A})$.

Definition 1.16 The category $K(\mathcal{A})$ of the complexes modulo homotopy (or homotopy category of complexes) has the same objects as $C(\mathcal{A})$ and as morphisms

$$\mathrm{Hom}_{K(\mathcal{A})}(A^{\cdot}, B^{\cdot}) = \mathrm{Hom}_{C(\mathcal{A})}(A^{\cdot}, B^{\cdot})/\{f \sim 0\}\,.$$

Analogously one defines $K^+(\mathcal{A}), K^-(\mathcal{A}), K^b(\mathcal{A})$. These are all additive categories, and $A^{\cdot} \rightsquigarrow A^{\cdot}[1]$ extends to a functor on $K^?(\mathcal{A})$.

Definition 1.17 For complexes $A^\cdot$ and $B^\cdot$ define the complex of **abelian groups** $\mathrm{Hom}^\cdot(A^\cdot, B^\cdot)$ by

$$\mathrm{Hom}^n(A^\cdot, B^\cdot) = \{\text{families } (f^m : A^m \to B^{m+n})\} = \prod_m \mathrm{Hom}_{\mathcal{A}}(A^m, B^{m+n}),$$

where the differential d is defined as follows: for $f = (f^m)$ set $df = (g^m)$, with

$$g^m = d_B f^m + (-1)^{m+1} f^{m+1} d_A \in \mathrm{Hom}_{\mathcal{A}}(A^m, B^{m+n+1}), .$$

Obviously we have

$$Z^0\mathrm{Hom}^\cdot(A^\cdot, B^\cdot) = \ker d^0 = \mathrm{Hom}_{\mathcal{C}(A)}(A^\cdot, B^\cdot)$$
$$B^0\mathrm{Hom}^\cdot(A^\cdot, B^\cdot) = im\, d^{-1} = \mathrm{Hom}^{\sim 0}_{C(A)}(A^\cdot, B^\cdot)$$
$$H^0(\mathrm{Hom}^\cdot(A^\cdot, B^\cdot)) = \ker d^0 / im\, d^{-1} = \mathrm{Hom}_{K(\mathcal{A})}(A^\cdot, B^\cdot)$$

Theorem 1.18 For an additive category $\mathcal{A}$, $K(\mathcal{A})$ becomes a triangulated category as follows: Define the translation functor by $T(A) := A[1]$ and call a triangle

$$X \to Y \to Z \to X[1]$$

distinguished (or exact), if it is isomorphic to a triangle

$$(*) \qquad A^\cdot \xrightarrow{f} B^\cdot \to Cone(f) \to A^\cdot[1]\,.$$

Proof We give full proof, indicating the shortcuts due to Lemma 1.6 and 1.7. (TR1): We only have to show that $X^\cdot \xrightarrow{id} X^\cdot \to 0 \to X[1]$ is exact, i.e., that $C := Cone(id_X) \cong 0$ in $K(\mathcal{A})$. This is (in an additive category) equivalent to $id_C = 0$ in $K(\mathcal{A})$, i.e. $id_C \sim 0$ (homotopic to 0). This follows from the diagram

$$\begin{tikzcd}
 & X^n \oplus X^{n+1} \arrow[dl, "h=(0,pr_1)"'] \arrow[d, "id"] \arrow[r, "(dpr_1+pr_2,-dpr_2)"] & X^{n+1} \oplus X^{n+2} \arrow[dl, "h=(0,pr_1)"] \\
X^{n-1} \oplus X^n \arrow[r, "(dpr_1+pr_2,-dpr_2)"'] & X^n \oplus X^{n+1} &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (x_n, x_{n+1}) \arrow[dl, "h"'] \arrow[r, mapsto, "d"] & (dx_n + x_{n+1}, -dx_{n+1}) \arrow[dl, "h"] \\
(0, x_n) \overset{d}{\mapsto} & (d_0 + x_n, -dx_n), (0, dx_n + x_{n+1})\,, &
\end{tikzcd}$$

where $(d0 + x_n, -dx_n) + (0, dx_n + x_{n+1}) = (x_n, x_{n+1})$, so that $id = dh + hd$.

(TR2): We can consider a distinguished triangle of the form

$$X \xrightarrow{f} Y \xrightarrow{i} Cone(f) \xrightarrow{p} X[1]\,.$$

Here we have

$$Cone(f)^n = Y^n \oplus X[1]$$

with differential

$$d(y_n, x_{n+1}) = (dy_n + fx_{n+1}, dx_{n+1})$$

and

$$i(y_n) = (y_n, 0) \quad , \quad p(y_n, x_{n+1}) = x_{n+1} \, .$$

We construct a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, equal] & Cone(f) \arrow[r, "p"] \arrow[d, equal] & X[1] \arrow[r, "-f[1]"] & Y[1] \arrow[d, equal] \\ Y \arrow[r, "i"] & Cone(f) \arrow[r, "\alpha"] & Cone(i) \arrow[r, "\beta"] \arrow[u, "\pi"'] & Y[1] \end{tikzcd} \tag{1}$$

with a (homotopy-) isomorphism π, in which α and β are the canonical morphisms for $Cone(i)$ mit $Cone(i)^n = Y^n \oplus X^{n+1} \oplus Y^{n+1}$:

$$\begin{aligned} \alpha(y_n, x_{n+1}) &= (y_n, x_{n+1}, 0) \, , \\ \beta(y_n, x_{n+1}, y_{n+1}) &= y_{n+1} \, . \end{aligned}$$

Then this shows that $Y \longrightarrow Cone(f) \xrightarrow{p} X[1] \xrightarrow{i[1]} Y[1]$ is a distinguished triangle.

We have

$$C = Cone(i) \; = \; Cone(f) \oplus Y[1] \; 0 \; Y \oplus X[1] \oplus Y[1]$$

with differential

$$d_C(y_n, x_{n+1}, y_{n+1}) = (dy_n + fx_{n+1} + y_{n+1}, -dx_{n+1}, -dy_n) \, .$$

Define π and a homotopic inverse ι of π by

$$\pi(y_n, x_{n+1}, y_{n+1}) = x_{n+1} \quad , \quad \iota x_{n+1} = (0, x_{n+1}, -fx_{n+1}) \, .$$

Then π and ι are morphisms of complexes:

$$\begin{aligned} \pi d_C(y_n, x_{n+1}, y_n) &= \pi(dy_n + fx_{n+1} + y_{n+1}, dx_{n+1}, -dy_n) \\ &= -dx_{n+1} = d_{X[1]}\pi(y_n, x_{n+1}, y_n) \end{aligned}$$

and

$$\begin{aligned} d_C\iota(x_{n+1}) &= d(0, x_{n+1}, -fx_{n+1}) \\ &= (fx_{n+1} - fx_{n+1}, -dx_{n+1}, dfx_{n+1}) \\ &= (0, -dx_{n+1}, -f(-dx_{n+1})) = \iota(d_{X[1]}x_{n+1}) \, . \end{aligned}$$

Moreover we have $\pi\iota = id_{X[1]}$:

$$\pi\iota(x_{n+1}) = \pi(0, x_{n+1}, -fx_{n+1}) = x_{n+1}$$

Furthermore we have $\iota\pi \sim id_{Cone(i)}$: Consider the homotopy

$$\begin{aligned} h : \qquad Cone(i)^n &\longrightarrow Cone(i)^{n-1} \\ (y_n, x_{n+1}, y_{n+1}) &\longmapsto (0, 0, y_n) \end{aligned}$$

Then we have

$$\begin{aligned}
& (id_{Cone(i)} - \iota\pi)(y_n, x_{n+1}, y_{n+1}) \\
= \ & (y_n, x_{n+1}, y_{n+1}) - \iota\pi(y_n, x_{n+1}, y_n) \\
= \ & (y_n, x_{n+1}, y_{n+1}) - \iota x_{n+1} \\
= \ & (y_n, x_{n+1}, y_{n+1}) - (0, x_{n+1}, -f x_{n+1}) \\
= \ & (y_n, 0, y_{n+1} + f x_{n+1})
\end{aligned}$$

and

$$\begin{aligned}
& (dh + hd)(y_n, x_{n+1}, y_{n+1}) \\
= \ & d(0, 0, y_n) + h(dy_n + f x_{n+1} + y_{n+1}, -dx_{n+1}, dy_{n+1}) \\
= \ & (y_n, 0, dy_n) + (0, 0, dy_n + f x_{n+1} + y_{n+1}) \\
= \ & (y_n, 0, y_{n+1} + f x_{n+1})
\end{aligned}$$

Finally we show that the two right squares of (1) are commutative.

We have $\pi\alpha = p$, so that the square in the middle is commutative:

$$\pi\alpha(y_n, x_{n+1}) = \pi(y_n, x_{n+1}, 0) = x_{n+1} = p(y_n, x_{n+1}) .$$

Furthermore $\beta \sim -f[1]\pi$, so that the square on the right hand side is commutative: Since π and ι are homotopy isomorphisms inverse to each other, it suffices to show that

$$\beta\iota \sim -f[1]\pi\iota \sim -f[1] .$$

But we have

$$\beta\iota(x_{n+1}) = \beta(0, x_{n+1}, -f(x_{n+1})) = -f(x_{n+1}) = -f[1](x_{n+1}) .$$

Now we show the backward direction in TR2. By Lemma 1.17 and (TR3) this is not necessary, but we get a short proof as follows. By applying the shown direction several times, by "rolling on" we obtain that

$$Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y) \xrightarrow{-T(v)} T(Z)$$

and then

$$T(X) \xrightarrow{-T(u)} T(Y) \xrightarrow{-T(u)} T(Z) \xrightarrow{-T(w)} T^2(X)$$

are distinguished triangles. By rolling on twice again we obtain that

$$T^2(X) \xrightarrow{T^2(u)} T^2(Y) \xrightarrow{T^2(u)} T^2(Z) \xrightarrow{T^2(w)} T^3(X)$$

is a distinguished triangle. This is up to renumbering $n \mapsto n-2$ the initial triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] .$$

(TR3): By Lemma 1.16 this follows from (TR4), but here is a direct proof. Without restriction we consider a diagram

(2)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "\alpha"] \arrow[d, "v"] & Cone(f) \arrow[r, "\beta"] \arrow[d, dotted, "?\ w"] & X[1] \arrow[d, "u"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "\alpha'"] & Cone(f') \arrow[r, "\beta'"] & X[1]
\end{tikzcd} ,$$

where the first square is commutative up to homotopy, and we need to find a morphism w which makes the diagram commutative (up to homotopy). By assumption there exists a homotopy

$$h^n : X^n \to (Y^{n-1})'$$

with

$$v^n f^n - (f')^n u^n = s^{n+1} d_X^n + d_{Y'}^{n-1} s^n \,.$$

Now define

$$w : Cone(f)^n = Y^n \oplus X^{n+1} \to (Y')^n \oplus (X')^{n+1} = Cone(f')^n$$

by

$$w^n = \begin{pmatrix} v^n & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} .$$

Then w is a morphism of complexes: We have

$$\begin{aligned}
d \begin{pmatrix} v^n & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} \begin{pmatrix} y_n \\ x_{n+1} \end{pmatrix} &= d \begin{pmatrix} v^n y_n + d^{n+1} x_{n+1} \\ u^{n+1} x_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} dv^n y_n + ds^{n+1} x_{n+1} + f' u^{n+1} x_{n+1} \\ -du^{n+1} x_{n+1} \end{pmatrix} \\
\begin{pmatrix} v^n & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} d \begin{pmatrix} y_n \\ x_{n+1} \end{pmatrix} &= \begin{pmatrix} v^n & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} \begin{pmatrix} dy_n + f x_{n+1} \\ -dx_{n+1} \end{pmatrix} \\
&= \begin{pmatrix} v^n dy_n + v^n f x_{n+1} - s^{n+1} dx_{n+1} \\ -u^{n+1} dx_{n+1} \end{pmatrix} .
\end{aligned}$$

Since $dv^n = v^n d, du^{n+1} = u^{n+1} d$ and $v^n f - (f') u^{n+1} = s^{n+1} d^n + d^{n-1} s^n$ we have $dw = wd$. Furthermore both last squares commute: We have

$$\begin{aligned}
w\alpha(y_n) &= \begin{pmatrix} v & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} \begin{pmatrix} y_n \\ 0 \end{pmatrix} = \begin{pmatrix} v y_n \\ 0 \end{pmatrix} \\
\alpha v(y_n) &= \alpha(v y_n) = \begin{pmatrix} v y_n \\ 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
u\beta \begin{pmatrix} y_n \\ x_{n+1} \end{pmatrix} &= u(x_{n+1}) \\
\beta' w \begin{pmatrix} y_n \\ x_{n+1} \end{pmatrix} &= \beta' \begin{pmatrix} v & s^{n+1} \\ 0 & u^{n+1} \end{pmatrix} \begin{pmatrix} y_n \\ x_{n+1} \end{pmatrix} = \beta' \begin{pmatrix} v y_n + s^{n+1} x_{n+1} \\ u^{n+1} x_{n+1} \end{pmatrix} = u(x_{n+1}) \,.
\end{aligned}$$

(TR4): We consider three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{i} Z' \longrightarrow X[1]$$

$$Y \xrightarrow{g} Z \longrightarrow X' \xrightarrow{j} Y[1]$$

$$X \xrightarrow{gf} Z \longrightarrow Y' \longrightarrow X[1]$$

and look for morphisms $Z' \xrightarrow{u} Y'$ and $Y' \xrightarrow{v} X'$, such that the diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "i"] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, "u"] & X[1] \arrow[d, "id"] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, "id"] & Y' \arrow[r] \arrow[d, "v"] & X[1] \arrow[d, "u[1]"] \\
Y \arrow[r, "g"] & Z \arrow[r] & X' \arrow[r, "j"] \arrow[d] & Y[1] \arrow[dl, "i[1]"] \\
 & & Z'[1] &
\end{tikzcd}
\tag{3}
$$

is commutative and the third column is a distinguished triangle. We can assume that

$$
\begin{array}{rcccl}
Z' & = & Cone(f) & = & Y \oplus X[1] \\
Y' & = & Cone(gf) & = & Z \oplus X[1] \\
X' & = & Cone(g) & = & Z \oplus Y[1]
\end{array}
$$

Define morphisms

$$
\begin{array}{rcccl}
u : Z' & = & Y \oplus X[1] & \to & Y' = Z \oplus X[1] \\
 & & (y_n, x_{n+1}) & \mapsto & (gy_n, x_{n+1}) \\
v : Y' & = & Z \oplus X[1] & \to & X' = Z \oplus Y[1] \\
 & & (z_n, x_{n+1}) & \mapsto & (z_n, fx_{n+1})
\end{array}
$$

These are morphisms of complexes:

$$
\begin{array}{rcl}
ud_{Z'}(y_n, x_{n+1}) & = & u(dy_n + fx_{n+1}, -dx_{n+1}) \\
 & = & (gdy_n + gfx_{n+1}, -dx_{n+1}) \\
d_{Y'}u(y_n, x_{n+1}) & = & d_{Y'}(gy_n, x_{n+1}) \\
 & = & (dgy_n + gfx_{n+1}, -dx_{n+1})
\end{array}
$$

$$
\begin{array}{rcl}
vd_{Y'}(z_n, x_{n+1}) & = & v(dz_n + gfx_{n+1}, -dx_{n+1}) \\
 & = & (dz_n + gfx_{n+1}, -fdx_{n+1}) \\
d_{X'}v(z_n, x_{n+1}) & = & d_{X'}(z_n, fx_{n+1}) = (dz_n + gfx_{n+1}, dfx_{n+1})
\end{array}
$$

Now define $X' \to Z'[1]$ as the composition

$$X' \xrightarrow{j} Y[1] \xrightarrow{i[1]} Z'[1]$$

Then it follows easily that the diagram (3) above is commutative:

$$\begin{tikzcd}
Y \arrow[rr] & & Z' = Y \oplus X[1] \arrow[rr] & & X[1] \\
y_n \arrow[r, mapsto] \arrow[d, mapsto, "g"] & (y_n, 0), \arrow[d, mapsto, "u"] & (y_n, x_{n+1}) \arrow[rr, mapsto] \arrow[d, mapsto, "u"] & & X^{n+1} \arrow[d, mapsto, "id"] \\
gy_n \arrow[r, mapsto] & (gy_n, 0), & (gy_n, x_{n+1}) \arrow[rr, mapsto] & & x_{n+1} \\
Z \arrow[rr] & & Y' = Z \oplus X[1] \arrow[rr] & & X[1] \\
z_n \arrow[r, mapsto] \arrow[d, mapsto, "id"] & (z_n, 0), \arrow[d, mapsto, "v"] & (z_n, x_{n+1}) \arrow[rr, mapsto] \arrow[d, mapsto, "v"] & & X^{n+1} \arrow[d, mapsto] \\
z_n \arrow[r, mapsto] & (z_n, 0), & (z_n, fx_{n+1}) \arrow[rr, mapsto] & & fx_{n+1} \\
Z \arrow[rr] & & X' = Z \oplus Y[1] \arrow[rr, "j"] \arrow[ddd, "w"] & & Y[1] \\
& (z_n, y_{n+1}) \arrow[d, mapsto] & & & y_{n+1} \arrow[dl, mapsto] \\
& (y_{n+1}, 0) & & (y_{n+1}, 0) & \\
& & Z'[1] = Y[1] \oplus X[2] & &
\end{tikzcd}$$

Now we show that the triangle

(4) $$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} Z'[1]$$

is distinguished.

For this we show that there are morphisms

$$\phi : Cone(u) \longrightarrow X' \quad , \quad \psi : X' \longrightarrow Cone(u)$$

which are homotopy inverse to each other, such that we have a diagram

$$\begin{tikzcd}
Z' \arrow[r, "u"] \arrow[d, equal] & Y' \arrow[r, "v"] \arrow[d] & X' \arrow[r, "w"] \arrow[d, shift left, "\psi"] & Z'[1] \arrow[d, equal] \\
Z' \arrow[r, "u"] & Y' \arrow[r, "\alpha(u)"] & Cone(u) \arrow[r, "\beta(u)"] \arrow[u, shift left, "\phi"] & Z'[1]
\end{tikzcd}$$

with $\phi \circ \alpha(u) = v$ and $\beta(u) \circ \psi = w$.

We have

$$\begin{aligned}
Z' = Cone(f) &= Y \oplus X[1], \text{ with } d_{Z'}(y_n, x_{n+1}) = (dy_n + fx_{n+1}, -dx_{n+1}) \\
Y' = Cone(gf) &= Z \oplus X[1] \text{ with } d_{Y'}(z_n, x_{n+1}) = (dz_n + gfx_{n+1}, -dx_{n+1}),
\end{aligned}$$

and

$C = Cone(u) = Y' \oplus Z'[1]$, with

$$
\begin{aligned}
& d_C(z_n, x_{n+1}; y_{n+1}, x_{n+2}) \\
= \;& (d(z_n, x_{n+1}) + u(y_{n+1}, x_{n+2}); -d(y_{n+1}, x_{n+2})) \\
= \;& ((dz_n + gfx_{n+1}, -dx_{n+1}) + (gy_{n+1}, x_{n+2}); (-dy_{n+1} - fx_{n+2}, dx_{n+2})) \\
= \;& (dz_n + gfx_{n+1} + gy_{n+1}, -dx_{n+1} + x_{n+2}; dy_{n+1} - fx_{n+2}, dx_{n+2})
\end{aligned}
$$

as well as

$$X' = Cone(g) = Z \oplus Y[1]$$

with

$$d_{X'}(z_n, y_{n+1}) = (dz_n + gy_{n+1}, -dy_{n+1})\,.$$

Define ϕ and ψ by

$$\phi^n = \begin{pmatrix} id_{Z^n} & 0 & 0 & 0 \\ 0 & f^{n+1} & id_{Y^{n+1}} & 0 \end{pmatrix}\,,$$

so that $\phi^n(z_n, x_{n+1}, y_{n+1}, x_{n+2}) = (z_n, fx_{n+1} + y_{n+1})$, and

$$\psi^n = \begin{pmatrix} id_{Z^n} & 0 \\ 0 & 0 \\ 0 & id_{Y^{n+1}} \\ 0 & 0 \end{pmatrix}\,,$$

so that $\psi^n(z_n, y_{n+1}) = (z_n, 0, y_{n+1}, 0)$.

Then ϕ and ψ are morphisms of complexes: We have

$$
\begin{aligned}
& d\phi(z_n, x_{n+1}, y_{n+1}, x_{n+2}) \\
= \;& d(z_n, fx_{n+1} + y_{n+1}) \\
= \;& (dz_n + gfx_{n+1} + gy_{n+1}, -dfx_{n+1} - dy_{n+1})
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi d(z_n, x_{n+1}, y_{n+1}, x_{n+2}) \\
& \phi(dz_n + gfx_{n+1} + gy_{n+1}, -dx_{n+1} + x_{n+2}, -dy_{n+1} - fx_{n+2}, +dx_{n+2}) \\
= \;& (dz_n + gfx_{n+1} + gy_{n+1}, -fdx_{n+1} + fx_{n+2} - dy_{n+1} - fx_{n+2}) \\
= \;& (dz_n + gfx_{n+1} + gy_{n+1}, -fdx_{n+1} - dy_{n+1})
\end{aligned}
$$

Furthermore we have

$$
\begin{aligned}
& d\psi(z_n, y_{n+1}) \\
= \;& d(z_n, 0, y_{n+1}, 0) \\
= \;& (dz_n + gy_{n+1}, 0, -dy_{n+1}, 0)
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi d(z_n, y_{n+1}) \\
= \;& \psi(dz_n + gy_{n+1}, -dy_{n+1}) \\
= \;& (dz_n + gy_{n+1}, 0, -dy_{n+1}, 0)
\end{aligned}
$$

Moreover we have $\phi \circ \psi = id_{X'}$:

$$\phi^n \psi^n(z_n, y_{n+1}) = \phi^n(z_n, 0, y_{n+1}, 0) = (z_n, y_{n+1})\,.$$

Finally we show that $\psi \circ \phi \sim id_{Cone(u)}$. Define the homotopy

$$s^n : Cone(u)^n \longrightarrow Cone(u)^{n-1}$$

$$s^n = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & id_X & 0 & 0 \end{pmatrix},$$

i.e.,

$$s(z_n, x_{n+1}, y_{n+1}, x_{n+2}) = (0, 0, 0, x_{n+1}).$$

Then $id_{Cone(u)} - \psi \circ \phi = sd + ds$:

$$\begin{aligned}
& (id_{Cone(u)} - \psi \circ \phi)(z_n, x_{n+1}; y_{n+1}, x_{n+2}) \\
= \; & (z_n, x_{n+1}; y_{n+1}, x_{n+2}) - \psi(\phi(z_n, x_{n+1}; y_{n+1}, x_{n+2})) \\
= \; & (z_n, x_{n+1}; y_{n+1}, x_{n+2}) - \psi(z_n, fx_{n+1} + y_{n+1}) \\
= \; & (z_n, x_{n+1}; y_{n+1}, x_{n+2}) - (z_n, 0, fx_{n+1} + y_{n+1}, 0) \\
= \; & (0, x_{n+1}, -fx_{n+1}, x_{n+2})
\end{aligned}$$

and

$$\begin{aligned}
& sd(z_n, x_{n+1}, y_{n+1}, x_{n+2}) + ds(z_n, x_{n+1}, y_{n+1}, x_{n+2}) \\
= \; & s(dz_n + gfx_{n+1} + gy_{n+1}, -dx_{n+1} + x_{n+2}, -dy_{n+1} - fx_{n+2}, dx_{n+2}) + d(0, 0\,0, x_{n+1}) \\
= \; & (0, 0, 0, -dx_{n+1} + x_{n+2} + (0, x_{n+1}, -fx_{n+1}, dx_{n+1}) \\
= \; & (0, x_{n+1}, -fx_{n+1}, x_{n+2})
\end{aligned}$$

Appendix 1.A

The following result is due to J.L. Verdier.

Lemma 1.15 (The 3x3 lemma) Consider a diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] \arrow[dr, phantom, "1"] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] \arrow[dr, phantom, "2"] & TX \arrow[d, "T(i)"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"] \arrow[dr, phantom, "3"] & Y' \arrow[r, "g'"] \arrow[d, "j'"] \arrow[dr, phantom, "4"] & Z' \arrow[r, "h'"] \arrow[d, dotted, "k'"] \arrow[dr, phantom, "5"] & TX' \arrow[d, "T(i')"] \\
X'' \arrow[r, dotted, "f''"] \arrow[d, "i''"] \arrow[dr, phantom, "6"] & Y'' \arrow[r, dotted, "g''"] \arrow[d, "j''"] \arrow[dr, phantom, "7"] & Z'' \arrow[r, dotted, "h''"] \arrow[d, dotted, "k''"] \arrow[dr, phantom, "8"] & TX'' \arrow[d, "-T(i'')"] \\
TX \arrow[r, "Tf"] & TY \arrow[r, "Tg"] & TZ \arrow[r, "-Th"] & T^2X \,,
\end{tikzcd}
$$

in which $jf = f'i$, and the two first rows and the two left columns are distinguished triangles. Then there is an object Z'' and the dotted arrows $f'', g'', h'', k, k', k''$ such that the diagram is commutative except for the bottom right square which commutes up to the sign -1, and all four rows and columns are distinguished.

Proof The bottom row is distinguished by the isomorphism of triangles

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] & TY \arrow[r, "Tg"] & TZ \arrow[r, "-Th"] & T^2X \\
TX \arrow[r, "-Tf"] \arrow[u, "id"] & TY \arrow[r, "-Tg"] \arrow[u, "-id"] & TZ \arrow[r, "-Th"] \arrow[u, "id"] & T^2X \arrow[u, "id"]
\end{tikzcd}
$$

where the bottom triangle is distinguished by (TR2). Similarly, the right column is distinguished. Now we start as in the proof of Lemma 1.6:

Applying (TR1), we have a distinguished triangle

$$X \xrightarrow{jf} Y' \xrightarrow{p} V \xrightarrow{q} TX\,.$$

By (TR4) we get a commutative diagram of distinguished triangles

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "id"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, "s"] & TX \arrow[d, "id"] \\
X \arrow[r, "jf"] \arrow[d, "f"] & Y' \arrow[r, "p"] \arrow[d, "id"] & V \arrow[r, "q"] \arrow[d, "t"] & TX \arrow[d, "T(f)"] \\
Y \arrow[r, "j"] & Y' \arrow[r, "j'"] & Y'' \arrow[r, "j''"] \arrow[d, "u"] & TY \arrow[dl, "T(g)"] \\
 & & TZ &
\end{tikzcd}
$$

so that $u = T(g)j''$. Similarly, for $f'i(= jf)$ we get the commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "id"] & X' \arrow[r, "i'"] \arrow[d, "f'"] & X'' \arrow[r, "i''"] \arrow[d, "s'"] & TX \arrow[d, "id"] \\
X \arrow[r, "f'i"] \arrow[d, "i"] & Y' \arrow[r, "p"] \arrow[d, "id"] & V \arrow[r, "q"] \arrow[d, "t'"] & TX \arrow[d, "T(i)"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] \arrow[d, "u'"] & TX' \arrow[dl, "T(i')"] \\
 & & TX'' &
\end{tikzcd}
$$

so that $u' = T(i')h'$, and

$$
\begin{array}{ccccccc}
pj = sg & , & tp = j' & , & qs = h & , & j''t = T(f)q \\
pf' = s'i' & , & t'p = g' & , & qs' = i'' & , & h't' = T(i)q
\end{array} \quad .
$$

Now we let $k = t's : Z \to Z'$. Then we have
$kg = t'sg = t'pj = g'j$, and
$h'k = h't's = T(i)qs = T(i)h$,
so that k makes the diagram (1) commutative.
Define $k = t's : Z \to Z'$. Then we get the commutativity of the first two squares:

1 $kg = t'sg = t'pj = g'j$

2 $h'k = h't's = T(i)qs = T(i)h.$

Corollary (TR3) holds.

Now define $f'' = t \circ s' : X'' \to Y''$, and apply (T1) to construct a distinguished triangle

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} TX''$$

Applying (TR4), we get a commutative diagram

$$\begin{tikzcd}
X'' \arrow[r, "s'"] \arrow[d, "id"] & V \arrow[r, "t'"] \arrow[d, "t"] & Z' \arrow[r, "T(i')h'"] \arrow[d, "k'"] & TX'' \arrow[d, "id"] \\
X'' \arrow[r, "f''"] \arrow[d, "s'"] & Y'' \arrow[r, "g''"] \arrow[d, "id"] & Z'' \arrow[r, "h''"] \arrow[d, "k''"] & TX'' \arrow[d, "T(s')"] \\
V \arrow[r, "t"] & Y'' \arrow[r, "T(g)j''"] & TZ \arrow[r, "-Ts"] \arrow[d] & TV \arrow[dl, "T(t')"] \\
 & & TZ' &
\end{tikzcd}$$

Now we get the claims on the remaining squares:

③ $f''i' = ts'i' = tpf' = j'f'$

④ $g''j' = g''tp = k't'p = k'g'$

⑤ $h''k' = T(i')h'$

⑥ $j''f'' = j''ts' = T(f)qs' = T(f)i''$

⑦ $k''g'' = T(g)j''$

⑧ $-T(h)k'' = -T(qs)k'' = -T(q)T(s)k'' = T(q)T(s')h'' = T(i'')h''$

(so that the squares ③ - ⑦ commute, and ⑧ anti-commutes).

2 Localization of categories and triangulated categories

Let $\mathcal{C}$ be a category and let S be a set of morphisms in $\mathcal{C}$. The localization $\mathcal{C}[S^{-1}]$ of $\mathcal{C}$ by S should have the following universal property:

There is a functor $a_S : \mathcal{C} \to \mathcal{C}[S^{-1}]$, so that we have: If $F : \mathcal{C} \to \mathcal{C}'$ is a functor in another category, so that for every $s \in S$ the morphism $F(s)$ is invertible $\mathcal{C}'$ (therefore has an inverse), then there is a uniquely determined functor $\tilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{C}'$, which makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "a_s"] \arrow[dr, "F"'] & & \mathcal{C}[S^{-1}] \arrow[dl, "\exists! \tilde{F}"] \\ & \mathcal{C}' & \end{tikzcd}$$

Such a category "always exists, modulo set theoretic difficulties". Existing constructions are (by proper conditions)

- a calculus of fractions (used here)
- closed model categories (Quillen)

Fix a set S of morphisms in $\mathcal{C}$ as above.

Definition 2.1 S allows calculus of fractions from the right if the following holds:

(FR1) For all $X \in Ob(\mathcal{C})$, $id_X \in S$, and S is closed under composition.

(FR2)^r Every diagram in $\mathcal{C}$ as on the left hand side can be continued to a commutative diagram on the right hand side:

$$\begin{tikzcd} & X' \arrow[d, "s"] \\ Y \arrow[r, "f"] & X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "s'"'] & X' \arrow[d, "s"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

where s and s' are in S.

(FR3)^r For every diagram

$$\begin{tikzcd} X' \arrow[r, dotted, "t"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Y' \end{tikzcd}$$

with $s \in S$ and $sf = sg$, there is a $t : X' \to X$ in S with $ft = gt$.

Dually (by reversing the arrows) one defines the calculus of fractions from the left. If S allows calculus of fractions from the right and the left, then one says that S allows calculus of fractions.

Theorem 2.2 If S allows calculus of fractions from the right, then $\mathcal{C}[S^{-1}]$ can be defined as follows. We have

$$ob(\mathcal{C}[S^{-1}]) = ob(\mathcal{C})\,.$$

For an object X in $\mathcal{C}$, let S/X be the category of the morphisms in S over X, i.e., the morphisms

$$s : X' \longrightarrow X$$

in $\mathcal{C}$ with $s \in S$. The morphisms are commutative diagrams

$$\begin{tikzcd} X'' \arrow[rr, "f"] \arrow[dr, "s''"'] & & X' \arrow[dl, "s'"] \\ & X & \end{tikzcd} \qquad (s', s'' \in S)$$

The category S/X is cofiltered: It is

(i) non-empty,

(ii) for objects $s : X' \to X$ and $s' : X'' \to X$ in S/X the diagram

$$\begin{tikzcd} & X' \arrow[d, "s'"] \\ X'' \arrow[r, "s''"] & X \end{tikzcd}$$

can be continued by (FR2)^r to a commutative diagram

$$\begin{tikzcd} X''' \arrow[r, "f"] \arrow[d, "s'''"'] & X' \arrow[d, "s'"] \\ X'' \arrow[r, "s''"] & X \end{tikzcd}$$

with $s''' \in S$, so that one has morphisms

$$\begin{tikzcd} & X' \\ X''' \arrow[ur] \arrow[dr] & \\ & X'' \end{tikzcd}$$

in S/X.

(iii) For morphisms

$$\begin{tikzcd} X'' \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "s''"'] & & X' \arrow[dl, "s'"] \\ & X & \end{tikzcd},$$

with $s'f = s'' = s'g$, there is, by (FR3)^r, a morphism $s''' : X''' \to X''$ with $fs''' = gs'''$.

Therefore the dual category $(S/X)^0$ is filtered, and we define

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) = \varinjlim_{X' \in (S/X)^0} \mathrm{Hom}(X', Y)\,.$$

This means: Every morphism in $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$ is given by a diagram

$$\begin{tikzcd}
X' \arrow[d, "s"'] \arrow[dr, "f"] & \\
X & Y
\end{tikzcd}$$

with $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $s \in S$. We also write fs^{-1} for this morphism. The composition of morphisms

$$\begin{tikzcd}
X' \arrow[d, "s"'] \arrow[dr, "f"] & Y' \arrow[d, "t"] \arrow[dr, "g"] & \\
X & Y & Z
\end{tikzcd} \qquad (s, t \in S)$$

is given by a commutative diagram

$$\begin{tikzcd}
X'' \arrow[d, "t'"'] \arrow[dr, "g"] & & \\
X \arrow[r, phantom, "*"] \arrow[d, "s"'] \arrow[dr, "f"] & Y' \arrow[d, "t"] \arrow[dr] & \\
X & Y & Z
\end{tikzcd}$$

where the commutative diagram $*$ with $t' \in S$ exists by $(\mathrm{FR2})^r$. Two morphisms

$$\begin{tikzcd}
X' \arrow[d, "s'"'] \arrow[dr, "f"] & \\
X & Y
\end{tikzcd} \qquad
\begin{tikzcd}
X'' \arrow[d, "s''"'] \arrow[dr, "g"] & \\
X & Y
\end{tikzcd}$$

with $s', s'' \in S$ coincide in $\mathcal{C}[S^{-1}]$, if and only if there is a commutative diagram

$$\begin{tikzcd}
 & X' \arrow[dl, "s'"'] \arrow[dr, "f"] & \\
X & X''' \arrow[l, "s'''"'] \arrow[r, "h"] \arrow[u] \arrow[d] & Y \\
 & X'' \arrow[ul, "s''"] \arrow[ur, "g"'] &
\end{tikzcd}$$

with $s''' \in S$.

Remark 2.3 If S allows calculus of fractions from the right and $\mathcal{C}$ is additive, then $\mathcal{C}[S^{-1}]$ is additive in a natural way and $\mathcal{C} \to \mathcal{C}[S^{-1}]$ is an additive functor. This is implied by the fact that $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$ as a filtered limit of abelian groups is an additive group in a natural way, and the bilinearity of compositions follows as well.

Remark 2.4 The concrete addition of morphisms can described as follows: Let fs^{-1} and gt^{-1} be morphisms from X to Y in $\mathcal{C}[S^{-1}]$. By $(\mathrm{FR2})^r$ we have morphisms f' and s' with

$s' \in S$

$$
\begin{tikzcd}
 & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X & Z \arrow[l, "u"'] \arrow[u, "f'"'] \arrow[d, "s'"] & Y \\
 & X'' \arrow[ul, "t"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

which make the left hand triangle without u commutative, so that $u = sf' = ts' \in S$. Then

$$fs^{-1} = ff'u^{-1} \quad \text{and} \quad gt^{-1} = gs'u^{-1}\,,$$

where ff' and gs' are morphisms in $\mathcal{C}$, so that one can add fs^{-1} and gt^{-1} in an obvious way by the common denominator u.

Now we show the universal property of $\mathcal{C}[S^{-1}]$. Let

$$F : \mathcal{C} \to \mathcal{C}'$$

be an exact functor into another triangulated category, such that $F(s)$ is invertible in $\mathcal{C}'$ for each $s \in S$. Define a functor

$$\tilde{F} : \mathcal{C}[S^{-1}] \to \mathcal{C}'$$

as follows. Since $\mathcal{C}[S^{-1}]$ has the same objects as $\mathcal{C}$, we necessarily have to define $\tilde{F}$ on objects by

$$\tilde{F}(X) = F(X)\,.$$

Next, for a morphism $fs^{-1} : X \to Y$ in $\mathcal{C}[S^{-1}]$ with f a morphism in $\mathcal{C}$ and s a morphism in S, we necessarily have to define

$$\tilde{F}(fs^{-1}) = F(f)F(s)^{-1}\,.$$

This is well-defined, since $F(s)$ is invertible by assumption on $\mathcal{C}'$. For a composition of morphisms

$$
\begin{tikzcd}
X'' \arrow[d, "t'"] \arrow[dr, "h"] & & \\
X' \arrow[d, "s"] \arrow[dr, "f"] & Y' \arrow[d, "t"] \arrow[dr, "g"] & \\
X & Y & Z
\end{tikzcd}
$$

in $\mathcal{C}[S^{-1}]$ (with $s, t, t' \in S$) we have

$$
\begin{aligned}
\tilde{F}(gt^{-1})\tilde{F}(fs^{-1}) &= F(g)F(t)^{-1}F(f)F(1)^{-1} \\
&= F(g)F(h)F(t')^{-1}F(s)^{-1} \\
&= F(gh)F(st')^{-1} \\
&= F(gh(st')^{-1})\,,
\end{aligned}
$$

which shows the compatibility with composition.

Finally we have $F = a_S\tilde{F}$ by construction.

Remark 2.5 Analogously one can define calculus of fractions from the left by reversing all arrows in the considerations above (thus passing to the dual category). In this case we have

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y) = \varinjlim_{Y'\in S\setminus Y} \mathrm{Hom}(X,Y'),$$

where $S \setminus Y$ is the category of morphisms under Y:

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[rr] & & Y_2 \end{tikzcd},$$

which is filtered. In some sources (see i.e. SGA 4 $\frac{1}{2}$, [C.D.] of J.L. Verdier), calculus of fractions is assumed from the left and from the right.

Now we apply the calculus of fractions to triangulated categories.

Definition 2.6 Let $\mathcal{D}$ be a triangulated category. A set S of morphisms with calculus of fractions from the right in $\mathcal{D}$ is called compatible with the translation if we have

(FR4) $s \in S \quad \Leftrightarrow \quad T(s) \in S$.

Theorem 2.7 Let $\mathcal{D}$ be a triangulated category, and let S be a set of morphisms with calculus of fractions from the right in $\mathcal{D}$, which is compatible with the translation. Then there is a unique structure of a triangulated category on $\mathcal{D}[S^{-1}]$ such that the following holds:

(a) $a_S : \mathcal{D} \longrightarrow \mathcal{D}[S^{-1}]$ is an exact functor.

(b) If $F : \mathcal{D} \to \mathcal{D}'$ is an exact functor into a triangulated category $\mathcal{D}'$, such that for all $s \in S$ the image $F(s)$ is an isomorphism in $\mathcal{D}'$, then there is a uniquely determined exact functor $\tilde{F} : \mathcal{D}[S^{-1}] \to \mathcal{D}'$, which makes the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "a_S"] \arrow[dr, "F"'] & & \mathcal{D}[S^{-1}] \arrow[dl, "\tilde{F}"] \\ & \mathcal{D}' & \end{tikzcd}$$

commutative.

Proof: (a) Define T on $\mathcal{D}[S^{-1}]$ by

$$\begin{aligned} T(C) &= T(C) \\ T(X \xleftarrow{s} X' \xrightarrow{f} Y) &= T(X) \xleftarrow{T(s)} T(X') \xrightarrow{T(f)} T(Y) \;. \end{aligned}$$

It follows from (FR4) that this is the unique functor with $Ta_S = a_S T$, and we define the distinguished triangles in $\mathcal{D}[S^{-1}]$ as those which are isomorphic to images of distinguished triangles under a_S.

Now we show that $\mathcal{D}[S^{-1}]$ is a triangulated category: The properties (TR1) (a), (TR1) (c) and (TR2) are obviously fulfilled. We show (TR1) (b): Let $fs^{-1} : X \xleftarrow{s^{-1}} X' \xrightarrow{f} Y$ be a morphism in $\mathcal{D}[S^{-1}]$. For f we have a distinguished triangle

$$X' \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX'$$

in $\mathcal{D}$. Then we obtain a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "g"] \arrow[d, "id"] & Z \arrow[r, "h"] \arrow[d, "id"] & TX' \arrow[d, "T(s)"] \\
X \arrow[r, "fs^{-1}"] & Y \arrow[r, "g"] & Z \arrow[r, "h"] & TX ,
\end{tikzcd}$$

such that the bottom triangle is isomorphic to the image of the distinguished triangle in $\mathcal{D}$ at the top, and therefore is distinguished in $\mathcal{D}[S^{-1}]$, by definition.

Now we first show (TR3). We use the following result.

Lemma 2.8 Consider a commutative diagram in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
X' \arrow[r, "u'"] & Y' \\
X \arrow[u, "m"] \arrow[r, "u"] & Y \arrow[u, "n"'] ,
\end{tikzcd}$$

where u und u' are morphisms in $\mathcal{D}$. Then there is a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "u'"] & Y' \\
X'' \arrow[u, "f"] \arrow[r, "u''"] \arrow[d, "s"'] \arrow[ur, phantom, "①"] \arrow[dr, phantom, "②"] & Y'' \arrow[u, "g"'] \arrow[d, "t"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}$$

in $\mathcal{D}$, with $s, t \in S$, such that $n = gt^{-1}$ and $m = fs^{-1}$.

Proof (1) Assume that $n = gt^{-1}$, with

$$Y \xleftarrow{t} Y'' \xrightarrow{g} Y' .$$

By $(\mathrm{FR2})^r$ we get a commutative diagram

$$\begin{tikzcd}
X_1'' \arrow[r, "u_1''"] \arrow[d, "s_1"'] & Y'' \arrow[d, "t"] \\
X \arrow[r, "u"] & Y
\end{tikzcd}$$

with $s_1 \in S$.

(2) Let $m = f_0 r^{-1}$ with $r \in S$. In the diagram in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
 & X' \arrow[r, "u'"] & Y'' \\
 & X''_2 \arrow[u, "f'"] \arrow[d, "s_2"] \arrow[dr, "u''_2"] & \\
T \arrow[uur, "f_0"] \arrow[dr, "r"'] & X''_1 \arrow[r, "u''_1"] \arrow[d, "s_1"] & Y'' \arrow[uu, "g"'] \arrow[d, "t"] \\
 & X \arrow[r, "u"] & Y
\end{tikzcd}$$

we can write $f_0 r^{-1} s_1$ as a morphism $f' s_2^{-1}$ as indicated, and we let $u''_2 = u''_1 s_2$. The diagram in $\mathcal{D}$

$$\begin{tikzcd}
X' \arrow[r, "u'"] & Y' \\
X''_2 \arrow[u, "f'"] \arrow[r, "u''_2"] & Y'' \arrow[u, "g"']
\end{tikzcd}$$

is only commutative in $\mathcal{D}[S^{-1}]$, but this means that there is a morphism $s_3 : X'' \to X''_2$ in S with $u' f' s_3 = g u''_2 s_3$. Setting $f = f' s_3$ and $u'' = u''_2 s_3$ and $s = s_1 s_2 s_3$, we get a commutative diagram

$$\begin{tikzcd}
X' \arrow[rr, "u'"] & & Y' \\
 & X'' \arrow[ul, "f"'] \arrow[dl, "s_3"] \arrow[ddr, "u''"] & \\
X''_2 \arrow[uu, "f'"] \arrow[d, "s_2"] \arrow[drr, "u''_2"] & & \\
X''_1 \arrow[rr, "u''_1"] \arrow[d, "s_1"] & & Y'' \arrow[uuu, "g"'] \arrow[d, "t"] \\
X \arrow[rr, "u"] & & Y
\end{tikzcd}$$

and hence the claim.

By Lemma 2.8, we can prove (TR3) for $\mathcal{D}[S^{-1}]$ as follows. Consider a diagram in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "m"] & Y \arrow[r, "v"] \arrow[d, "n"] & Z \arrow[r, "w"] & X[1] \arrow[d, "m[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1]
\end{tikzcd}$$

where the rows are distinguished triangles and the first square is commutative. We may assume that the triangles are triangles in $\mathcal{D}$, so that $u, v, w, u', v', w' \in \mathcal{D}$. By Lemma 2.8 we may assume that we have a diagram in $\mathcal{D}$

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & X[1] \\
X'' \arrow[u, "s"] \arrow[d, "f"] \arrow[r, "u''"] & Y'' \arrow[u, "t"] \arrow[d, "g"] \arrow[r] & Z'' \arrow[u, dotted, "u"] \arrow[d, dotted, "h"] \arrow[r] & X''[1] \arrow[u, "{s[1]}"] \arrow[d, "{f[1]}"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

with morphisms f and g in $\mathcal{D}$, and s and t in S, where the first two squares are commutative, and where we have completed u'' to a distinguished triangle. By (TR3) in $\mathcal{D}$ we have a morphism h in $\mathcal{D}$ as indicated giving a morphism (f, g, h) of triangles. On the other hand, we get a morphism u as indicated, such that (s, t, u) is a morphism of triangles. Now u might not be in S, but s and t become isomorphisms in $\mathcal{D}[S^{-1}]$, and by Lemma 1.10, u becomes an isomorphism in $\mathcal{D}[S^{-1}]$ as well, so that hu^{-1} is a well-defined morphism in $\mathcal{D}[S^{-1}]$.

Finally we prove that (TR4) holds in $\mathcal{D}[S^{-1}]$. Let $a = a_S : \mathcal{D} \to \mathcal{D}[S^{-1}]$ be the canonical functor, and consider a diagram in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & U \arrow[r] & X[1] \arrow[d, equal] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, equal] & V \arrow[r] & X[1] \arrow[d, "{f[1]}"] \\
Y \arrow[r, "g"] & Z \arrow[r] & W \arrow[r] & Y[1]
\end{tikzcd}$$

in which the rows are distinguished triangles and the two left squares are commutative.

By assumption there is an isomorphism of triangles in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r] & U \arrow[r] & X[1] \\
a\tilde{X} \arrow[u, "\wr"] \arrow[r, "a\tilde{f}"] & a\tilde{Y} \arrow[u, "\wr"] \arrow[r] & a\tilde{U} \arrow[u, "\wr"] \arrow[r] & a\tilde{X}[1] \arrow[u, "\wr"]
\end{tikzcd}$$

for a distinguished triangle

$$\tilde{X} \xrightarrow{\tilde{f}} \tilde{Y} \longrightarrow \tilde{U} \longrightarrow \tilde{X}[1]$$

in $\mathcal{D}$, and an isomorphism of triangles in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
Y \arrow[r, "g"] & Z \arrow[r] & W \arrow[r] & Y[1] \\
aY' \arrow[u, "\wr"] \arrow[r] & aZ' \arrow[u, "\wr"] \arrow[r] & aW' \arrow[u, "\wr"] \arrow[r] & aY'[1] \arrow[u, "\wr"]
\end{tikzcd}$$

for a distinguished triangle

$$Y' \longrightarrow Z' \longrightarrow W' \longrightarrow Y'[1]$$

in $\mathcal{D}$. Now we use the following

Lemma 2.9 Given a diagram in $\mathcal{D}[S^{-1}]$

$$\begin{tikzcd}
 & aY' \\
a\tilde{X} \arrow[r, "a\tilde{f}"] & a\tilde{Y} \arrow[u, "\wr"']
\end{tikzcd}$$

where $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a morphism in $\mathcal{D}$ and the vertical morphism is an isomorphism in $\mathcal{D}[S^{-1}]$, then there exists a morphism $f' : X' \to Y'$ in $\mathcal{D}$ and an isomorphism $a\tilde{X} \to aX'$ in $\mathcal{D}[S^{-1}]$ such that the diagram

$$\begin{tikzcd}
aX' \arrow[r, "af'"] & aY' \\
a\tilde{X} \arrow[r, "a\tilde{f}"] \arrow[u, "\wr"'] & a\tilde{Y} \arrow[u, "\wr"']
\end{tikzcd}$$

commutes.

Proof: The morphism $a\tilde{Y} \to aY'$ is given by a morphism $\tilde{Y} \xleftarrow{s} Y \xrightarrow{g} Y'$. Then we get a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
\hat{X} \arrow[r] \arrow[u, equal] \arrow[d, "t"'] & \hat{Y} \arrow[u, "g"'] \arrow[d, "s"] \\
X \arrow[r] & Y
\end{tikzcd}$$

with $s, t \in S$, where the lower square exists by (FR2)^r.

We now conclude as follows. Since we have an isomorphism $Y' \xrightarrow{\sim} Y \xleftarrow{\sim} \tilde{Y}$ in $\mathcal{D}[S^{-1}]$, we get an isomorphism of distinguished triangles in $\mathcal{D}[S^{-1}]$

(1)

$$\begin{tikzcd}
aX' \arrow[r, "af'"] & aY' \arrow[r] & aU' \arrow[r] \arrow[dd, dotted] & aX'[1] \arrow[dd, "\wr"'] \\
a\tilde{X} \arrow[r, "a\tilde{f}"] \arrow[u, "\wr"'] \arrow[d, "\wr"'] & a\tilde{Y} \arrow[u, "\wr"'] \arrow[d, "\wr"'] & & \\
X \arrow[r] & Y \arrow[r] & U \arrow[r] & X[1]
\end{tikzcd}$$

where we have used the property (TR3) proved before.

Similarly we get an isomorphism of distinguished triangles

(2)

$$\begin{tikzcd}
aY' \arrow[r, "ag'"] & aZ' \arrow[r] & aW' \arrow[r] & aY'[1] \\
Y \arrow[r, "g"] \arrow[u, "\wr"'] & Z \arrow[r] \arrow[u, "\wr"'] & W \arrow[r] \arrow[u, "\wr"'] & Y[1] \arrow[u, "\wr"']
\end{tikzcd}$$

and an isomorphism of distinguished triangles

$$
(3) \qquad \begin{tikzcd}
aX' \arrow[r, "a(g'f')"] & aZ' \arrow[r] & aV' \arrow[r] & aX'[1] \\
Y \arrow[u, "\wr"] \arrow[r, "gf"] & Z \arrow[u, "\wr"] \arrow[r] & V \arrow[u, "\wr"] \arrow[r] & X[1] \arrow[u, "\wr"]
\end{tikzcd}
$$

Since (TR4) holds for the three distinguished triangles in $\mathcal{D}$, it also holds for the given distinguished triangles in $\mathcal{D}[S^{-1}]$.

I am indebted to Johannes Sprang for suggesting the above proof. As far as I know, this is the first full proof of (TR4) for a localized triangulated category in the literature.

Now we consider the properties (a) and (b) in Theorem 2.7. Here (a) is fulfilled by definition of the distinguished triangles in $\mathcal{D}[S^{-1}]$. For (b) let

$$\mathcal{D} \xrightarrow{F} \mathcal{D}'$$

be an exact functor into another triangulated category, such that all morphisms in S become invertible in $\mathcal{D}'$. Forgetting the triangulated structures we get a unique functor $\tilde{F} : \mathcal{D}[S^{-1}] \to \mathcal{D}'$ which makes the diagram

$$
\begin{tikzcd}
\mathcal{D} \arrow[rr, "a_S"] \arrow[dr, "F"'] & & \mathcal{D}[S^{-1}] \arrow[dl, "\tilde{F}"] \\
& \mathcal{D}' &
\end{tikzcd}
$$

commutative. In addition, $\tilde{F}$ is an exact functor, because it maps all distinguished triangles in $\mathcal{D}[S^{-1}]$, which by definition are isomorphic to images of distinguished triangles in $\mathcal{D}$ under a_S, by construction to triangles in $\mathcal{D}'$ which are isomorphic to images of distinguished triangles in $\mathcal{D}$ under a_S:

More precisely, if we have an isomorphism of triangles in $\mathcal{D}[S^{-1}]$

$$
(i) \qquad \begin{tikzcd}
a_S X \arrow[r, "f"] & a_S Y \arrow[r, "g"] & a_S Z \arrow[r, "h"] & a_S X[1] \\
X' \arrow[u, "\alpha r^{-1}"] \arrow[r, "f'a^{-1}"] & Y' \arrow[u, "\beta s^{-1}"] \arrow[r, "g'b^{-1}"] & Z' \arrow[u, "\gamma t^{-1}"] \arrow[r, "h'c^{-1}"] & X'[1]\,, \arrow[u, "\alpha s^{-1}[1]"]
\end{tikzcd}
$$

with $r, s, t, a, b, c \in S$, where the upper triangle is the image under a_S of a distinguished triangle in $\mathcal{D}$, then we get an isomorphism of triangles in $\mathcal{D}'$,

$$
(ii) \qquad \begin{tikzcd}
F(X) \arrow[r, "F(f)"] & F(Y) \arrow[r, "F(g)"] & F(Z) \arrow[r, "F(h)"] & F(X)[1] \\
F(X') \arrow[u, "F(\alpha)F(r^{-1})"] \arrow[r, "F(f')F(a)^{-1}"] & F(Y') \arrow[u, "F(\beta)F(s^{-1})"] \arrow[r, "F(g')F(b)^{-1}"] & F(Z') \arrow[u, "F(\gamma)F(t^{-1})"] \arrow[r, "F(h')F(c)^{-1}"] & F(X')[1]\,, \arrow[u, "F(\alpha)F(s)^{-1}[1]"]
\end{tikzcd}
$$

so that the bottom triangle in (i) is mapped to a triangle isomorphic to the top triangle in (ii) and hence is distinguished in $\mathcal{D}'$.

Remark 2.8 In the literature, the above result is sometimes shown under the following additional assumption

(FR5) If

$$\begin{tikzcd} X \arrow[r] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] & T(X) \arrow[d, "T(r)"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X') \end{tikzcd}$$

is a diagram with $r, s \in S$, where the first square is commutative and the rows are distinguished triangles, we get a morphism $t : Z \to Z'$ in S, which makes the diagram commutative.

As we have seen, this assumption was not necessary. In fact, in the situation of (FR5), in any case, by (TR3) we get a morphism $t : Z \to Z'$ which makes the diagram commutative, and by Lemma 1.10, t becomes an isomorphism in $\mathcal{D}[S^{-1}]$, since r and s become isomorphisms. Therefore t is an isomorphism in $\mathcal{D}'$ as well.

In a triangulated category, a set of morphisms with calculus of fractions from both sides often arises by the following construction.

Definition 2.9 Let $(\mathcal{D})$ be a triangulated category. A triangulated subcategory $\mathcal{N}$ of $\mathcal{D}$ is a full subcategory which satisfies the following conditions:

N1: $0 \in \mathcal{N}$.

N2: $X \in \mathcal{N} \Leftrightarrow TX \in \mathcal{N}$.

N3: If $X, Y \in \mathcal{N}$ and $X \to Y \to Z \to X[1]$ is a distinguished triangle, then $Z \in \mathcal{N}$.

Sometimes such a subcategory is also called a null system.

Theorem 2.10 Every triangulated subcategory $\mathcal{N}$ of $\mathcal{D}$ defines a multiplicative system S with calculus of fractions which satisfies (FR4) and (FR5).

Proof: For a triangulated subcategory $\mathcal{N}$ of $\mathcal{D}$ define the set $S = S(\mathcal{N})$ by

$$s \in S(\mathcal{N}) \quad \Leftrightarrow \quad \text{there is a distinguished triangle } X \xrightarrow{s} Y \longrightarrow Z \longrightarrow X[1] \text{ with } Z \in \mathcal{N}\,.$$

We check the conditions (FR1) to (FR5).

(FR1): $id_X \in S$ by the distinguished triangle

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow X[1]$$

and the fact that $0 \in \mathcal{N}$ by N1.

Next let s, t the morphisms in $S(\mathcal{N})$, such that the composition ts exists. Then, by assumption, there are distinguished triangles

$$X \xrightarrow{s} Y \longrightarrow Z' \longrightarrow X[1]$$

$$Y \xrightarrow{t} Z \longrightarrow X' \longrightarrow Y[1]$$

with Z', X' in $\mathcal{N}$. By the axiom (TR1) for triangulated categories there is a distinguished triangle

$$X \xrightarrow{ts} Z \longrightarrow Y' \longrightarrow X[1]\,.$$

By the axiom (TR4) there is a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1]\,.$$

By applying (TR2) twice, we get a distinguished triangle

$$X' \longrightarrow Z'[1] \longrightarrow Y'[1] \longrightarrow X'[1]\,.$$

Since X' and $Z'[1]$ are in $\mathcal{N}$, we have $Y'[1] \in \mathcal{N}$ by (N3), and hence $Y' \in \mathcal{N}$ by (N2). This shows that $ts \in S(\mathcal{N})$.

$(\mathrm{FR2})^r$: We have to complete the left diagram to a commutative diagram as on the right:

$$\begin{tikzcd} & X' \arrow[d, "s\in S"] \\ Z \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t\in S"'] & X' \arrow[d, "s"] \\ Z \arrow[r, "f"] & Y\,. \end{tikzcd}$$

By assumption there is a distinguished triangle

$$X' \xrightarrow{s} Y \xrightarrow{k} Z' \longrightarrow X'[1]$$

with $Z' \in \mathcal{N}$. By (TR1) there is a distinguished triangle

$$Z \xrightarrow{kf} Z' \longrightarrow W \longrightarrow Z[1]\,,$$

and by (TR2) a distinguished triangle

$$W[-1] \longrightarrow Z \xrightarrow{kf} Z' \longrightarrow W$$

Since the diagram

$$\begin{tikzcd} Z \arrow[r, "kf"] \arrow[d, "f"] & Z' \arrow[d, "id_{Z'}"] \\ Y \arrow[r, "k"] & Z' \end{tikzcd}$$

is commutative, it can be extended to a morphism of distinguished triangles

$$\begin{tikzcd} Z \arrow[r, "kf"] \arrow[d, "f"] & Z' \arrow[r] \arrow[d, "id_{Z'}"] & W[1] \arrow[r] \arrow[d, "g"] & Z[1] \arrow[d, "f[1]"] \\ Y \arrow[r, "k"] & Z' \arrow[r] & X[1] \arrow[r] & Y[1] \end{tikzcd}$$

by (TR3).

By (TR2) we obtain a morphism of distinguished triangles

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"] & Z \arrow[r, "kf"] \arrow[d, "f"] & Z' \arrow[r] \arrow[d, "id_{Z'}"] & W[1] \arrow[d, "g[1]"] \\ X \arrow[r, "s"] & Z' \arrow[r, "k"] & Z' \arrow[r] & X[1]\,. \end{tikzcd}$$

Since $Z' \in \mathcal{N}$, we have $s \in S = S(\mathcal{N})$. This gives the wanted diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"] & X \arrow[d, "s"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

with $t \in S$.

$(\mathrm{FR2})^\ell$: Here we have to complete the following left diagram to a commutative diagram as on the right:

$$\begin{tikzcd} X' & \\ X \arrow[u, "t\in S"] \arrow[r, "g"] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, "f"] & Y' \\ Z \arrow[u, "t\in S"] \arrow[r, "g"] & Y \arrow[u, "s\in S"'] \end{tikzcd}$$

By assumption, there is a distinguished triangle

$$X \xrightarrow{t} X' \xrightarrow{k} Z' \xrightarrow{\ell} X[1]$$

with $Z' \in \mathcal{N}$. Furthermore, by (TR3), the commutative diagram

$$\begin{tikzcd} Z' \arrow[r, "\ell"] \arrow[d, "id_{Z'}"] & X[1] \arrow[d, "g[1]"] \\ Z' \arrow[r, "g[1]\ell"] & Y[1] \end{tikzcd}$$

can be extended to a morphism of distinguished triangles

$$\begin{tikzcd} Z' \arrow[r, "\ell"] \arrow[d, "id_{Z'}"] & X[1] \arrow[r, "-t[1]"] \arrow[d, "g[1]"] & X'[1] \arrow[r, "-k[1]"] \arrow[d, "f[1]"] & Z'[1] \arrow[d, "id_{Z'[1]}"] \\ Z' \arrow[r, "g[1]\ell"] & Y[1] \arrow[r] & Y' \arrow[r] & Z'[1]\,. \end{tikzcd}$$

By (TR2) we obtain a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "t"] \arrow[d, "g"] & X' \arrow[r, "k"] \arrow[d, "f"] & Z' \arrow[r] \arrow[d, "id_{Z'}"] & X[1] \arrow[d, "g[1]"] \\ Y \arrow[r, "s"] & Y' \arrow[r] & Z' \arrow[r] & Y[1] \end{tikzcd}$$

Here $t \in S$, since $Z' \in \mathcal{N}$. This gives the claim.

(FR3): In an additive category this means, that for a morphism $f : X \to Y$ we have:

$$sf = 0 \ \text{ for an } \ s \in S \quad \Leftrightarrow \quad ft = 0 \ \text{ for an } \ t \in S\,.$$

Let $sf = 0$. By definition we have a distinguished triangle

$$Z \xrightarrow{g} Y \xrightarrow{s} Y' \longrightarrow Z[1]$$

with $Z' \in \mathcal{N}$, and by assumption we have a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "①"] & 0 \arrow[d] \\ Y \arrow[r, "s"] & Y' . \end{tikzcd}$$

By (TR1), (TR3) and (TR2) this can be extended to a morphism of distinguished triangles

(∗)
$$\begin{tikzcd} X \arrow[r, "id_x"] \arrow[d, "h"] \arrow[dr, phantom, "②"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d] \\ Z'[-1] \arrow[r, "g"] & Y \arrow[r, "s"] & Z \arrow[r] & Z' . \end{tikzcd}$$

By (TR1) and (TR2) the morphism h can be extended to a distinguished triangle

$$W \xrightarrow{t} X \xrightarrow{h} Z'[-1] \longrightarrow W[1] .$$

Here $t \in S$, since $Z'[-1] \in \mathcal{N}$. As above, the commutative diagram ($gh = f$)

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, phantom, "③"] & Z'[-1] \arrow[d, "g"] \\ Y \arrow[r, "id_Y"] & Y \end{tikzcd}$$

can be embedded into a morphism of distinguished triangles

(∗∗)
$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "④"] & X \arrow[r, "h"] \arrow[d, "f"] \arrow[dr, phantom, "③"] & Z'[-1] \arrow[r] \arrow[d, "g"] & W[1] \arrow[d] \\ 0 \arrow[r] & Y \arrow[r, "id_Y"] & Y \arrow[r] & 0 \end{tikzcd}$$

This implies $ft = 0$, where $t \in S$.

The converse direction starts with ④, obtains (∗∗) from it, since $t \in S$, and from this the commutativity of ③. This is equivalent to $gh = f$, hence to ②, and hence to (∗), which gives ①.

(FR4): If $s \in S$, we have a distinguished triangle

(∗ ∗ ∗)
$$X \xrightarrow{s} Y \xrightarrow{f} Z \xrightarrow{g} X[1]$$

with $Z \in \mathcal{N}$. By (TR2) we have distinguished triangles

$$Y \xrightarrow{f} Z \xrightarrow{g} X[1] \xrightarrow{-Ts} Y[1]$$

$$Z \xrightarrow{g} X[1] \xrightarrow{-Ts} Y[1] \xrightarrow{-Tf} Z[1]$$

$$X[1] \xrightarrow{-Ts} Y[1] \xrightarrow{-Tf} Z[1] \xrightarrow{-Tg} X[2]\,.$$

Since $Z[1] \in \mathcal{N}$, we get $-Ts \in S$. By Lemma 1.4 and $0 \in \mathcal{N}$ we have $-id_{X[1]} \in S$ and we conclude that $(-Ts) \circ (-id_{X[1]}) = Ts \in S$ by (FR1) proved above. By going back, we get the converse direction.

(FR5): Consider a diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] & X[1] \arrow[d, "{r[1]}"] \\
X' \arrow[r, "f'"] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

with $r, s \in S$, where the rows are distinguished triangles and the first square is commutative. By assumption we have distinguished triangles

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & Z_1 \arrow[r] & X[1] \\
Y \arrow[r, "s"] & Y' \arrow[r] & Z_2 \arrow[r] & Y[1]
\end{tikzcd}$$

with $Z_1, Z_2 \in \mathcal{N}$, which are connected by the morphisms f and f'. By Lemma 1.15 (3×3 lemma), the diagram of solid arrows below can be completed by the dotted arrows

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"] & Y \arrow[r] \arrow[d, "s"] & Z \arrow[r] \arrow[d, dotted, "t"] & X[1] \arrow[d] \\
X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, dotted] & X'[1] \arrow[d] \\
Z_1 \arrow[r, dotted] \arrow[d] & Z_2 \arrow[r, dotted] \arrow[d] & Z_3 \arrow[r, dotted] \arrow[d, dotted] & Z_1[1] \arrow[d] \\
X[1] \arrow[r, "{f[1]}"] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2]
\end{tikzcd}$$

such that all rows and all columns are distinguished triangles. By assumption we have $Z_1, Z_2 \in \mathcal{N}$, and hence $Z_3 \in \mathcal{N}$ by axiom N3 of definition 2.9. Hence $t \in S(\mathcal{N})$ by definition.

3 t-structures

Definition 3.1 Let $\mathcal{D}$ be a triangulated category.

(a) A t-structure on $\mathcal{D}$ consists of two strictly full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ in $\mathcal{D}$, such that with the definition $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ we have

(i) $Hom(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.

(ii) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$, and $\mathcal{D}^{\geq 1} \subseteq \mathcal{D}^{\geq 0}$.

(iii) For every object X in $\mathcal{D}$ there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with A in $\mathcal{D}^{\leq 0}$ and B in $\mathcal{D}^{\geq 1}$.

(b) The heart of the t-structure is the full subcategory

$$\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}\,.$$

Example 3.2 Let $\mathcal{A}$ be an abelian category. The natural t-structures on its derived category $D(\mathcal{A}) = \mathcal{K}(\mathcal{A})/\mathcal{K}^{\emptyset}(\mathcal{A}) = \mathcal{K}(\mathcal{A})[Quis^{-1}]$ is given by

$$\mathcal{D}(\mathcal{A})^{\leq n} = \text{ complexes } C^{\cdot} \text{ with } H^i(C^{\cdot}) = 0 \text{ for } i > n\,.$$

$$\mathcal{D}(\mathcal{A})^{\geq n} := \text{ complexes } B^{\cdot} \text{ with } H^i(B^{\cdot}) = 0 \text{ for } i < n$$

This is in fact a t-structure:

(iii): For a complex $X^{\cdot}$ in $\mathcal{D}(\mathcal{A})$ define the short exact sequence

$$0 \longrightarrow \tau_{\leq 0}X^{\cdot} \longrightarrow X^{\cdot} \longrightarrow \tau_{\geq 1}X^{\cdot} \longrightarrow 0$$

by

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & 0 \arrow[r] & X^2 \arrow[r, "id"] & X^2 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & X^1 \arrow[r, "id"] \arrow[u] & X^1 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] \arrow[u] & \ker d^0 \arrow[r] \arrow[u] & X^0 \arrow[r] \arrow[u] & im\, d^0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] \arrow[u] & X^{-1} \arrow[r, "id"] \arrow[u] & X^{-1} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] \arrow[u] & X^{-2} \arrow[r, "id"] \arrow[u] & X^{-2} \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u] \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{tikzcd}$$

Then one has $H^i(\tau_{\leq 0}X^\cdot) = 0$ for $i > 0$, so that $\tau_{\leq 0}X^\cdot \in \mathcal{D}^{\leq 0}$, and $H^i(\tau_{\geq 1}X^\cdot) = 0$ for $i \leq 0$, so that $\tau_{\geq 1}X^\cdot \in \mathcal{D}^{\geq 1}$.

(i) If we have $C^\cdot$ with $H^i(C^\cdot) = 0$ for $i > 0$ and $B^\cdot$ with $H^i(B^\cdot) = 0$ for $i \leq 0$ then we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(A^\cdot, B^\cdot) = 0\,.$$

In fact, a morphism in $\mathcal{D}(\mathcal{A})$ is represented by a diagram

$$\begin{tikzcd} X^\cdot \arrow[d, "s"'] \arrow[dr, "f"] & \\ A^\cdot & B^\cdot \end{tikzcd}$$

with a quasi-isomorphism s and a morphism f in $C(\mathcal{A})$. By assumption,

$$t : \tau_{\leq 0}X^\cdot \longrightarrow X^\cdot$$

is a quasi-isomorphism, and so is

$$\tau : B^\cdot \longrightarrow B^\cdot/\tau_{\leq 0}B^\cdot\,.$$

Then obviously $rft = 0$ in $\mathcal{C}(\mathcal{A})$, and this implies $f = 0$ in $\mathcal{D}(\mathcal{A})$, and hence $fs^{-1} = 0$.

(ii) is obvious.

The heart of this t-structure is given by the complexes which are concentrated in degree zero, and is equivalent to the category $\mathcal{A}$ itself, and hence an abelian category.

Remark 3.3 If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure, then the shift by n, $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$, is a t-structure as well.

Let $\mathcal{D}$ be t-category (i.e., a triangulated category with a t-structure).

Proposition 3.4 (i) The inclusion $i_{\leq n} : \mathcal{D}^{\leq n} \hookrightarrow \mathcal{D}$ has a right adjoint $\tau_{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$.

(ii) The inclusion $j_{\geq n} : \mathcal{D}^{\geq n} \hookrightarrow \mathcal{D}$ has a left adjoint $\tau_{\geq n} : \mathcal{D} \to \mathcal{D}^{\geq n}$.

(iii) For each X in $\mathcal{D}$ there is a unique morphism $d \in \mathrm{Hom}^1(\tau_{\geq 1}X, \tau_{\leq 0}X)$ such that

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \xrightarrow{d} \tau_{\leq 0}X[1]$$

is a distinguished triangle. Up to unique isomorphism this is the only distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

We use the following lemma.

Lemma 3.5 Consider a diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, dotted, "f"] \arrow[dr, phantom, "1"] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "2"] & Z \arrow[r, "d"] \arrow[d, dotted, "h"] & X[1] \arrow[d, dotted, "f[1]"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "d'"] & X[1] \end{tikzcd}$$

where the rows are distinguished triangles and $g : Y \to Y'$ is a morphism. Then the following conditions are equivalent

(a) $v'gu = 0$.

(b) There exists an f as indicated making the square ① commutative.

(b') There exists an h as indicated making the square ② commutative.

(c) There exists a morphism (f, g, h) of triangles.

If these conditions hold, and if $\mathrm{Hom}^{-1}(X, Z') = 0$, then the morphism f of (b) (resp. h of (b')) is unique.

Proof 1) The exactness of the sequence

$$\begin{array}{ccccccc} \mathrm{Hom}^{-1}(X, Z') & \longrightarrow & \mathrm{Hom}(X, X') & \xrightarrow{(u')_*} & \mathrm{Hom}(X, Y') & \xrightarrow{(v')_*} & \mathrm{Hom}(X, Z') \\ & & f & \longmapsto & u'f, \; gu & \longmapsto & v'gu \end{array}$$

shows that (a) $\Leftrightarrow$ (b), and the unicity of f, if $\mathrm{Hom}^{-1}(X, Z') = 0$.

The implication (b) $\Rightarrow$ (c) follows from (TR3): If f satisfies (b), then there exists h such that (f, g, h) is a morphism of triangles. The converse is trivial.

2) The exactness of the sequence

$$\begin{array}{ccccccc} \mathrm{Hom}(X[1], Z') & \longrightarrow & \mathrm{Hom}(Z, Z') & \xrightarrow{v^*} & \mathrm{Hom}(Y, Z') & \xrightarrow{u^*} & \mathrm{Hom}(X, Z') \\ & & h & \longmapsto & hv, \; v'g & \longmapsto & v'gu \end{array}$$

shows that (a) $\Leftrightarrow$ (b'), and the unicity of h if $\mathrm{Hom}^{-1}(X, Z') = 0$.

Proof of 3.4 By translation it suffices to show (i) for $i_{\leq 0} : \mathcal{D}^{\leq 0} \to \mathcal{D}$.

Let

$$A \xrightarrow{f} X \xrightarrow{g} B \xrightarrow{d} A[1]$$

be as in 5.1 (iii), with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$. Then, for every T in $\mathcal{D}^{\leq 0}$ we have an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(T, B[-1]) \arrow[r] \arrow[d, equal, "(1)"] & \mathrm{Hom}(T, A) \arrow[r, "f_*"] & \mathrm{Hom}(T, X) \arrow[r] & \mathrm{Hom}(T, B) \arrow[d, equal, "(2)"] \\ 0 & & & 0 \end{tikzcd}$$

Here (1) holds, since $B[-1] \in \mathcal{D}^{\geq 2} \subseteq \mathcal{D}^{\geq 1}$; and (2) holds, since $B \subseteq \mathcal{D}^{\geq 1}$. Hence f_* is an isomorphism, and we can set $\tau_{\leq 0} X := A$, so that we get an adjunction isomorphism

$$\mathrm{Hom}(T, \tau_{\leq 0} X) \xrightarrow{\sim} \mathrm{Hom}(i_{\leq 0} T, X)$$

for all T in $\mathcal{D}^{\leq 0}$. This isomorphism is functorial in T and X: The functoriality in T is obvious, and if $f : X \to X'$ is a morphism, then we get a diagram

$$\begin{tikzcd} \tau_{\leq 0}X \arrow[r, "i"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] & \tau_{\leq 0}X[1] \\ \tau_{\leq 0}X' \arrow[r] & X' \arrow[r, "j"] & B & \end{tikzcd}$$

in which $jfi = 0$, since $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$. By Lemma 3.5, and the fact that $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$, we get a unique morphism of distinguished triangles

$$\begin{tikzcd} \tau_{\leq 0}X \arrow[r] \arrow[d, "\tau_{\leq 0}f"] & X \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & \tau_{\leq 0}X[1] \arrow[d, "\tau_{\leq 0}f[1]"] \\ \tau_{\leq 0}X' \arrow[r] & X' \arrow[r] & B \arrow[r] & \tau_{\leq 0}X'[1]\,. \end{tikzcd}$$

(ii) For $T \in \mathcal{D}^{\geq 1}$ we have an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(\tau_{\leq 0}X, T) \arrow[r] \arrow[d, equal, "(1)"] & \mathrm{Hom}(X, T) \arrow[r] & \mathrm{Hom}(B, T) \arrow[r] & \mathrm{Hom}(\tau_{\leq 0}X[1], T) \arrow[d, equal, "(2)"] \\ 0 & & & 0 \end{tikzcd}$$

Here (1) holds since $\tau_{\leq 0}X \in \mathcal{D}^{\leq 0}$ and $T \in \mathcal{D}^{\geq 1}$, and (2) holds, since $\tau_{\leq 0}X[1] \in \mathcal{D}^{\leq 0}[1] = \mathcal{D}^{\leq -1}$ and $T \in \mathcal{D}^{\geq 1}$. Hence we get a functorial isomorphism

$$\mathrm{Hom}(X, j_{\geq 1}T) \xrightarrow{\sim} \mathrm{Hom}(B, T)\,,$$

so that we can define $\tau_{\geq 1}X := B$. The functoriality of $\tau_{\geq 1}$ is shown as above.

Corollary 3.6 Let $X \xrightarrow{u} Y \longrightarrow Z \xrightarrow{d} X[1]$ be a distinguished triangle. If $\mathrm{Hom}^{-1}(X, Z) = 0$, then the following holds:

(i) The cone of u is unique up to unique isomorphism.

(ii) d is the unique morphism $Z \to X[1]$ making the above triangle distinguished.

Proof (i) For a diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dotted, "h"] & X[1] \arrow[d, "f[1]"] \\ X \arrow[r, "u"] & Y \arrow[r] & Z' \arrow[r] & X[1] \end{tikzcd}$$

with isomorphisms f, g the morphism h is uniquely determined by Lemma 3.5.

(ii) Apply Lemma 3.5 to

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "d"] \arrow[d, dotted, "h"] & X[1] \arrow[d, equal] \\ X \arrow[r, "u"] & Y \arrow[r, "v"] & Z' \arrow[r, "x"] & X[1]\,. \end{tikzcd}$$

Necessarily we have $h = id_Z$, so that $x = d$.

Remark 3.7 (i) The distinguished triangle

$$\tau_{\leq 0}X \longrightarrow X \longrightarrow \tau_{\geq 1}X \longrightarrow \tau_{\leq 0}X[1] \tag{3.7.1}$$

shows that the following conditions are equivalent:

(a) $\tau_{\leq 0}X = 0$

(a') $\mathrm{Hom}(T, X) = 0$ for every $T \in \mathcal{D}^{\leq 0}$.

(b) $X \xrightarrow{\sim} \tau_{\geq 1}X$ is an isomorphism.

(b') $X \in \mathcal{D}^{\geq 1}$.

The equivalence between (a') and (b') can be stated as follows

(c) $\mathcal{D}^{\geq 1}$ is the right orthogonal of $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\leq 0}$ is the left orthogonal of $\mathcal{D}^{\geq 1}$.

(ii) Call a subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ stable by extension, if for any distinguished triangle (X, Y, Z) in $\mathcal{D}$ with $X, Z \in \mathcal{D}'$ we also have $Y \in \mathcal{D}'$.

Then $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are stable by extension. In fact, for a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

with $X, Z \in \mathcal{D}^{\geq 1}$ and any $T \in \mathcal{D}^{\leq 0}$ we have an exact sequence

$$Hom(T, X) \longrightarrow \mathrm{Hom}(T, Y) \longrightarrow \mathrm{Hom}(T, Z)\,,$$

where $\mathrm{Hom}(T, X) = 0 = \mathrm{Hom}(T, Z)$ by (i), so that $\mathrm{Hom}(T, Y) = 0$ and hence $Y \in \mathcal{D}^{\geq 1}$. The case of $\mathcal{D}^{\leq 0}$ is similar (dual). In particular, $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are stable under finite direct sums.

(iii) For $m \leq n$ we have $\mathcal{D}^{\leq m} \subset \mathcal{D}^{\leq n}$, and therefore a commutative diagram for each X in $\mathcal{D}$

$$\begin{tikzcd}
\tau_{\leq m}X \arrow[rr] \arrow[dr, "i_m"'] & & \tau_{\leq n}X \\
& X \arrow[ur, "i_n"'] &
\end{tikzcd}$$

which identifies $\tau_{\leq m}X$ with $\tau_{\leq m}\tau_{\leq n}X$, or more precisely with

$$\tau_{\leq m}|_{\mathcal{D}^{\leq n}}\tau_{\leq n}X$$

in the commutative diagram

$$\begin{tikzcd}
& \mathcal{D} \\
\mathcal{D} \arrow[r, "\tau_{\leq n}"] \arrow[dr, "\tau_{\leq m}"'] & \mathcal{D}^{\leq n} \arrow[u, hook, "i_{\leq n}"'] \\
& \mathcal{D}^{\leq m} \arrow[u, hook, "i_{m,n}"'] \arrow[uu, bend right=60, "i_m"']
\end{tikzcd}$$

Dually, we have a morphism $\tau_{\geq n}X \to \tau_{\geq m}X$, identifying $\tau_{\geq n}\tau_{\geq m}X$ with $\tau_{\geq n}X$.

(iv) Let $a \leq b$. Then the adjunction morphisms for 3.4 (i) and (ii) induce a commutative diagram

(∗)
$$\begin{tikzcd}
\tau_{\leq b}X \arrow[r, "v"] \arrow[d] & X \arrow[r, "v'"] & \tau_{\geq a}X \arrow[d] \\
\tau_{\geq a}\tau_{\leq b}X \arrow[rr, "\varphi"] & & \tau_{\leq b}\tau_{\geq a}X
\end{tikzcd}$$

with a uniquely determined isomorphism φ. Here we have omitted the canonical inclusions $i_b : \mathcal{D}^{\leq b} \hookrightarrow \mathcal{D}$ and $j_a : \mathcal{D}^{\geq a} \hookrightarrow \mathcal{D}$.

Proof: By the adjunction $\mathrm{Hom}(\tau_{\leq b}X, j_a\tau_{\geq a}X) \cong \mathrm{Hom}(\tau_{\geq a}\tau_{\leq b}X, \tau_{\geq a}X)$, the upper morphism induces a unique morphism

$$\varphi_1 : \tau_{\geq a}\tau_{\leq b}X \longrightarrow \tau_{\geq a}X$$

making

$$\begin{tikzcd}
\tau_{\leq b}X \arrow[r] \arrow[d] & \tau_{\geq a}X \\
\tau_{\geq a}\tau_{\leq b}X \arrow[ur, "\varphi_1"'] &
\end{tikzcd}$$

commutative. On the other hand, by adjunction again, there is a unique morphism

$$\varphi : \tau_{\geq a}\tau_{\leq b}X \longrightarrow \tau_{\leq b}\tau_{\geq a}X$$

making the diagram

$$\begin{tikzcd}
& \tau_{\geq a}X \arrow[d] \\
\tau_{\geq a}\tau_{\leq b}X \arrow[ur, "\varphi_1"] \arrow[r, "\varphi"] & \tau_{\leq b}\tau_{\geq a}X
\end{tikzcd}$$

commutative, and hence the diagram (∗) commutative.

To show that φ is an isomorphism, we apply (TR4) to the composition

$$\tau_{<a}X \xrightarrow{u} \tau_{\leq b}X \xrightarrow{v} X\,.$$

We get a commutative diagram

(∗)
$$\begin{tikzcd}
\tau_{<a}X \arrow[r, "u"] \arrow[d, "id"] \arrow[dr, phantom, "1"] & \tau_{\leq b}X \arrow[r, "\alpha"] \arrow[d, "v"] & Y \arrow[r] \arrow[d, dotted, "\beta"] & {} \\
\tau_{<a}X \arrow[r, "w=vu"] \arrow[d, "u"] & X \arrow[r, "v'"] \arrow[d, "id"] & \tau_{\geq a}X \arrow[r] \arrow[d, dotted] & {} \\
\tau_{\leq b}X \arrow[r, "v"] & X \arrow[r] & \tau_{>b}X \arrow[r] & {}
\end{tikzcd}$$

where the rows are distinguished triangles, and where the column with Y is a distinguished triangle and makes the diagram commutative. Here the square 1 is commutative by 3.7 (iii), and v and v' are as in $(*)$ above.

Since $\tau_{<a}X \cong \tau_{<a}\tau_{\leq b}X$, we have a morphism of distinguished triangles by (TR3)

$$\begin{tikzcd}
\tau_{<a}\tau_{\leq b}X \arrow[r] \arrow[d, equal, "\wr"] & \tau_{\leq b}X \arrow[r] \arrow[d, equal] & \tau_{\geq a}\tau_{\leq b}X \arrow[r] \arrow[d, "\varphi_1"] & \tau_{<a}\tau_{\leq b}X[1] \arrow[d, equal, "\wr"] \\
\tau_{<a}X \arrow[r] & \tau_{\leq b}X \arrow[r] & Y \arrow[r] & \tau_{<a}X[1]
\end{tikzcd}$$

and hence an isomorphism $\tau_{\geq a}\tau_{\leq b}X \xrightarrow[\sim]{\varphi_1} Y$.

Similarly, since $\tau_{>b}X = \tau_{>b}\tau_{\geq a}X$, we have an isomorphism of distinguished triangles

$$\begin{tikzcd}
\tau_{\leq b}\tau_{\geq a}X \arrow[r] & \tau_{\geq a}X \arrow[r] & \tau_{>b}\tau_{\geq a}X \arrow[r] & \tau_{\leq b}\tau_{\leq a}X[1] \arrow[d, "{\varphi_2[1]}"] \\
Y \arrow[r] \arrow[u, "\wr"', "\varphi_2"] & \tau_{\geq a}X \arrow[r] \arrow[u, equal] & \tau_{>b}X \arrow[r] \arrow[u, equal] & Y[1] \, ,
\end{tikzcd}$$

and hence an isomorphism $Y \xrightarrow[\sim]{\varphi_2} \tau_{\leq b}\tau_{\leq a}X$.

Now, in the diagram $(*)$ we see that $Y \in \mathcal{D}^{\geq a}$ and $Y \in \mathcal{D}^{\leq b}$. Hence the morphism α factors through $\tau_{\geq a}\tau_{\leq b}X$, and the morphism β factors through $\tau_{\leq b}\tau_{\geq a}X$. This gives a commutative diagram

$$\begin{tikzcd}
\tau_{\leq b}X \arrow[r, "v"] \arrow[d] \arrow[dr, "\alpha"] & X \arrow[r, "v'"] & \tau_{\geq a}X \arrow[d] \\
\tau_{\geq a}\tau_{\leq b}X \arrow[r, "\varphi_1", "\sim"'] & Y \arrow[r, "\varphi_2", "\sim"'] \arrow[ur, "\beta"] & \tau_{\leq b}\tau_{\geq a}X \, ,
\end{tikzcd}$$

so that the unique morphism at the bottom is an isomorphism.

Now we show

Theorem 3.8 (a) The heart

$$\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$

of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ (see 3.1 (b)) is naturally an abelian category.

(b) The functor

$$\begin{array}{rccl} H^0 : & \mathcal{D} & \to & \mathcal{C} \\ & X & \mapsto & H^0(X) = \tau_{\geq 0}\tau_{\leq 0}X \end{array}$$

is a cohomological functor.

Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{C}$ be a full subcategory. Assume that the following holds.

(3.8.1) $$\mathrm{Hom}^i(X,Y) := \mathrm{Hom}(X, Y[i])$$

is zero for $i < 0$ and $X, Y \in \mathcal{C}$.

Example 3.9 Let $\mathcal{D} = \mathcal{D}(\mathcal{A})$ for an abelian category $\mathcal{A}$, and let $\mathcal{C}$ be the image of the functor

$$\begin{array}{rcl} \mathcal{A} & \to & \mathcal{D} \\ A & \mapsto & A \text{ placed in degeree zero.} \end{array}$$

Proposition 3.10 Let $f : X \to Y$ be in $\mathcal{C}$ and extend it to a distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow S \longrightarrow X[1]\,.$$

Suppose that we have a commutative diagram

(3.10.1)
$$\begin{tikzcd}
 & & K[2] & \\
 & & C \arrow[u] & \\
X \arrow[r, "f"] & Y \arrow[ur, "\beta"] \arrow[r, "g"] & S \arrow[u] \arrow[r] & X[1] \\
 & & K[1] \arrow[u] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

where K and C are in $\mathcal{C}$, and the column is a distinguished triangle.
Then $\beta : Y \to C$ is a cokernel of f, and $\alpha[-1] : K \to X$ is a kernel of f.

Proof For each object Z in $\mathcal{C}$ we have exact sequences

$$\begin{tikzcd}
\mathrm{Hom}(X[1], Z) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(S, Z) \arrow[r, "g^*"] & \mathrm{Hom}(Y, Z) \arrow[r, "f^*"] & \mathrm{Hom}(X, Z) \\
0 & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(K[2], Z) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(C, Z) \arrow[r] & \mathrm{Hom}(S, Z) \arrow[r] & \mathrm{Hom}(K[1], Z)\,, \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

since $Z, X, K \in \mathcal{C}$. Taking this together, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, Z) \xrightarrow{\beta^*} \mathrm{Hom}(Y, Z) \xrightarrow{f^*} \mathrm{Hom}(X, Z)$$

for each Z in $\mathcal{C}$, which shows the first claim.

On the other hand, we get exact sequences for each $T \in \mathcal{C}$

$$\begin{tikzcd} \mathrm{Hom}(T, Y[-1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(T, S[-1]) \arrow[r] & \mathrm{Hom}(T, X) \arrow[r, "f^*"] & \mathrm{Hom}(T, Y) \\ 0 & & & \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(T, C[-2]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(T, K) \arrow[r] & \mathrm{Hom}(T, S[-1]) \arrow[r] & \mathrm{Hom}(T, C[-1]) \arrow[d, equal] \, , \\ 0 & & & 0 \end{tikzcd}$$

since $T, C, K \in \mathcal{C}$. Taking this together, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(T, K) \longrightarrow \mathrm{Hom}(T, X) \xrightarrow{f_*} \mathrm{Hom}(T, Y)$$

for each T in $\mathcal{C}$, which shows the second claim.

Example 3.11 In the case of example 3.9, where we consider $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, by sending $A \in \mathcal{A}$ to the complex $\ldots \to 0 \to A \to 0 \to \ldots$ where A is placed in degree zero, the cone S of $f : X \to Y$ is the complex $\ldots \to 0 \to X \xrightarrow{f} Y \to \ldots$ with X in degree -1 and Y in degree 0. It has the subcomplex $\ker(f)[1] = H^{-1}(S)[1]$ and the quotient $\mathrm{coker}(f) = X/\ker(f) \xrightarrow{\sim} H^0(S)$ in degree zero. One gets a diagram as in Proposition 3.10.

Definition 3.12 Let $\mathcal{C} \subset \mathcal{D}$ be as in (3.8.1). Call a morphism $f : X \to Y$ in $\mathcal{C}$ admissible (or $\mathcal{C}$-admissible) if it is the basis of a diagram (3.10.1).

Remark 3.13 (a) If f is admissible and a monomorphism, then by 3.10 one has $K = 0$ and hence $S \xrightarrow{\sim} C$, so that S is in $\mathcal{C}$, and is a cokernel of f in $\mathcal{C}$ by 3.10. Moreover (3.10.1) reduces to the distinguished triangle

$$X \xrightarrow{f} Y \longrightarrow C \longrightarrow$$

with X, Y, C in $\mathcal{C}$.

(b) If f is admissible and an epimorphism, then $C = 0$, and hence $K[1] \xrightarrow{\sim} S$, and (3.10.1) reduces to a distinguished triangle

$$K \longrightarrow X \longrightarrow Y \longrightarrow$$

with $K, X, Y \in \mathcal{C}$.

(c) Conversely, for every distinguished triangle

$$(*) \qquad X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{d} X[1]$$

with X, Y, Z in $\mathcal{C}$, f and g are admissible, f is a kernel of g and g is a cokernel of f. By the assumption that $\mathrm{Hom}(X, Y[i])$ is zero for $i < 0$, so that $\mathrm{Hom}^{-1}(X, Y) = 0$, d is uniquely determined by f and g, by Corollary 3.6 (ii).

In fact, by Lemma 1.9 we have an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(T, Z[-1]) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(T, X) \arrow[r, "f_*"] & \mathrm{Hom}(T, Y) \arrow[r, "g_*"] & \mathrm{Hom}(T, Z) \\ 0 \end{tikzcd}$$

for every T in $\mathcal{C}$, which means that f is a kernel of g. On the other hand, we have an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(Z[-1], V) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Z, V) \arrow[r, "g_*"] & \mathrm{Hom}(Y, V) \arrow[r, "f_*"] & \mathrm{Hom}(X, V) \\ 0 \end{tikzcd}$$

for every V in $\mathcal{C}$, which means that g is a cokernel of f.

By this, the distinguished triangle

$$(*) \qquad X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

shows that f is admissible. In fact, since g is a cokernel of f, in (3.10.1) $S \to C$ is an isomorphism, and (3.10.1) becomes $(*)$. On the other hand, since g is an epimorphism, the cokernel of g is zero, and the diagram (3.10.1) becomes

$$\begin{tikzcd} & & X[2] & \\ & & 0 \arrow[u] & \\ Y \arrow[r, "g"] & Z \arrow[r] \arrow[ur] & X[1] \arrow[r] \arrow[u] & Y[1] \\ & & X[1] \arrow[u, equal, "\wr"] \arrow[ur, "f[1]"'] & \end{tikzcd}$$

so that g is admissible.

The last claim is clear.

Definition 3.14 A sequence $X \to Y \to Z$ in $\mathcal{C}$ is called admissible, if it is obtained by a distinguished triangle in $\mathcal{D}$ by omitting the arrow $Z \xrightarrow{d} X[1]$.

Proposition 3.15 Assume that $\mathcal{C}$ is closed under finite direct sums in $\mathcal{D}$. Then the following conditions are equivalent:

(i) $\mathcal{C}$ is an abelian category, and the short exact sequences in $\mathcal{C}$ are admissible.

(ii) Every morphism in $\mathcal{C}$ is $\mathcal{C}$-admissible.

Proof (ii) $\Rightarrow$ (i): If $f : X \to Y$ is admissible, then by Proposition 3.10 it has a kernel and a cokernel. To show that $\mathcal{C}$ is an abelian category, we have to show that, in the diagram

$$\begin{tikzcd}
K \arrow[r, "{\alpha[-1]}"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[dd] & Y \arrow[r, "g"] & S \arrow[d, equal] \\
\ker(f) & & & \operatorname{coker}(f) \\
& coim(f) \arrow[r] \arrow[d, equal] & im(f) \arrow[uu] \arrow[d, equal] & \\
& \operatorname{coker}(\alpha[-1]) & \ker(g)\,, &
\end{tikzcd}$$

the natural morphism $coim(f) \to im(f)$ is an isomorphism.

But by (TR4) we obtain a commutative diagram with distinguished triangles as rows and third column:

$$\begin{tikzcd}
K \arrow[r] \arrow[d, equal] & S[-1] \arrow[r] \arrow[d] & C[-1] \arrow[r] \arrow[d, dotted] & K[1] \arrow[d, equal] \\
K \arrow[r, "{\alpha[-1]}"] \arrow[d] & X \arrow[r] \arrow[d, equal] & I \arrow[r] \arrow[d, dotted] & K[1] \arrow[d] \\
S[-1] \arrow[r] & X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[r] \arrow[d, dotted, "\beta"] & S \\
& S \arrow[r] & C &
\end{tikzcd}$$

Since $\alpha[-1]$ is a monomorphism in $\mathcal{C}$, and admissible by assumption, we have that $I \in \mathcal{C}$, and is a cokernel of $\alpha[-1]$ by 3.13 (a), so that canonically $\operatorname{coker}(\alpha[-1]) \xrightarrow{\sim} I$. On the other hand, the exact sequence

$$\begin{tikzcd}
\operatorname{Hom}(T, C[-1]) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(T, I) \arrow[r] & \operatorname{Hom}(T, Y) \arrow[r, "\beta_*"] & \operatorname{Hom}(T, C) \\
0 & & &
\end{tikzcd}$$

for any $T \in \mathcal{C}$ shows that canonically $I \xrightarrow{\sim} \ker(\beta)$.

(i) $\Rightarrow$ (ii): Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The two short exact sequences in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & X \arrow[r] & I \arrow[r] \arrow[d, equal] & 0 \\
& \ker(f) & & im(f) &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, equal] & Y \arrow[r] & C \arrow[r] \arrow[d, equal] & 0 \\
& im(f) & & \operatorname{coker}(f) &
\end{tikzcd}$$

give, by (TR4), a commutative diagram of distinguished triangles

$$\begin{tikzcd}
 & I[1] \arrow[r] & K[2] & \\
 & C \arrow[u] \arrow[r, equal] & C \arrow[u, dotted] & \\
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[u] \arrow[r] \arrow[ur, dotted, "\beta"] & S \arrow[r] \arrow[u, dotted] & X[1] \arrow[d, equal] \\
X \arrow[r] & I \arrow[r] \arrow[u] & K[1] \arrow[r] \arrow[u, dotted] \arrow[ur, dotted, "\alpha"'] & X[1]\,,
\end{tikzcd}$$

so that f is admissible.

Definition 3.16 A full subcategory $\mathcal{C}$ of a triangulated category is called admissible, if it satisfies (3.8.1) and the equivalent properties of Proposition 3.15.

Now we prove Theorem 3.8. Let $\mathcal{D}$ be a triangulated category, let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure on $\mathcal{D}$, and let $\mathcal{C} = \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0}$ be the heart of the t-structure. Then we have

(3.8.1) $$\mathrm{Hom}(X, Y[i]) = 0 \quad \text{for} \quad i < 0 \quad \text{and} \quad X, Y \in \mathcal{C}\,.$$

In fact, we have $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 0}$ so that the claim follows from 3.1(i).

Therefore we can apply all results in this chapter.

By Proposition 3.15 ist suffices to show that every morphism $f : X \to Y$ in $\mathcal{C}$ is admissible. Consider a diagram

$$\begin{tikzcd}
 & & \tau_{\leq -1}S[1] & \\
 & & \tau_{\geq 0}S \arrow[u] & \\
X \arrow[r, "f"] & Y \arrow[r] & S \arrow[r] \arrow[u] & X[1] \\
 & & \tau_{\leq -1}S \arrow[u] &
\end{tikzcd}$$

where the row and the column are distinguished triangles. By assumption, Y and $X[1]$ are in $\mathcal{D}^{\geq -1} \cap \mathcal{D}^{\leq 0}$, and by 3.7 (ii) this also holds for the extension S. Then in the vertical distinguished triangle we must have $\tau_{\leq -1}S \in \mathcal{D}^{\geq -1} \cap \mathcal{D}^{\leq -1} = \mathcal{C}[1]$, and $\tau_{\geq 0}S \in \mathcal{D}^{\geq 0} \cap \mathcal{D}^{\leq 0} = \mathcal{C}$ (compare Proposition 3.7 (iv)). Hence f is admissible.

3.17 Finally we show that for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

the sequence $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact, where $H^0(X) = \tau_{\geq o}\tau_{\leq 0}X$.

Case 1 If X, Y and Z are in $\mathcal{D}^{\leq 0}$, then the sequence $H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.

Proof For any U in $\mathcal{D}^{\leq 0}$ one has $H^0(U) = \tau_{\geq 0}U$, and for any V in $\mathcal{D}^{\geq 0}$ one has $H^0(V) = \tau_{\leq 0}V$, and isomorphisms

$$\mathrm{Hom}(H^0(U), H^0(V)) \cong \mathrm{Hom}(U, H^0(V)) \cong \mathrm{Hom}(U, V)\,. \tag{1}$$

For $T \in \mathcal{C}$ (and hence in $\mathcal{D}^{\geq 0}$), the long exact Hom-sequence gives an exact sequence

$$0 = \mathrm{Hom}(X[1], T) \longrightarrow \mathrm{Hom}(Z, T) \longrightarrow \mathrm{Hom}(Y, T) \longrightarrow \mathrm{Hom}(X, T)\,,$$

where $\mathrm{Hom}(X[1], T) \cong \mathrm{Hom}(X, T[-1]) = 0$, because $X \in \mathcal{D}^{\leq 0}$ and $T[-1] \in \mathcal{D}^{\geq 1}$. By (1) this gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(H^0(Z), T) \longrightarrow \mathrm{Hom}(H^0(Y), T) \longrightarrow \mathrm{Hom}(H^0(X), T)\,.$$

Since this holds for all $T \in \mathcal{D}^{\geq 0}$, the wanted exactness follows.

Case 2 If X is in $\mathcal{D}^{\leq 0}$, the the sequence $H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.

Proof For all T in $\mathcal{D}^{\geq 1}$ the long exact Hom-sequence gives an isomorphism

$$0 = \mathrm{Hom}(X[1], T) \longrightarrow \mathrm{Hom}(Z, T) \xrightarrow{\sim} \mathrm{Hom}(Y, T) \longrightarrow \mathrm{Hom}(X, T) = 0$$

since $X \in \mathcal{D}^{\leq 0}$, $X[1] \in \mathcal{D}^{\leq -1}$, and $T \in \mathcal{D}^{\geq 1}$, and hence an isomorphism

$$\tau_{\geq 1}Y \xrightarrow{\sim} \tau_{\geq 1}Z\,.$$

By applying TR4 to the composition $X \to \tau_{\leq 0}Y \to Y$ we obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, equal] & \tau_{\leq 0}Y \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d, dotted] & X[1] \arrow[d, equal] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\beta"] & X[1] \\
\tau_{\leq 0}Y \arrow[r] & Y \arrow[r] & \tau_{\geq 1}Y &
\end{tikzcd}$$

Since the morphism $Y \to Z$ induces an isomorphism $\tau_{\geq 1}Y \xrightarrow{\sim} \tau_{\geq 1}Z$, we get a canonical isomorphism $Z' \cong \tau_{\leq 0}Z$ and hence a distinguished triangle

$$X \longrightarrow \tau_{\leq 0}Y \longrightarrow \tau_{\leq 0}Z \longrightarrow\,,$$

which can be treated by Case 1.

Case 3 If Z lies in $\mathcal{D}^{\geq 0}$, then

$$0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$$

is exact. This case is dual to Case 2.

The general Case: Let the distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

be arbitrary. By (TR4) we get a commutative diagram

(1)
$$\begin{tikzcd}
\tau_{\leq 0}X \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \tau_{\geq 1}X \arrow[r] \arrow[d, dotted] & {} \\
\tau_{\leq 0}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d, dotted] & {} \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & {}
\end{tikzcd}$$

Case 2 and triangle (1) imply an exact sequence

$$\begin{tikzcd}
H^0(X) \arrow[r] \arrow[d, equal] & H^0(Y) \arrow[r] & H^0(U) \arrow[r] & 0 \\
H^0(\tau_{\leq 0}X) & & &
\end{tikzcd}$$

Case 3 and the triangle
$$U \longrightarrow Z \longrightarrow (\tau_{\geq 1}X)[1]$$
imply that $0 \to H^0(U) \to H^0(Z)$ is exact. Hence $H^0(X) \to H^0(Y) \to H^0(X)$ is exact.

Remark 3.18 A t-structure is called non-degenerate, if $\bigcap\limits_{n\geq 0} \mathcal{D}^{\leq n} = 0 = \bigcap\limits_{n\geq 0} \mathcal{D}^{\geq n}$. In this case the family of functors ${}^tH^i$ is conservative and we have

$$\begin{array}{lcl}
X \in \mathcal{D}^{\geq 0} & \Leftrightarrow & {}^tH^i(X) = 0 \text{ for all } i < 0 \\
X \in \mathcal{D}^{\leq 0} & \Leftrightarrow & {}^tH^i(X) = 0 \text{ for all } i > 0\,.
\end{array}$$

4 Derived functors

Definition 4.1 For triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ let $\mathrm{Hom}_{ex}(\mathcal{D}, \mathcal{D}')$ be the group of exact functors $F : \mathcal{D} \to \mathcal{D}'$.

Definition 4.2 Let $F, G : \mathcal{D} \to \mathcal{D}'$ be exact functors between triangulated categories. A morphism of exact functors $\eta : F \to G$ is a morphism of functors such that for every object X of $\mathcal{D}$ the diagram

$$\begin{tikzcd} \eta_{TX} : & F(TX) \arrow[r] \arrow[d, "\wr"] & G(TX) \arrow[d, "\wr"] \\ T'(\eta_X) : & T'F(X) \arrow[r] & T'G(X) \end{tikzcd}$$

commutes.

Definition 4.3 Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and let

$$F : K^?(\mathcal{A}) \to K^{?'}(\mathcal{B})$$

be an exact functor.

(a) Consider the canonical diagram

$$\begin{tikzcd} K^?(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & K^{?'}(\mathcal{B}) \arrow[d, "Q'"] \\ \mathcal{D}^?(\mathcal{A}) & \mathcal{D}^{?'}(\mathcal{B}) \end{tikzcd}$$

An exact functor $RF : \mathcal{D}^?(\mathcal{A}) \to \mathcal{D}^{?'}(\mathcal{B})$ is called right derivative of F, if it represents the functor

$$\begin{array}{rcl} \mathrm{Hom}_{ex}(\mathcal{D}^?(\mathcal{A}), \mathcal{D}^{?'}(\mathcal{B})) & \to & Ab \\ G & \mapsto & \mathrm{Hom}_{ex}(Q'F, GQ)\,. \end{array}$$

This means that there is a morphism of exact functors

$$\eta_0 : \qquad Q'F \longrightarrow RFQ$$

$$\begin{tikzcd} K^?(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"] & K^{?'}(\mathcal{B}) \arrow[d, "Q'"] \arrow[dl, Rightarrow, "\eta_0"'] \\ \mathcal{D}?(\mathcal{A}) \arrow[r, "RF"] & \mathcal{D}^{?'}(\mathcal{B})\,, \end{tikzcd}$$

and for each exact functor $\mathcal{D}^?(\mathcal{A}) \xrightarrow{G} \mathcal{D}^{?'}(B)$ and each morphism of functors $\eta : Q'F \to GQ$ there is a unique morphism of exact functors $\eta : RF \to G$ such that η equals the composition

$$Q'F \xrightarrow{\eta_0} RFQ \xrightarrow{\eta \circ Q} GQ\,.$$

(b) Dually one defines left derived functors via $\zeta : LFQ \Rightarrow Q'F$.

Theorem 4.4 Let $\mathcal{D}' \subset K^?(\mathcal{A})$ be a full triangulated subcategory such that the following holds:

(1) Each object X in $K^?(\mathcal{A})$ possesses a quasi-isomorphism

$$X \to X' \quad (\text{resp. } X' \to X)$$

with X' in $\mathcal{D}'$.

(2) If $X' \in \mathcal{D}'$ and is acyclic (i.e., in $\mathcal{D}' \cap K^{\emptyset}(\mathcal{A})$), then $F(X')$ is acyclic.

Then F has a right derivative (resp. left derivative), and with the notations of (1) one has $RF(X) = QF(X')$ (resp. $LF(X) = Q(X')$).

Proof (1) $\widetilde{\mathcal{D}'} = \mathcal{D}'/\mathcal{D}' \cap K(\mathcal{A}) \xrightarrow{\iota} \mathcal{D}^?(\mathcal{A})$ is an equivalence of categories.

(2) The restriction of F to $\mathcal{D}'$ induces

$$\mathcal{D}'/\mathcal{D} \cap K^{\emptyset}(\mathcal{A}) \xrightarrow{\tilde{F}} K^{?'}(\mathcal{A})$$

(3) Define $RF = \tilde{F}\rho$, where ρ is a quasi-inverse of ι. (4) We have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{D}'} \arrow[drr, "\iota"', "\sim"] \arrow[drrr, bend right, "\tilde{F}"'] & \mathcal{D}' \arrow[l] \arrow[r, hook, "u"] & K^?(\mathcal{A}) \arrow[r] \arrow[d, "Q"] & K^{?'}(\mathcal{B}) \arrow[d, "Q'"] \\
 & & \mathcal{D}^?(\mathcal{A}) \arrow[r, dotted, "G"] & \mathcal{D}^{?'}(\mathcal{B})
\end{tikzcd}$$

If X is an object in $K^?(\mathcal{A})$, take a quasi-isomorphism $X \to X'$ with $X' \in \mathcal{D}'$. Then one has a commutative diagram

$$\begin{tikzcd}
Q'F(X) \arrow[r] \arrow[d] & GQ(X) \arrow[d, "\wr"] \\
Q'F(uX') \arrow[r] & GQ(uX')\,,
\end{tikzcd}$$

and

$$\mathrm{Hom}(Q'F, GQ) = \mathrm{Hom}(Q'Fu, GQu) = \mathrm{Hom}(\tilde{F}v, G\iota v) = \mathrm{Hom}(\tilde{F}, G\iota) = \mathrm{Hom}(\tilde{F}\rho, G)\,.$$

Note: $\rho(X) = X'$ implies that $X \backsimeq \iota\rho(X) = \iota X'$ is a quasi-isomorphism.

Alternatively: use the Ind-object $(Q'FX')_{X \to X' Quis}$; this is essentially constant, isomorphic to $(Q'FX')_{\substack{X \to X' Quis \\ X' \in \mathcal{D}'}}$.

Lemma 4.5 Let $\mathcal{A}$ be an abelian category, and $\mathcal{I} \subseteq ob(\mathcal{A})$.

(a) If any object in $\mathcal{A}$ has a monomorphism $A \hookrightarrow I$ for an object in $\mathcal{I}$, then for every complex $A^\cdot$ in $\mathcal{C}^{\geq n}(\mathcal{A})$ there is a quasi-isomorphism

$$A^\cdot \to I^\cdot$$

into a complex $I^\cdot$ in $\mathcal{C}^{\geq n}(\mathcal{A})$ with components in $\mathcal{I}$.

(b) In particular, if A has enough injectives, then right derivatives exist for any functor on $\mathcal{D}^{\geq n}(\mathcal{A})$.

(c) Dually, similar results hold. For example, one has left derivates if A has enough projectives.

5 The six functors

Following Grothendieck, these are the functors

$$Rf_*,\ Rf_!,\ f^*,\ Rf^!,\ R\,\underline{\mathrm{Hom}},\ \otimes^L\,.$$

Here $R\,\underline{\mathrm{Hom}}$ is the right derivative of the sheaf-Hom, i.e., $R\,\underline{\mathrm{Hom}}(F,G)$ is the value at G of the right derivative of the functor $G \mapsto \underline{\mathrm{Hom}}(F,G)$. Furthermore $\otimes^L$ is the left derivative of the tensor product, i.e., the value at G the left derivative of the functor $G \mapsto F \otimes G$. In both cases one can also consider modules over a ring sheaf $\mathcal{A}$.

We consider a fixed base scheme S, and the category $\mathcal{C}_S$ of quasi-compact, quasi-separated S-schemes with morphisms that are compactifiable over S:

$$\begin{tikzcd}
 & & Y \times_S P \arrow[dl] \arrow[dd] \\
X \arrow[r, "f"] \arrow[urr, bend left, "\text{quasi-finite, separated}"] \arrow[ddr] & Y \arrow[dd] & \\
 & & P \arrow[dl, "\text{proper}"] \\
 & S &
\end{tikzcd}$$

This implies that f is separated, of finite type. If $S = \mathrm{Spec}(k)$ for a field k, one can consider all separated k-schemes of finite type.

Endow all sheaves in $\mathcal{C}_S$ with the étale topology, and let $\mathcal{A}$ be an étale torsion ring-sheaf on S (for example $\mathcal{A} =$ constant sheaf $\mathbb{Z}/n$). Let $\mathcal{A}_X = a_X^*\mathcal{A}$ for $a_X : X \to S$.

By the previous consideration, for any morphism $f : X \to Y$ in $\mathcal{C}_S$ we have a morphism

$$Rf_* : D^+(X, \mathcal{A}_X) \to D^+(Y, \mathcal{A}_Y)$$

Theorem 5.1 (SGA 4, XVII 5.18) There is, up to canonical isomorphism, a unique way to associate to each morphism $f : X \to Y$ in $\mathcal{C}_S$ a functor

$$Rf_! : D(X, \mathcal{A}_X) \to D(Y, \mathcal{A}_X)$$

such that the following holds:

(i) If f is proper, then $Rf_! = Rf_*$.

(ii) If j is an open immersion, then $Rj_! = j_!$.

(iii) $f \mapsto Rf_!$ is "functorial": One has canonical isomorphisms

$$R(gf)_! \xrightarrow{\sim} Rg_! Rf_!$$

with "cocycle condition".

(iv) For a commutative diagram

$$\begin{tikzcd} U \arrow[r, hook, "j_1"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ V \arrow[r, hook, "j_2"] & Y \end{tikzcd}$$

with open immersions j_1 and j_2 and proper morphisms g, f one has a canonical morphism

$$j_{2!}Rg_* \longrightarrow Rf_*j_1! \,.$$

Theorem 5.2 (SGA 4 XVII 3.1.4) $Rf_!$ has a partial right adjoint $Rf^!$, i.e., for K in $\mathcal{D}(X, \mathcal{A}_X)$ and $L \in \mathcal{D}^+(Y, \mathcal{A}_Y)$ one has a functorial isomorphism

$$\mathrm{Hom}(Rf_!K, L) \cong \mathrm{Hom}(K, Rf^!L) \,.$$

Theorem 5.3 (Poincaré duality, SGA XVIII 3.2.5) If f is smooth and purely of relative dimension d, then for n invertible on S and $n\mathcal{A} = 0$ one has a canonical isomorphism

$$Rf^!K \xrightarrow{\sim} f^*K(d)[2d] \,,$$

where $L(d) := L \otimes_{\mathbb{Z}/n} \mathbb{Z}/n(d)$. In particular, $j^! = j^*$ for an open immersion.

Corollary 5.4 Let X be smooth, separated over a separably closed field k, of pure dimension d, and F is a locally constant constructible $\mathbb{Z}/n$-sheaf with n invertible in k. Then one has a canonical isomorphism

$$H^{2d-i}(X, F^\vee(d)) \xrightarrow{\sim} H^i_C(X, F) \,.$$

"Proof" One has

$$R\,\underline{\mathrm{Hom}}(F, \mathbb{Z}/n) = \underline{\mathrm{Hom}}(F, \mathbb{Z}/n) \quad \text{(placed in degree 0)} \,,$$

since $\mathbb{Z}/n$ is an injective $\mathbb{Z}/n$-sheaf. Then we have

$$\begin{tikzcd} R\Gamma R\,\underline{\mathrm{Hom}}(F, \mathbb{Z}/n(d)[2d]) \arrow[r, "\sim"] \arrow[d, equal] & R\,\mathrm{Hom}(R\Gamma_C(F), \mathbb{Z}/n) \\ R\Gamma(F^\vee(d)[2d]) & \end{tikzcd}$$

Theorem 5.5 (Deligne's finiteness theorem, SGA 4 $\frac{1}{2}$, "Théoremes de finitude on cohomologie ℓ-adique"). Let S be a noetherian scheme of dimension 0 or 1, let $f : X \to Y$ be a morphism of S-schemes of finite type, and let F be an étale sheaf in the full subcategory of constructible $\mathbb{Z}/n$-schemes on X. Then R^if_*F is constructible for all $i \geq 0$, if n is invertible on S.

Corollary 5.6 If X is of finite type over a separably closed field k, and n is invertible in k, then $H^i(X, F)$ is finite for every constructible $\mathbb{Z}/n$-sheaf F on X.

Definition 5.7 For a noetherian scheme X and $n \in \mathbb{N}$ let

$$D_c^?(X_{\text{ét}}, \mathbb{Z}/n) \subseteq D^?(X_{\text{ét}}, \mathbb{Z}/n)$$

be the full subcategory of constructible complexes $K^\cdot$, i.e., those complexes whose cohomology sheaves are constructible.

Corollary 5.8 (a) Rf_* maps D_c^+ into D_c^+.

(b) Rf_* has finite cohomological dimension.

(c) Rf_* maps D_c^b to D_c^b.

Corollary 5.9 The 6 functors respect

$$D^h_{ctf} = \begin{array}{l}\{\text{constructible complexes with finite}\\ \text{Tor-dimension as } \mathbb{Z}/n\text{-modules}\}\end{array}$$

Lemma/Definition 5.10 (compare SGA 4 $\frac{1}{2}$ [finitude] 4.7) Let X be separated of finite type over a field k, and let $n \in \mathbb{N}$ be invertible in k. Let $f : X \to \mathrm{Spec}(k)$ be the structural morphism.

(a) The **dualizing complex** of X in $D_c^b(X_{\text{ét}}, \mathbb{Z}/n)$ is

$$K_X \ := \ Rf^!\mathbb{Z}/n\,.$$

(b) The Verdier dual of a complex K in $D_c^b(X_{\text{ét}}, \mathbb{Z}/n)$ is defined by

$$D_X K \ := \ R\,\underline{\mathrm{Hom}}(K, K_X)\,.$$

(c) (Biduality) There is a canonical isomorphism in $D_c^b(X_{\text{ét}}, \mathbb{Z}/n)$

$$K \xrightarrow{\sim} D_X D_X K\,.$$

Corollary 5.11 Let $h : X \to Y$ be a morphism of separated algebraic k-schemes. Then D exchanges Rf_* and $Rf_!$, and also f^* and $Rf^!$, i.e., there are canonical functorial isomorphisms

$$\begin{aligned} Rh_* D_X K &\cong D_Y Rh_! K \\ Rh_! D_X K &\cong D_Y Rh_* K \\ D_X h^* K &\cong Rh^! D_Y K \\ D_X Rh^! L &\cong h^* D_Y L \end{aligned}$$

for K in $D_c^b(X_{\text{ét}}, \mathbb{Z}/n)$ and L in $D_c^b(Y_{\text{ét}}, \mathbb{Z}/n)$.

Proof For

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

we have

$$\begin{aligned} D_Y Rh_! K &= R\,\underline{\mathrm{Hom}}(Rh_! K, Rg^! \mathbb{Z}/n) \\ &\overset{(1)}{\cong} Rh_* R\,\underline{\mathrm{Hom}}(K, Rh^! Rg^! \mathbb{Z}/n) \\ &= Rh_* R\,\underline{\mathrm{Hom}}(K, Rf^! \mathbb{Z}/n) \\ &= Rh_* D_X K\,, \end{aligned}$$

where (1) comes from "local" Verdier duality.

Applied to $D_X K$ this gives

$$D_Y Rh_! D_X K \;\cong\; Rh_* K$$

via biduality on X. Hence, by applying D_Y and biduality on Y we also get

$$Rh_! D_X K \;\cong\; D_Y Rh_* K\,.$$

The two remaining isomorphisms follow from the "induction formula" (SGA XVIII 3.1.12.2)

$$R\,\underline{\mathrm{Hom}}(h^* L_1, Rh^! L_2) \;\cong\; Rh^! R\,\underline{\mathrm{Hom}}(L_1, L_2)$$

for L_1, L_2 in $D_c^b(Y_{\text{ét}}, \mathbb{Z}/n)$.

6 Glueing of t-structures

6.1 Let $\mathcal{D}, \mathcal{D}_U$ and $\mathcal{D}_F$ be three triangulated categories and let

$$\mathcal{D}_F \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_U$$

be exact functors such that the following holds.

(1) i_* has an exact left adjoint i^* and an exact right adjoint $i^!$.

(2) j^* has an exact left adjoint $j_!$ and an exact right adjoint j_*.

(3) One has $j^*i_* = 0$, and hence $i^*j_! = 0$ and $i^!j_* = 0$, by adjunction. For A in $\mathcal{D}_F$ and B in $\mathcal{D}_U$ we hence have

$$\mathrm{Hom}(j_!B, i_*A) = 0 \quad \text{and} \quad \mathrm{Hom}(i_*A, j_*B) = 0\,.$$

(4) For each object K in $\mathcal{D}$ there is a morphism

$$d : i_*i^*K \longrightarrow j_!j^*K, \quad \text{resp.} \quad d' : j_*j^*K \longrightarrow i_*i^![K1]$$

such that

$$j_!j^*K \longrightarrow K \longrightarrow i_*i^*K \xrightarrow{d} j_!j^*K[1]\,,$$

resp.

$$i_*i^!K \longrightarrow K \longrightarrow j_*j^*K \xrightarrow{d'} i^*i^!K[1]$$

are distinguished triangles.

(5) $i_*, j_!$ and j_* are fully faithful, and the adjunction morphisms

$$i^*i_* \longrightarrow id \longrightarrow i^!i_*$$

and

$$j^*j_* \longrightarrow id \longrightarrow j^*j_!$$

are isomorphisms.

Remark 6.2 The morphisms d and d' in (4) are uniquely determined by (3): Consider

$$\begin{tikzcd} K \arrow[r] \arrow[d, equal] & i_*i^*K \arrow[r, "d"] \arrow[d, dotted, "h"] & j_!j^*K[1] \arrow[d, equal] \\ K \arrow[r] & i_*i^*K \arrow[r, "\tilde{d}"] & j_!j^*K[1] \end{tikzcd}$$

where h exists by (TR3). By the exact sequence

$$\begin{tikzcd} \mathrm{Hom}^{-1}(j_!j^*K, i_*i^*K) \arrow[r] \arrow[d, equal, "(3)"] & \mathrm{Hom}(i_*i^*K, i_*i^*K) \arrow[r] & \mathrm{Hom}(K, i_*i^*K) \\ 0 & & \end{tikzcd}$$

the morphism h is unique, hence the identity, which implies $d = \tilde{d}$.

Example 6.3 (a) Let X be a scheme, $j : U \hookrightarrow X$ an open immersion, and $i : F \hookrightarrow X$ a closed immersion with $X = i(F) \bigcup j(U)$. Then the morphisms

$$\mathcal{D}^+(F_{\text{ét}}, \mathbb{Z}/n) \xrightarrow{i_*} \mathcal{D}^+(X_{\text{ét}}, \mathbb{Z}/n) \xrightarrow{j^*} \mathcal{D}^+(U_{\text{ét}}, \mathbb{Z}/n)$$

satisfy the conditions in 6.1:

$$\begin{array}{rcl} i^* & = & \text{usual } i^* \\ i^! & = & \text{usual } Ri^! \text{ (right derivative of } i^!) \\ j_! & = & \text{usual } j_! \\ j_* & = & \text{usual } Rj_* \end{array}$$

The adjunction properties hold by the formalism of the 6 functors.

(b) If X is of finite type over a field k, then one can replace $\mathcal{D}^+$ by $\mathcal{D}_c^+, \mathcal{D}_c$ or $\mathcal{D}_c^b$, by Deligne's finiteness theorem.

Theorem 6.4 (glueing theorem) In the situation of Theorem 6.1 let $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ and $(\mathcal{D}_F^{\leq 0}, \mathcal{D}_F^{\geq 0})$ be t-structures on $\mathcal{D}_U$ and $\mathcal{D}_F$, respectively. Then the categories

$$D^{\leq 0} = \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\leq 0} \text{ and } i^*K \in \mathcal{D}_F^{\leq 0}\}$$

and

$$\mathcal{D}^{\geq 0} = \{K \in \mathcal{D} \mid j^*K \in \mathcal{D}_U^{\geq 0} \text{ and } i^!K \in \mathcal{D}_F^{\geq 0}\}$$

define t-structures on $\mathcal{D}$ (We say that these are obtained by glueing).

Proof We verify the conditions in 3.1 (a).

(i) Let K in $\mathcal{D}^{\leq 0}$ and L in $\mathcal{D}^{\geq 1}$. By 6.1 (4) we obtain an exact sequence

$$\begin{tikzcd} \mathrm{Hom}(i_*i^*K, L) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(K, L) \arrow[r] & \mathrm{Hom}(j_!j^*K, L) \arrow[d, equal] \\ \mathrm{Hom}(i^*K, i^!L) \arrow[d, equal] & & \mathrm{Hom}(j^*K, j^*L) \arrow[d, equal] \\ 0 & & 0 \end{tikzcd}$$

and hence $\mathrm{Hom}(K, L) = 0$.

(ii) We have $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$ by the same properties for $\mathcal{D}_U$ and $\mathcal{D}_F$.

(iii) Let K in $\mathcal{D}$, and form the distinguished triangles

$$Y \longrightarrow X \xrightarrow{\alpha} j_*\tau_{\geq 1}j^*X \longrightarrow,$$

where α comes by adjunction from $j^*X \to \tau_{\geq 1}j^*X$, and

$$A \longrightarrow Y \xrightarrow{\beta} i_*\tau_{\geq}i^*Y \longrightarrow,$$

where β comes by adjunction from $i^*Y \to \tau_{\geq 1} i^*Y$.

By (TR4) we get a commutative diagram of distinguished triangles

$$\begin{tikzcd}
A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & i_*\tau_{\geq 1} i^*Y \arrow[r] \arrow[d, dotted] & {} \\
A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, dotted] & {} \\
Y \arrow[r] & X \arrow[r] & j_*\tau_{\geq 1} j^*X \arrow[r] \arrow[d] & {} \\
 & & i_*\tau_{\geq 1} i^*Y[1] &
\end{tikzcd}$$

Applying j^* to the third column we get

$$j^*B \xrightarrow{\sim} \tau_{\geq 1} j^*X \,,$$

since j^* is exact and $j^*i_* = 0$ and $j^*j_* = id$.

Applying j^* to the second row we thus get

$$j^*A \xrightarrow{\sim} \tau_{\leq 0} j^*X \,.$$

Applying i^* to the first row we get

$$i^*A \xrightarrow{\sim} \tau_{\leq 0} i^*Y \,,$$

since i^* is exact and $i^*i_* = id$.

Applying $i^!$ to the third column we get

$$\tau_{\geq 1} i^*Y \xrightarrow{\sim} i^!B \,,$$

since $i^!i_* = id$ and $i^!j_* = 0$.

This implies $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$, by definition, and hence 3.1 (iii).

Remark 6.5 The t-structure on $\mathcal{D}$ is non-degenerate if and only if this holds for the t-structures on $\mathcal{D}_U$ and $\mathcal{D}_F$.

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8

9 t-exactness and the intermediate direct image

Definition 9.1 Let $f : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor between triangulated categories which are endowed with t-structures. Then f is called t-right exact if $f(\mathcal{D}_1^{\leq 0}) \subseteq \mathcal{D}_2^{\leq 0}$, and t-left exact if $f(\mathcal{D}_1^{\geq 0}) \subseteq \mathcal{D}_2^{\geq 0}$, and f is called t-exact if f is t-right exact and t-left exact.

Proposition 9.2 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the hearts of (the t-structures of) $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

(a) If f is t-left exact (respectively, t-right exact), then the additive functor

$$ {}^t f := {}^t H^0 f|_{\mathcal{C}_1} : \mathcal{C}_1 \to \mathcal{C}_2 $$

is left exact (respectively, right exact), and for K in $\mathcal{D}_1^{\leq 0}$ (respectively, in $\mathcal{D}_1^{\geq 0}$) one has an isomorphism

$$ {}^t f\, {}^t H^0 K \xrightarrow{\sim} {}^t H^0 f K $$

(respectively, ${}^t H^0 f K \xrightarrow{\sim} {}^t f\, {}^t H^0 K$).

(b) Let (f^*, f_*) be a pair of adjoint exact functors $f_* : \mathcal{D}_1 \to \mathcal{D}_2$, $f^* : \mathcal{D}_2 \to \mathcal{D}_1$, where f^* is left adjoint to f_*.

Then f^* is t-right exact if and only if f_* is t-left exact, and in this case $({}^t f^*, {}^t f_*)$ is a pair of adjoint functors

$$ \mathcal{C}_1 \underset{{}^t f_*}{\overset{{}^t f^*}{\rightleftarrows}} \mathcal{C}_2 $$

(${}^t f^*$ left adjoint to ${}^t f_*$).

(c) If $f : \mathcal{D}_1 \to \mathcal{D}_2$ and $g : \mathcal{D}_2 \to \mathcal{D}_3$ are t-left exact (respectively, t-right exact), then the same holds for gf, and one has

$$ {}^t(gf) \;=\; {}^t g \circ {}^t f\,. $$

Proof (a): Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{C}_1$. Then there is a distinguished triangle

$$ X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] $$

in $\mathcal{D}_1$ (which is uniquely determined, see), and hence we get a distinguished triangle

$$ fX \longrightarrow fY \longrightarrow fZ \longrightarrow fX[1] $$

in $\mathcal{D}_1$. If f is t-right exact, then $fX \in \mathcal{D}_2^{\leq 0}$, hence $fX[1] \in \mathcal{D}_2^{\leq -1}$, and therefore

$$ {}^t H^0(fX[1]) = \tau_{\leq 0}\tau_{\geq 0}(fX[1]) = 0\,. $$

Therefore we get an exact sequence

$$ \begin{array}{ccccccc} {}^t H^0 X & \longrightarrow & {}^t H^0 fY & \longrightarrow & {}^t H^0 fZ & \longrightarrow & 0 \\ \| & & & & & & \\ {}^t fX & & & & & & \end{array} $$

If X is in $\mathcal{D}_1^{\leq 0}$, then we have a distinguished triangle

$$\begin{tikzcd} \tau_{<0}X \arrow[r] & X \arrow[r] & {}^tH^0X \arrow[r] \arrow[d, equal] & {,} \\ & & \tau_{\geq 0}X & \end{tikzcd}$$

and hence a distinguished triangle

$$f\tau_{<0}X \longrightarrow fX \longrightarrow f({}^tH^0X) \longrightarrow$$

with $f\tau_{<0}X \in \mathcal{D}_2^{\leq 0}$, so that we get an isomorphism

$${}^tH^0(fX) \xrightarrow{\sim} f({}^tH^0X)\,.$$

The other case is dual.

(b): If f_* is t-left exact, then for U in $\mathcal{D}_1^{>0}$ and V in $\mathcal{D}_2^{\leq 0}$ one has

$$\mathrm{Hom}(f^*V, U) \;=\; \mathrm{Hom}(V, f_*U) = 0\,,$$

because $f_*U \in \mathcal{D}_2^{>0}$. Since this holds for any U, we have $\tau_{>0}f^*V = 0$, i.e., $f^*V \in \mathcal{D}_1^{\leq 0}$, so that f^* is t-right exact.

If now $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, we have

$$H^0(f^*B) \;=\; \tau_{\geq 0}(f^*B) \quad \text{und} \quad H^0(f_*A) \;=\; \tau_{\leq 0}(f_*B)\,,$$

and hence a functorial isomorphism

$$\mathrm{Hom}(H^0(f^*B), A) \xrightarrow{\sim} \mathrm{Hom}(f^*B, A) = \mathrm{Hom}(B, f_*A) \xleftarrow{\sim} \mathrm{Hom}(A, H^0(f_*B))\,.$$

This, together with the dual arguments, proves (b).

(c): If f and g are t-left exact, and $A \in \mathcal{C}_1$, then $fA \in \mathcal{D}^{\geq 0}$, and ${}^t(gf)A = H^0(g)H^0(f)A$ by the second part of (a). This, together with the dual arguments, proves (c).

Theorem 9.3 Let X be a scheme of finite type over a field k, and let ℓ be a prime invertible in k.

Let $i : Y \hookrightarrow X$ be a closed immersion, and let $j : U \hookrightarrow X$ be the open immersion of the complement. Consider the triangulated categories $\mathcal{D}_c^b(?, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-constructible bounded complexes for $? = X, Y$ and U, endowed with the perverse t-structure p. Then the following holds.

(a) $j_!$ and i^* are p-right exact.

(b) j^* and i_* are p-exact.

(c) Rj_* and $Ri^!$ are p-left exact.

(d) $({}^pj_!, {}^pj^* = j^*, {}^pRj_*)$ and $({}^pi^*, {}^pi_* = i_*, {}^pRi^!)$ form triples of adjoint functors (where for f, g f is the left adjoint to g).

Proof By definition of the perverse t-structure p, j^* is p-exact: For $x \in U$ one has a factorization

$$i_x : \; x \xhookrightarrow{\tilde{i}_x} U \xhookrightarrow{j} X$$

so that $\tilde{i}_x^* j^* = i_x^*$ and $R\tilde{i}_x^! j^* = Ri_x^!$. Furthermore, by definition i_* is p-right exact (respects ${}^p\mathcal{D}^{\leq 0}$) and $Ri^!$ is p-left exact (respects ${}^p\mathcal{D}^{\geq 0}$). The remaining claims follow from the adjunctions

$$(j_!, j^*, Rj_*) \quad , \quad (i^*, i_*, Ri^!)\,.$$

In fact, $j_!$ is left adjoint to j^* so that $j_!$ is p-right exact, and Rj_* is right adjoint to j^* so that Rj_* is p-left exact, by 9.2 (b).

Finally (d) follows from 9.2 (b).

We have a canonical morphism of functors from $D_c^b(U, \mathbb{Q}_\ell)$ to $D_c^b(X, \mathbb{Q}_\ell)$

$$j_! \longrightarrow Rj_*\,,$$

given by the canonical inclusion

$$j_! I^{\cdot} \longrightarrow j_* I^{\cdot}$$

for any complex $I^{\cdot}$ with injective components.

This induces a morphism of functors

$$\begin{array}{rcl} Perv(U_{\text{ét}}, \mathbb{Q}_\ell) & \to & Perv(X_{\text{ét}}, \mathbb{Q}_\ell) \\ {}^p j_! & \to & {}^p Rj_* \end{array}$$

by applying ${}^pH^0$ to the previous morphism of functors.

Definition 9.4 The intermediate direct image $j_{!*}F$ of a perverse sheaf P in $Perv(U, \mathbb{Q}_\ell)$ is by definition

$$j_{!*}F \;:=\; im({}^p j_! F \to {}^p Rj_* F) \in Perv(X, \mathbb{Q}_\ell)\,.$$

This gives a functor $Perv(U, \mathbb{Q}_\ell) \to Perv(X, \mathbb{Q}_\ell)$, and obviously one has $j^* j_{!*} F = F$.

10 Laumon's ℓ-adic Fourier transformation

Let $q = p^n$ for a prime p, and let $\mathbb{F}_q$ be the finite field with q elements. Let V be a vector space of dimension d over $\mathbb{F}_q$, and let

$$J := \mathbb{V}(V) := \mathrm{Spec}(Sym(V)) \cong \mathbb{A}^d_{\mathbb{F}_q} .$$

Let $Frob_q : J \to J$ be the q-Frobenius morphism: it is the identity on the underlying topological space and the map $x \mapsto x^q$ on $\mathcal{O}_J$. This is a morphism of algebraic groups.

We get an exact sequence

$$0 \longrightarrow J(\mathbb{F}_q) \longrightarrow J \xrightarrow{Frob_q - id} J \longrightarrow 0$$

(Artin-Schreier sequence). $J \to J$ is an étale covering, from which we get a continuous morphism

$$\pi_1(J) \longrightarrow J(\mathbb{F}_q) .$$

Let

$$\chi : J(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell^\times$$

be a character, where ℓ is a prime $\neq p$. This gives a continuous homomorphism

$$\pi_1(J) \longrightarrow J(\mathbb{F}_q) \xrightarrow{\chi^{-1}} \overline{\mathbb{Q}}_\ell^\times ,$$

and hence a smooth $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{L}_\chi$ on J.

Remark 10.1 (1) One has a canonical isomorphism

$$\mathcal{L}_\chi|_{\{0\}} \ \cong\ \overline{\mathbb{Q}}_{\ell,0} .$$

(2) One has a canonical isomorphism

$$s^*\mathcal{L}_X \ \cong\ p_1^*\mathcal{L}_\chi \otimes p_2^*\mathcal{L}_\chi$$

where $s : J \times J \to J$ is the addition morphism and $p_i : J \times J \to J$ are the projections to the i-th factor, $i = 1, 2$.

(3) One has

$$R\Gamma_c(J \otimes_{\mathbb{F}_q} \overline{k}, \mathcal{L}_\chi) = 0 = R\Gamma(J \otimes_{\mathbb{F}_q} \overline{k}, \mathcal{L}_\chi)$$

if $\overline{k}/\mathbb{F}_q$ is an algebraically closed field extension and χ is a **non-trivial** character.

Let k be a perfect field of characteristic p, and let $\overline{k}$ be an algebraic closure of k. Fix a non-trivial character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell$, and let $\mathcal{L}_\psi$ be the associated smooth $\overline{\mathbb{Q}}_\ell$-sheaf on $J = \mathbb{G}_{a,\mathbb{F}_p}$.

Let S/k be of finite type, let $E \xrightarrow{\pi} S$ be a vector bundle of rank r, and let $E' \xrightarrow{\pi'} S$ be the dual vector bundle.

Then we have a canonical pairing

$$\langle\ ,\ \rangle : E \times_S E' \longrightarrow \mathbb{G}_{a,k} .$$

Let $\mathcal{L}(\langle\ ,\ \rangle)$ be the pull-back of $\mathcal{L}_\psi/\mathbb{G}_{a,k}$ via $\langle\ ,\ \rangle$.

Definition 10.2 The Fourier transformation associated to ψ on $E \xrightarrow{\pi} S$ is the exact functor

$$\mathcal{F}_\psi : D_c^b(E, \mathbb{Q}_\ell) \longrightarrow D_c^b(E', \mathbb{Q}_\ell)$$

defined by

$$\mathcal{F}_\psi(K) = Rpr'_!(pr^*K \otimes \mathcal{L}_\psi(\langle\ ,\ \rangle))[r]$$

where

$$\begin{tikzcd} & E \times E' \arrow[dl, "pr"'] \arrow[dr, "pr'"] & \\ E & & E' \end{tikzcd}$$

are the projections.

Let $E'' \xrightarrow{\pi''} S$ be the vector bundle which is bi-dual to $E \to S$, and define the isomorphism

$$a : E \xrightarrow{\sim} E''$$

by $a(e) = -\langle e\,,\,\cdot\rangle$.

Theorem 10.3 Let $\mathcal{F}'$ be the Fourier transformation associated to ψ on $E' \xrightarrow{\pi'} S$. Then there is a functorial isomorphism

$$(\mathcal{F}' \circ \mathcal{F})(K) \xrightarrow{\sim} a_*K(r)$$

for K in $D_c^b(E, \overline{\mathbb{Q}}_\ell)$.

Proof: We have a diagram

$$\begin{tikzcd}
 & & E' \times_S E'' \arrow[rrrr, "\beta"] \arrow[dddll, bend right, "pr_1"'] \arrow[dddrr, bend left, "pr_2"] & & & & E'' \\
 & & E \times_S E' \times E'' \arrow[u, "pr_{1,3}"'] \arrow[rrrr, "\alpha"] \arrow[dl, "pr_{12}"'] \arrow[dr, "pr_{23}"] & & & & E' \times_S E'' \arrow[u, "pr''"'] \\
 & E \times_S E' \arrow[dl, "pr"'] \arrow[dr, "pr'_2"'] & & E' \times_S E'' \arrow[dl, "pr'_1"] \arrow[dr, "pr''"] & & & \\
E & & E' & & E'' & &
\end{tikzcd}$$

where

$$\begin{aligned} \alpha(e, e', e'') &= (e', e'' - a(e)) \\ \beta(e, e'') &= e'' - a(e) \end{aligned}$$

Claim 10.3.1 $pr_{12}^*\mathcal{L}(\langle\ ,\ \rangle) \otimes pr_{23}^*\mathcal{L}(\langle\ ,\ \rangle) = \alpha^*\mathcal{L}(\langle\ ,\ \rangle)$

Proof We have a commutative diagram

$$
\begin{tikzcd}
 & E \times E' \times E'' \arrow[dl, "pr_{12}"'] \arrow[dr, "pr_{23}"] \arrow[dd, "f"] & \\
E \times E' \arrow[dd, "\langle\,,\rangle"'] & & E' \times E'' \arrow[dd, "\langle\,,\rangle"] \\
 & \mathbb{G}_a \times \mathbb{G}_a \arrow[dl, "p_1"] \arrow[d, "s"] \arrow[dr, "p_2"'] & \\
\mathbb{G}_a & \mathbb{G}_a & \mathbb{G}_a
\end{tikzcd}
$$

with the obvious projections, $f(e, e', e'') = (\langle\, e,\, e'\rangle, \langle\, e',\, e''\rangle)$, and the sum map s.
By 9.1 (2) we have

$$p_1^* \mathcal{L}_\psi \otimes p_2^* \mathcal{L}_\psi = s^* \mathcal{L}_\psi$$

and hence

$$p_{12}^* \mathcal{L}_\psi(\langle\ ,\ \rangle) \otimes pr_{23}^* \mathcal{L}_\psi(\langle\ ,\ \rangle) = f^* s^* \mathcal{L}_\psi$$

Since

$$s \circ f(e, e', e'') = \langle\, e,\, e'\rangle + \langle\, e',\, e''\rangle = \langle\, e', e'' - a(e)\rangle = \langle\ ,\ \rangle \circ \alpha(e, e', e'')$$

we get the claim.

Now by definition we have

$$
\begin{aligned}
& (\mathcal{F}' \circ \mathcal{F})(K) \\
= \ & Rpr''_!((pr'_1)^* \circ R(pr'_2)_!)(pr^* K \otimes \mathcal{L}(\langle\ ,\ \rangle))[r] \otimes \mathcal{L}(\langle\ ,\ \rangle)[r]
\end{aligned}
$$

By proper base change (SGA 4, XVII 5.2.6) we have

$$(pr'_1)^* \circ R(pr'_2)_! = R(pr_{23})_! \circ pr_{12}^*$$

Hence we get

$$
\begin{aligned}
& \mathcal{F}' \circ \mathcal{F}(K) \\
= \ & Rpr''_!(R(pr_{23})_! \circ pr_{12}^*(pr^* K \otimes \mathcal{L}(\langle\ ,\ \rangle)) \otimes \mathcal{L}(\langle\ ,\ \rangle)[2r] \\
\overset{(1)}{=} \ & R(pr'' \circ pr_{23})_!(pr \circ pr_{12})^* K \otimes pr_{12}^* \mathcal{L}(\langle\ ,\ \rangle) \otimes pr_{23}^* \mathcal{L}(\langle\ ,\ \rangle)[2r] \\
= \ & R(pr_2 \circ pr_{13})_!(pr_{13}^* \circ pr_1^* K \otimes \alpha^* \mathcal{L}(\langle , \rangle) \\
\overset{(2)}{=} \ & R(pr_2)_!(pr_1^* K \otimes R(pr_{13})_! \alpha^* \mathcal{L}(\langle\ ,\ \rangle)[2r] \\
\overset{(3)}{=} \ & R(pr_2)_!(pr_1^* K \otimes \beta^* Rpr''_! \mathcal{L}(\langle\ ,\ \rangle)[2r]
\end{aligned}
$$

where (1) and (2) follow from the projection formula and (3) follows from base change.

Now we use

Proposition 10.4 For L in $D^b(S, \mathbb{Q}_\ell)$ one has a functorial isomorphism

$$\mathcal{F}(\pi^* L[r]) \cong \sigma'_* L(-r)\,,$$

where $\pi : E \to S$ and $\pi' : E' \to S$ are as above, and $\sigma : S \to E'$ is the zero section.

Proof: later.

With this we proceed as follows. By definition of $\mathcal{F}'$ we have $Rpr''_!\mathcal{L}(\langle\ ,\ \rangle)[r] = \mathcal{F}'(\mathbb{Q}_{\ell,E'})$. Applying 10.4 to $\pi' : E' \longrightarrow S$, $\pi'' :$ $\begin{tikzcd} E'' \arrow[r, shift left] & S \arrow[l, bend left, "\sigma''"] \end{tikzcd}$, and $\mathcal{L} = \overline{\mathbb{Q}}_\ell$ we get

$$\mathcal{F}'(\mathbb{Q}_{\ell,E'}[r]) = \sigma''_*\overline{\mathbb{Q}}_{\ell,S}(-r)\,.$$

Hence we get

$$\mathcal{F} \circ \mathcal{F}(K) \cong R(pr_2)_!(pr_1^*K \otimes \beta^*\sigma''_*\overline{\mathbb{Q}}_{\ell,S}(-r))\,.$$

Since

$$\begin{tikzcd} E \arrow[r, "id\times a"] \arrow[d, "\pi"'] & E \times E'' \arrow[d, "\beta"] \\ S \arrow[r, "\sigma''"] & E'' \end{tikzcd}$$

is cartesian, proper base change implies

$$\beta^*\beta''_* = (id \times a)_*\pi^*\,.$$

Hence we get

$$\begin{aligned} \mathcal{F}' \circ \mathcal{F}(K) &= R(pr_2)_!(pr_1^*K \otimes (id \times a)_*\overline{\mathbb{Q}}_{\ell,E}(-r)) \\ &\overset{(1)}{=} R(pr_2)_!((id \times a)_*pr_1^*K(-r)) \\ &\overset{(2)}{=} a_*K(-r)\,, \end{aligned}$$

where (1) holds by the projection formula, and (2) holds since $pr_2 \circ (id \times a) = a$ and $pr_1 \circ (id_E \times a) = id_E$.

Proof of Proposition 10.4 We have a diagram

$$\begin{tikzcd} & E \times_S E' \arrow[dl, "pr"'] \arrow[dr, "pr'"] \arrow[rr, "\langle\,,\rangle"] & & \mathbb{A}^1_k \\ E \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] & \\ & S \arrow[ur, bend right, "\sigma'=0-\text{section}"'] & & \end{tikzcd}$$

and use the Fourier transform F for $\mathcal{L} = \mathcal{L}_\psi$. By definition we have

(10.4.1)
$$\begin{aligned} & \mathcal{F}(\pi^*L[r]) \\ = \ & Rpr'_!(pr^*\pi^*L[r] \otimes \mathcal{L}(\langle\,,\,\rangle))[r] \\ = \ & Rpr'_!((pr')^*(\pi')^*L[r] \otimes \mathcal{L}(\langle\,,\,\rangle))[r] \quad (\text{since } \pi pr = \pi' pr') \\ = \ & (\pi')^*L \otimes Rpr'_!\mathcal{L}(\langle\,,\,\rangle)[2r] \quad (\text{projection formula}) \end{aligned}$$

(10.4.2) On the other hand, by the cartesian diagram

$$\begin{tikzcd} E = E \times_S S \arrow[r, "id\times\sigma'"] \arrow[d, "\pi"'] & E \times E' \arrow[d, "pr'"] \\ S \arrow[r] & E' \end{tikzcd}$$

and proper base change we have

$$\begin{aligned} (\sigma')^* Rpr'_! \mathcal{L}(\langle\, , \,\rangle) &= R\pi_!(id \times \sigma')^* \mathcal{L}(\langle\, , \,\rangle) \\ &= R\pi_!(id \times \sigma')^* \langle\, , \,\rangle^* \mathcal{L}\,. \end{aligned}$$

Since

$$\begin{tikzcd} E \arrow[r, "id_E\times\sigma'"] \arrow[dr, "\pi"'] & E \times E' \arrow[r, "\langle\,,\,\rangle"] & \mathbb{G}_a \\ & S \arrow[ur, "0"'] & \end{tikzcd}$$

commutes, we get

$$(id \times \sigma')^* \langle\, , \,\rangle^* \mathcal{L}_\psi = \overline{\mathbb{Q}_{\ell,E}}$$

(trivial sheaf) by (10.1.1).

Finally we have $Rpr'_!\mathcal{L}(\langle\, , \,\rangle)|_{E'\setminus\sigma(S)} = 0$, because for $s' \in E'$ we have, by proper base change,

$$\begin{aligned} (R^i pr'_! \mathcal{L}(\langle\, , \,\rangle))_{\overline{s'}} &= H^i_c(E \times_S \overline{s'}, \mathcal{L}(x, s')) \\ &= H^i_c(\mathbb{A}^r_k \times k(\overline{s'}), \mathbb{Q}_\ell(\chi))\,, \end{aligned}$$

where $\chi : \pi_1(\mathbb{A}^r_k \times k(\overline{s'})) \to \overline{\mathbb{Q}_\ell}^\times$ is a character. Here χ is trivial if and only if $s' = 0$.

In fact, let $s' \neq 0'$. The cartesian diagram of Galois covers

$$\begin{tikzcd} X' \arrow[r, phantom, "\cong"] & \mathbb{A}^{r-1}_{k(s')} \times_{k(s')} \mathbb{A}^1_{k(s')} \arrow[r] \arrow[d, "G'\cong\mathbb{F}_p"'] & \mathbb{A}^1_k \times_k k(\overline{s'}) \arrow[d, "G=\mathbb{F}_p"] \\ X \arrow[r, phantom, ":="] & \mathbb{A}^r \times_k k(\overline{s'}) \arrow[r, "\psi(x,s')"] & \mathbb{A}^1_k \times_k k(\overline{s'}) \end{tikzcd}$$

(where the bottom map is without restriction the projection to the last factor, since $s' \neq 0$) shows that χ factors through G'. For the étale cohomology of affine space we have

$$H^i(X', \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & , \quad i = 0\,, \\ 0 & , \quad i \neq 0\,, \end{cases}$$

and we obtain an isomorphism of G'-modules

$$H^i(X', \overline{\mathbb{Q}_\ell}(\chi^{-1})) = \begin{cases} \overline{\mathbb{Q}_\ell}(\chi^{-1}) & , \quad i = 0\,, \\ 0 & , \quad i \neq 0\,. \end{cases}$$

Since $H^i(X, \mathbb{Q}_\ell(\chi^{-1})) \cong H^i(X', \mathbb{Q}_\ell(\chi^{-1}))^{G'}$ (general property of Galois covers), we obtain

$$H^i(X, \overline{\mathbb{Q}_\ell}(\chi^{-1}) = 0 \quad \text{for all} \quad i \neq 0$$

Hence Poincaré duality gives

$$H_c^i(X, \overline{\mathbb{Q}_\ell}(\chi)) = \begin{cases} \mathbb{Q}_\ell(-r) & , \quad i = 2r, s' = 0 \\ 0 & , \quad \text{otherwise}. \end{cases}$$

This implies

$$Rpr'_! \mathcal{L}(\langle\ ,\ \rangle)|_{E' \setminus \sigma(S)} = 0$$

and hence

$$\begin{aligned} Rpr'_! \mathcal{L}(\langle\ ,\ \rangle) &\cong \sigma'_* \sigma'^* \mathcal{L}(\langle\ ,\ \rangle) \\ &= \sigma'_* \overline{\mathbb{Q}_\ell}(-r)[-2r] \end{aligned}$$

as wanted.

11 Properties of perverse sheaves

In the following we consider perverse sheaves in $\mathcal{D}(X) = D^b_c(X, \overline{\mathbb{Q}_\ell})$, where X is a variety over the field k. Let $i : Y \hookrightarrow X$ be a closed subscheme, and let $j : U \hookrightarrow X$ be the open complement.We write $\mathrm{Perv}(X)$ for the category $\mathrm{Perv}(X, \overline{\mathbb{Q}_\ell})$ of perverse $\overline{\mathbb{Q}_\ell}$-sheaves on X.

Remark 11.1 Obviously the perverse t-structure p on $\mathcal{D}_X = D^b_c(X, \overline{\mathbb{Q}_\ell})$ is obtained by glueing the perverse t-structures on U and Y. The morphism of functors $j_! \to j_*$ is obtained by the canonical distinguished triangle for $K = j_*L$:

$$j_!j^*K \xrightarrow{Ad_j} K \xrightarrow{ad_i} i_*i^*K \xrightarrow{d} ,$$

where Ad_j and ad_i are the adjunction morphisms.

Lemma 11.2 (a) The inclusion of the full subcategory

$$\mathcal{D}' := \{K \in \mathcal{D}_X \mid i^*K \in {}^p\mathcal{D}^{\leq 0}_Y\} \hookrightarrow \mathcal{D}_X$$

has a right adjoint $\tau^Y_{\leq 0}$.

Proof (a) $\mathcal{D}' = \mathcal{D}^{\leq'}$ is obtained by glueing the perverse t-structure on Y and the degenerate t-structure $(\mathcal{D}_U, 0)$ (i.e., $\mathcal{D}^{\leq'}_U = \mathcal{D}_U$, $\mathcal{D}^{\geq'}_U = 0$) on U.

By 3.4 (i), the inclusion $\mathcal{D}' \hookrightarrow \mathcal{D}_X$ has a right adjoint.

(b) $\mathcal{D}'' = \mathcal{D}^{\geq''}$ is obtained by glueing the perverse t-structure on $\mathcal{D}_Y$ and the degenerate t-structure $(0, \mathcal{D}_U)$, i.e. $(\mathcal{D}^{\leq'} = 0, \mathcal{D}^{\geq'} = \mathcal{D}_U)$

By 3.4 (ii), we have a left adjoint of $\mathcal{D}'' \hookrightarrow \mathcal{D}_X$.

Lemma 11.3 (a) By the constructions of Theorem 6.1 (glueing of t-structures), we obtain the following description of $\tau^Y_{\leq 0} = {}^p\tau^Y_{\leq 0}$: Form distinguished triangles

$$L \longrightarrow K \longrightarrow j_*\tau'_{\geq 1}j^*K \longrightarrow$$

and

$$A \longrightarrow L \longrightarrow i_*\tau_{\geq 1}i^*K \longrightarrow .$$

Then we have

$$\tau^Y_{\leq 0}K = A .$$

But we have $\tau'_{\geq 1}j^*K = 0$, hence $L = K$, hence $\tau^Y_{\leq 0}K$ is given by a distinguished triangle

$$\tau^Y_{\leq 0}K \longrightarrow K \longrightarrow i_*\tau_{\geq 1}i^*K \longrightarrow .$$

(b) For $\tau^Y_{\geq 0} = {}^p\tau^Y_{\geq 0}$ we form distinguished triangles

$$L \longrightarrow K \longrightarrow j_*\tau''_{\geq 1}j^*K \longrightarrow$$

$$A \longrightarrow L \longrightarrow i_*\tau_{\geq 1}i^*L \longrightarrow$$

$$A \longrightarrow K \longrightarrow B \longrightarrow$$

Then we have $B = \tau^Y_{\geq 1}$. However, we have $\tau''_{\geq 1} j^* K = j^* K$, hence $L = i_* i^! K$; and from the distinguished triangle

$$A \longrightarrow i_* i^! K \longrightarrow i_* \tau_{\geq 1} i^! K \longrightarrow$$

we obtain $A = i_* \tau_{\leq 0} i!K$, and get a distinguished triangle

(11.3.1) $$i_* \tau_{\leq 0} i^! K \longrightarrow K \longrightarrow \tau^Y_{\geq 1} K \longrightarrow .$$

Definition 11.4 Let $L \in \mathcal{D}_U$. A complex K in $\mathcal{D}_X$ with an isomorphism $j^* K \xrightarrow{\sim} L$ is called an extension of L (to X).

Proposition 11.5 Let $L \in \mathcal{D}_U$ and $n \in \mathbb{Z}$. Then up to unique isomorphism there is exactly one extension $K \in \mathcal{D}_X$ of L such that $i^* K \in \mathcal{D}_Y^{\leq n-1}$ and $i^! K \in \mathcal{D}_Y^{\geq n+1}$.
In fact, the extension is

$$K = L^{[n]} := \tau^Y_{\leq n-1} j_* L = \tau_{\geq n+1} j_! L\,.$$

Proof Let K be an extension of L. By the canonical distinguished triangle (6.1(4), 6.3)

(11.5.1) $$i_* i^! K \longrightarrow K \longrightarrow j_* j^* K \xrightarrow{d'}$$

(where $j^* K = L$) and applying i^* and shifting the triangle, we obtain the distinguished triangle

(11.5.2) $$i^* K \longrightarrow i^* j_* L \longrightarrow i^! K[1] \longrightarrow .$$

This implies that the following properties are equivalent:
(a) $i^* K \in \mathcal{D}_Y^{\leq n+1}$ and $i^! K \in \mathcal{D}_Y^{\geq n+1}$ ($\Leftrightarrow i^! K[1] \in \mathcal{D}_Y^{\geq n}$).
(b) $i^! K[1] = \tau_{\geq n} i^* j_* L$.
(b') $i^* K = \tau_{\leq n-1} i^* j_* L$,
(c) $K = \tau^Y_{\leq n-1} i^* j_* L$,
(c') $K = \tau^Y_{\geq n+1} j_! L$.
In fact, the equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (b') are obvious, and the distinguished triangle

(11.5.3) $$K \longrightarrow j_* L \longrightarrow i_* i^! K[1] \longrightarrow$$

shows the equivalence between (b) and (c) , because of the distinguished triangle

(11.5.4) $$\tau^Y_{\leq 0} j_* L \longrightarrow j_* L \longrightarrow i_* \tau_{\geq 1} i^* j_* L \longrightarrow$$

coming from 11.3 (a) for $K = j_* L$.

Exercise: Identify the morphisms

$$\begin{tikzcd}
j_* L \arrow[r] \arrow[d] & i_* i^* j_* L \arrow[r] \arrow[d] & i_* \tau_{\geq n} i^* j_* L \arrow[d] \\
i_* i^! K[1] \arrow[r] \arrow[rr, bend right, "id"'] & i_* i^* i_* i^! K[1] \arrow[r] & i_* i^! K[1]
\end{tikzcd}$$

Dually note that

$$i^*j_*L \cong i^!j_!L[1]\,.$$

In fact, we have a distinguished triangle

$$j_!j^*K \longrightarrow K \longrightarrow i_*i^*K \longrightarrow\ , \tag{11.5.5}$$

hence for $K = j_*L$ a distinguished triangle

$$j_!L \longrightarrow j_*L \longrightarrow i_*i^*j_*L \longrightarrow\ , \tag{11.5.6}$$

and after shifting and applying $i^!$ a distinguished triangle

$$\begin{tikzcd}
{} \arrow[r] & i^!j_*L \arrow[r] \arrow[d, equal] & i^!i_*i^*j_*L \arrow[r] \arrow[d, equal, "\wr"] & i^!j_!L[1] \arrow[r] & i^!j_*L[1] \arrow[d, equal] \\
& 0 & i^*j_*L & & 0
\end{tikzcd}$$

and hence the claim.

The distinguished triangle

$$i_*i^*K[-1] \longrightarrow j_!L \longrightarrow K \longrightarrow \tag{11.5.7}$$

then implies (b') $\Leftrightarrow$ (c'), by the distinguished triangle

$$i_*\tau_{\leq 0}i^!j_!L \longrightarrow j_!L \longrightarrow \tau^Y_{\geq 1}j_!L \longrightarrow 0 \tag{11.5.8}$$

from 11.3 (b).

Uniqueness: (a) If $i^*K \in \mathcal{D}_Y^{\leq n-1}$, then we have isomorphisms

$$\mathrm{Hom}(K, \tau^Y_{\leq n-1}j_*L) \overset{(1)}{\cong} \mathrm{Hom}(K, j_*L) \overset{(2)}{\cong} \mathrm{Hom}(j^*K, L)\,,$$

where (1) comes from 11.2 (a), and (2) is the adjunction map. The chosen isomorphism $j^*K \xrightarrow{\sim} L$ gives a unique morphism $\beta : K \to \tau^Y_{\leq n-1}j_*L$, which is an isomorphism.

(b) The claims in (c) and (c') follow similarly.

Proposition 11.6 For B in $\mathcal{C}_U$ $(= \mathrm{Perv}(U, \overline{\mathbb{Q}_\ell}))$ we have:

(a) $p_{j_!}B = \tau^Y_{\geq 0}j_!B = \tau^Y_{\leq -2}j_*B \;=\; B[-1]$

(b) $j_{!*}B = \tau^Y_{\geq 1}j_!B = \tau^Y_{\leq -1}j_*B \;=\; B[0]$

(c) $p_{j_*}B = \tau^Y_{\geq 2}j_!B = \tau^Y_{\leq 0}j_*B \;=\; B[1]\,.$

Proof All nine terms are extensions of B to X ($\tau^Y_{\leq n}$ and $\tau^Y_{\geq n}$ only change on Y). In particular, one has $j^*C = B$ for them. Further we have $j_!B \in {}^p\mathcal{D}_X^{\leq 0}$ (t-right-exactness of $j_!$).

Consider the distinguished triangle

$$i_*\tau_{\leq -1}i^!j_!B \longrightarrow j_!B \longrightarrow \tau^Y_{\geq 0}j_!B \longrightarrow$$

(obtained in a similar way as (11.5.8)).

Here $i_*\tau_{\leq -1}i^! j_! B \in {}^p\mathcal{D}_X^{\leq -1}$, since i_* is t-exact. This shows that

$$ {}^p j_! B = {}^p H^0(j_! B) = {}^p\tau_{\geq 0} j_! B = \tau^Y_{\geq 0} j_! B\,, $$

and by 11.5 this is equal to $\tau^Y_{\leq -2} j_* B = B^{[-1]}$. The equalities in (c) follow in a similar way.

(b): By definition, $i^* B^{[0]} \in \mathcal{D}^{\leq -1}$, $i^! B \in \mathcal{D}_Y^{\geq 1}$, and by $j^* B = B \in \mathcal{C}_U$ we have

$$ B^{[0]} \in \mathcal{C}_X\,. $$

(TR4) and 7.6. (b) give the following commutative diagram for $K = j_! B$, with distinguished triangles in the two bottom rows

$$\begin{tikzcd}
 & {} & {} \\
 & \tau^Y_{\geq 1}K \arrow[u] \arrow[r, equal] & \tau^Y_{\geq 1}K \arrow[u, dotted] \\
i_*\tau_{\leq -1}i^! K \arrow[r] & K \arrow[u] \arrow[r] & \tau^Y_{\geq 1}K \arrow[u, dotted] \arrow[r] & {} \\
i_*\tau_{\leq -1}i^! K \arrow[u, equal] \arrow[r] & i_*\tau^Y_{\geq 0}K \arrow[u] \arrow[r] & i_*\, {}^pH^0 K \arrow[u, dotted] \arrow[r] & {}
\end{tikzcd}$$

and hence a distinguished triangle

$$ i_*\, {}^pH^0(i^! K) \longrightarrow \tau^Y_{\geq 0}K \longrightarrow \tau^Y_{\geq 1} \longrightarrow\ . $$

Here all three terms are in $\mathcal{C}_X$ (note that i_* is t-exact). Hence we obtain an exact sequence in $\mathcal{C}_X$

$$\begin{tikzcd}
0 \arrow[r] & i_*\, {}^pH^0(i^! K) \arrow[r] & B^{[-1]} \arrow[r] \arrow[d, equal] & B^{[0]} \arrow[r] & 0 \\
 & & {}^p j_! B & &
\end{tikzcd}$$

Correspondingly, 11.3 (a) gives a distinguished triangle and hence an exact sequence in $\mathcal{C}_X$

$$\begin{tikzcd}
0 \arrow[r] & \tau^Y_{\leq -1} j_* B \arrow[r] \arrow[d, equal] & \tau^Y_{\leq 0} j_* B \arrow[r] \arrow[d, equal] & i_*\, {}^pH^0 i^* j_* B \arrow[r] & 0 \\
 & B^{[0]} & B^{[1]} = {}^p j_* B & &
\end{tikzcd}$$

This shows that

$$ j_{!*}B = im({}^p j_! B \to {}^p j_* B) = B^{[0]}\,. $$

Theorem 11.7 Let $B \in \mathcal{C}_U = \mathrm{Perv}(U_{\text{ét}}, \overline{\mathbb{Q}_\ell})$. Then $j_{!*}B$ is the unique extension C of B in $\mathcal{C}_X = \mathrm{Perv}(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ with $i^* C$ in $\mathcal{D}_Y^{\leq -1}$ (i.e., in ${}^p\mathcal{D}_c^{\leq -1}(Y_{\text{ét}}, \overline{\mathbb{Q}_\ell})$) and $i^! C$ in $\mathcal{D}_Y^{\geq 1}$ (i.e., in ${}^p\mathcal{D}_C^{\geq 1}(Y_{\text{ét}}, \overline{\mathbb{Q}_\ell})$).

Proof This follows immediately from 11.6 (b) and 11.8. Correspondingly, one obtains characterizations of ${}^p j_! B$ and ${}^p j_* B$ with 11.6 and 11.8.

Lemma 11.8 (a) The compositions ${}^p j^* \, {}^p i_*$, ${}^p i^* \, {}^p j_!$ and ${}^p i^! \, {}^p j_*$ are zero, and for A in $\mathcal{C}_Y$ and B in $\mathcal{C}_U$ one has

$$\mathrm{Hom}({}^p j_! B,\, {}^p i_* A) \;=\; 0 \;=\; \mathrm{Hom}({}^p i_* A,\, {}^p j_* B)\,.$$

(b) For each C in $\mathcal{C}_X$ the sequences

$${}^p j_! \, {}^p j_* C \longrightarrow C \longrightarrow {}^p i_* \, {}^p i^* C \longrightarrow 0$$

and

$$0 \longrightarrow {}^p i_* \, {}^p i^! C \longrightarrow C \longrightarrow {}^p j_* \, {}^p j^* C$$

are exact.

(c) ${}^p i_*$, ${}^p j_!$ and ${}^p j_*$ are fully faithful, and the adjunction morphisms

$${}^p i^* \, {}^p i_* \longrightarrow id \longrightarrow {}^p i^! \, {}^p i_*$$

$${}^p j^* \, {}^p j_! \longrightarrow id \longrightarrow {}^p j_* \, {}^p j^*$$

are isomorphisms.

Proof This follows immediately from Theorem 9.3 and the corresponding properties for the functors $j^*, j_!$ etc. themselves. For example, we show

(a): The functors $i^!$ and j_* are both t-left exact by 9.3 (c). Hence we have

$${}^p i^! \, {}^p j_* = {}^p(i^! j_*) = 0\,,$$

since $i^! j_* = 0$.

(b): We have a distinguished triangle

$$j_! j^* C \longrightarrow C \longrightarrow i_* i^* C \longrightarrow\,,$$

and $j_!$ and j^* are t-right exact. Hence $j_! j^* C \in {}^p\mathcal{D}_X^{\leq 0}$, so that ${}^p H^1(j_! j^* C) = 0$. Now the claim follows from the long exact ${}^p H^{\cdot}$-sequence.

(c): $i^* i_* = id \Rightarrow {}^p i^* \, {}^p i_* = {}^p(i^* i_*) \xrightarrow{\sim} {}^p id = id$.

We are now ready to prove the following characterization of the intermediate direct image.

Theorem 11.9 Let $B \in \mathcal{C}_U = \mathrm{Perv}(U_{\text{ét}}, \overline{\mathbb{Q}_\ell})$. Then $j_{!*} B$ is the uniquely determined extension of B to $\mathcal{C}_X = \mathrm{Perv}(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ which does not have any subobject or quotient in the essential image $\overline{\mathcal{C}_Y}$ of $i_* = {}^p i_* : \mathcal{C}_Y \to \mathcal{C}_X$ for $\mathcal{C}_Y = \mathrm{Perv}(Y_{\text{ét}}, \overline{\mathbb{Q}_\ell})$.

Proof By definition, $j_{!*} B$ lies in $\mathcal{C}_X$. If C is an extension of B in $\mathcal{X}$, then $i^* C \in \mathcal{D}_Y^{\leq 0}$, and hence $i^* C \in \mathcal{D}_Y^{\leq -1}$ if and only if ${}^p i^* C = 0$.

Dually we have $i^!C \in \mathcal{D}_Y^{\geq 0}$, and $i^!C \in \mathcal{D}_Y^{\geq 1}$ if and only if ${}^p i^! C = 0$.

Now $C = j_{!*}B$ is just characterized by $i^*C \in \mathcal{D}_Y^{\leq -1}$ and $i^!C \in \mathcal{D}_Y^{\geq 0}$ (see Theorem 11,7).

On the other hand, by the exact sequence from 11.8 (b),

$$ {}^p j_! \, {}^p j_* C \longrightarrow C \longrightarrow {}^p i_* \, {}^p i^* C \longrightarrow 0 \, , $$

and since $\mathrm{Hom}({}^p j_! \, {}^p j_* C, \, {}^p i_* \, {}^p i^* C) = 0$ by 11.8 (a), ${}^p i_* \, {}^p i^* C$ is the maximal quotient of C which lies in $\overline{\mathcal{C}_Y}$.

Dually, ${}^p i_* \, {}^p i^! C$ is just the maximal sub-object of C which lies in $\overline{\mathcal{C}_Y}$.

11.10 Let again X be a scheme of finite type over a field k.

Let now $j : U \hookrightarrow X$ be the inclusion of a *locally closed* subscheme. Again we have the functors $j_*(= Rj_*), j^*, j_!$, between the categories $\mathcal{C}_X = \mathrm{Perv}(X_{et}, \overline{\mathbb{Q}_\ell})$ and $\mathcal{C}_U = \mathrm{Perv}(U_{et}, \overline{\mathbb{Q}_\ell})$, and the corresponding functors ${}^p j_*$, ${}^p j^*$, ${}^p j_!$ between them.

If $k : V \hookrightarrow U$ is a further such immersion, then one has similar transition morphisms, and transivity isomorphisms $(jk)_* = j_* k_*$, and similarly perverse variants ${}^p j$, ${}^p k$, with ${}^p(jk) = {}^p j \, {}^p k$ etc.

For an *open immersion* j the four functors are $j_*, j^*, j_!, j^*$, and for a *closed immersion* $j = i$ the four functors are $i_*, i^*, i_*, i^!$.

From this we get the following in the general case:

Lemma 11.11 (a) $(j_!, j^!), (j^*, j_*), ({}^p j^*, \, {}^p j_*)({}^p j_!, \, {}^p j^!)$ and $({}^p j^*, \, {}^p j_*)$ are pairs of adjoint functors.

(b) $j_!$ and j^* are p-right exact, and $j^!$ and j_* are t-left exact.

Proof: Obvious.

Definition 11.12 Let $B \in \mathcal{D}_U$. Then define

$$ j_{!*}B = im({}^p j!B \to {}^p j_* B) \in \mathcal{D}_X \, . $$

Here the arrow is induced by the canonical morphism $j!B \to j_*B$, which comes from $id : B \cong j^* j_! B \to B$ by adjunction.

Remark 11.13 We have a factorization

$$ \begin{array}{l} j_!B \longrightarrow {}^p j_! B = {}^p\tau_{\geq 0} j_! B \longrightarrow {}^p j_* B = {}^p\tau_{\leq 0} j_* B \longrightarrow j_* B \\ \Cap \\ {}^p D^{\leq 0} \end{array} $$

Theorem 11.14 (a) The category $\mathrm{Perv}(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ is noetherian and artinian: every object has finite length.

(b) Let $j : V \hookrightarrow X$ be the immersion of a locally closed irreducible subvariety of X for which $(V \times_k \overline{k})_{\mathrm{red}}$ is smooth, and let L be an irreducible smooth $\overline{\mathbb{Q}_\ell}$-sheaf on V. Then $j_{!*}$ $(L[\dim V])$‘ is a simple perverse sheaf on X.

(c) Every simple perverse sheaf on X is of this form.

Since the étale topology is not changed if we replace k by its perfect closure, we may assume that k is perfect. Then we can replace "$(V \times_k \overline{k})_{\mathrm{red}}$ is smooth" by "V is smooth". We start with

Lemma 11.15 Let X be irreducible and smooth, and let L be a smooth $\overline{\mathbb{Q}_\ell}$-sheaf on X. If $j : U \hookrightarrow X$ is a non-empty open subset of X, then for the perverse sheaf $F := L[\dim X]$ we have

$$F = j_{!*} j^* F\,.$$

Proof: By Theorem 11.7 we have to show that for each $x \in X \smallsetminus U$ we have

$$\mathcal{H}^i(i_x^* F) = 0 \quad \text{ for all } i \geq -\dim(x)\,,$$

and

$$\mathcal{H}^i(i_x^! F) = 0 \quad \text{ for all } i \leq -\dim(x)\,.$$

Since $\dim x < \dim X$, the first claim holds, because we obviously have

$$\mathcal{H}^i(i_x^* F) = \mathcal{H}^i(i_x^* L[\dim X]) = \mathcal{H}^{i+\dim X}(i_x^* L) = 0$$

for $i + \dim X > i + \dim(x) \geq 0$.

On the other hand, by cohomological purity we have

$$\mathcal{H}^i(i_x^! F) = 0 \text{ for } i < 2\operatorname{codim}(x) - \dim(X)\,,$$

and we have $-\dim(x) < -\dim(x) + (\dim X - \dim x) = 2\operatorname{codim}(x) - \dim X$.

Lemma 11.16 If X is irreducible and smooth and L is an irreducible smooth $\overline{\mathbb{Q}_\ell}$-sheaf on X, then the perverse sheaf $G = L[\dim X]$ is irreducible.

Proof Let $G \subseteq F$ be a subobject. There is a non-empty Zariski open subset $j : U \hookrightarrow X$ such that $j^* G$ is of the form $M[\dim X]$, for a smooth subsheaf M of $L|_U$. Since X is normal,the homomorphism of fundamental groups

$$\pi_1(U) \twoheadrightarrow \pi_1(X)$$

is surjective, because both groups are quotients of $\mathrm{Gal}(k(x)^{\mathrm{sep}}/k(x))$. Hence $L|_U$ remains irreducible (as a smooth sheaf). Hence we have $M = 0$ or $M = L|_U$.

If M=0, then $j^* G = 0$, and hence

$$G \cong i_* {}^p i^* G$$

for the closed immersion $i : Y = X \smallsetminus U \hookrightarrow X$ (compare 11.8 (b)).

If $M = L|_U$, then $j^*(F/G) = 0$, so that F/G has support in Y, i.e., it lies in the essential image of

$$i_* : \mathrm{Perv}(Y_{\text{ét}}, \overline{\mathbb{Q}_\ell}) \longrightarrow \mathrm{Perv}(X_{\text{ét}}, \overline{\mathbb{Q}_\ell}) .$$

On the other hand, by Lemma 11.15 we have

$$F = j_{!*}j^*F ,$$

and by Theorem 11.9 every subobject or quotient of F in $\mathrm{Perv}(X_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ with support in Y is trivial, which gives a contradiction.

Now we prove Theorem 11.14: To show that the perverse sheaf $j_{!*}(L[\dim V])$ in 11.14 (b) is simple, we use the factorization (where $\overline{V}$ is the closure of V in X)

$$j : \qquad V \underset{\text{open}}{\overset{\mu}{\hookrightarrow}} \overline{V} \underset{\text{closed}}{\overset{\kappa}{\hookrightarrow}} X$$

with an open immersion μ and a closed immersion κ. Then for $F \in \mathrm{Perv}(V_{\text{ét}}, \overline{\mathbb{Q}_\ell})$ we have

$$\begin{array}{rcl} j_{!*}(F) & = & \mathrm{Im}\big({}^pj_!F \longrightarrow {}^pj_*F \big) \\ & & \phantom{\mathrm{Im}(}\Big\| \qquad\qquad\quad \Big\| \\ & & \phantom{\mathrm{Im}}\kappa_*\,{}^p\mu_!F \qquad \kappa_*\,{}^p\mu_*F \\ & = & \kappa_*\mathrm{Im}({}^p\mu_!F \to {}^p\mu_*F) \\ & = & \kappa_*\mu_{!*}F , \end{array}$$

since $\kappa_* = {}^p\kappa_*$ is exact.

Step 1: If F is simple (as a perverse sheaf on V), then the same holds for $\mu_{!*}F$: If $G \subseteq \mu_{!*}F$ is a perverse subsheaf, then $\mu^*G \subseteq F$ is a perverse subsheaf; hence

$$\mu^*G = 0 \qquad \text{or} \quad \mu^*(\mu_{!*}F/G) = 0 .$$

Since F has no non-trivial subobjects or quotients with support in $\overline{V} \smallsetminus V$, we deduce that $G = 0$ or $G = \mu_{!*}F$.

Step 2: If $F \in \mathrm{Perv}(\overline{V_{\text{ét}}}, \overline{\mathbb{Q}_\ell})$ is simple, then κ_*F is simple, too:

If $G \subseteq \kappa_*F$ and $\rho : X \smallsetminus \overline{V} \hookrightarrow X$ is the open immersion, then $\rho^*G \subseteq \rho^*\kappa_*F = 0$, and hence $G \cong \kappa_*\,{}^p\kappa^*G$ for the perverse sheaf ${}^p\kappa^*G$ on $\overline{V}$. By applying ${}^p\kappa^*$ we have ${}^p\kappa^*G = 0$ or ${}^p\kappa^*G = F$, and hence also $G = 0$ or $G = \kappa_*F$.

Combining this with Lemma 11.16, we see that $j_{!*}(L[\dim V])$ is simple.

It remains to show that any perverse sheaf F on X has a finite composition series with quotients of the form $j_{!*}(L[\dim V])$ as in Theorem 11.14 (b).

In any case, there is a dense open $U \overset{j}{\hookrightarrow} X$ such that U is smooth and equidimensional, and $F|_U$ is of the form $L[\dim U]$ for a smooth sheaf L on U. By noetherian induction we can assume that the claim holds on the closed complement $i : Y \hookrightarrow X$ of U.

Now we have the following

Lemma 11.17 Let C be a perverse sheaf on X.

(a) There are exact sequences of perverse sheaves

$$0 \longrightarrow {}^p i_* \, {}^p H^{-1} i^* C \longrightarrow {}^p j_! \, {}^p j^* C \longrightarrow C \longrightarrow {}^p i_* \, {}^p i^* C \longrightarrow 0$$

$$0 \longrightarrow {}^p i_* \, {}^p i^! C \longrightarrow C \longrightarrow {}^p j_* \, {}^p j^* C \longrightarrow {}^p i_* \, {}^p H^1 i^! C \longrightarrow 0$$

(b) C is in the essential image $\overline{\mathrm{Perv}(Y)}$ of

$${}^p i_* : \mathrm{Perv}(Y) \longrightarrow \mathrm{Perv}(X)$$

if and only if ${}^p j^* C = 0$.

(c) $\overline{\mathrm{Perv}(Y)}$ is a Serre subcategory of $\mathrm{Perv}(X)$, and ${}^p j^*$ induces an equivalence of categories

$$\mathrm{Perv}(X)/\overline{\mathrm{Perv}(Y)} \xrightarrow{\sim} \mathrm{Perv}(U)\,.$$

Proof (a) follows immediately from the long ${}^p H^{\cdot}$-sequences associated to the distinguished triangles in 6.1 (4).

(b) is obvious by the fact that ${}^p j^* \, {}^p i_* = 0$ and each of the sequences in (a).

(c) The first part follows from (b) and, the exactness of ${}^p j^*$. For the second part see Astérisque 100, Prop. 1.4.18.

Now we continue with the proof of 11.14. From Lemma 11.13 we see that there is a filtration by perverse sheaves

$$F_2 \subseteq F_1 \subseteq F$$

such that $F_1/F_2 = j_{!*} j^* F = j_{!*} L[\dim U]$, and such that F/F_1 and F_2 are in the essential image of ${}^p i_*$. If L is simple, then we are done.

If $L \twoheadrightarrow L'$ is a simple quotient with kernel $L'' \neq 0$, then the morphism

$$j_{!*} L[\dim U] \longrightarrow\!\!\!\!\!\rightarrow j_{!*} L'[\dim U]$$

is surjective (since ${}^p j_!$ is right exact), and the kernel F'' is an extension of $L''[\dim U]$. The claim now follows by induction on the length of L. This finishes the proof of Theorem 11.14.