Universität Regensburg
NWF I - Mathematik
Prof. Dr. U. Jannsen, Dr. P. Schlagheck, Dr. I. Kausz

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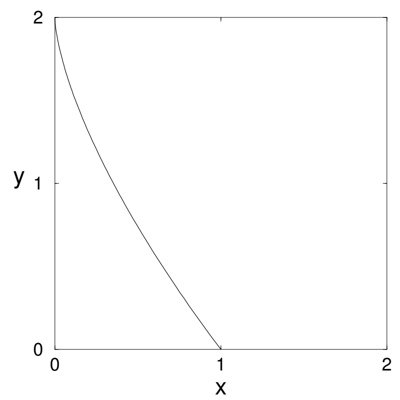
Klausur zur Analysis II für Physiker - Musterlösung

1) $\gamma : [0,1] \to \mathbb{R}^2,\ t \mapsto (t^3, 2-2t^2)$.

a)
$$\dot\gamma(t) = (3t^2, -4t)$$
$$\text{Länge } L = \int_0^1 \|\dot\gamma(t)\| \, dt = \int_0^1 \sqrt{9t^4+16t^2}\,dt$$
$$= \int_0^1 t\sqrt{9t^2+16}\,dt \overset{u=9t^2+16}{\underset{du=18tdt}{=}} \frac{1}{18}\int_{16}^{25} \sqrt{u}\,du$$
$$= \frac{1}{18}\cdot\frac{2}{3}[u^{3/2}]_{16}^{25} = \frac{1}{27}(125-64) = \frac{61}{27}$$

b)
$$\dot\gamma(t_0) = (3t_0^2, -4t_0) = \alpha(1,-2)$$
$$\Rightarrow\ 3t_0^2 = 2t_0 \Rightarrow t_0 = \frac{2}{3}$$
$$\Rightarrow\ x_0 = \left(\frac{2}{3}\right)^3 = \frac{8}{27},\ y_0 = 2-2\cdot\left(\frac{2}{3}\right)^2 = \frac{10}{9}$$

c)



d)
$$\text{Fläche} = \int_0^1 dx \int_0^{y(x)} dy = \int_0^1 y(x)dx$$
$$y(x) = 2-2\big(t(x)\big)^2 \quad \text{mit} \quad x = t^3 \Rightarrow t = x^{1/3}$$
$$= 2-2x^{2/3}$$
$$\Rightarrow\ \text{Fläche} = 2-2\cdot\frac{3}{5} = \frac{4}{5}$$

2) $f : \mathbb{R}^3 \to \mathbb{R}^3,\ (r,\theta,\varphi) \mapsto (ar\ \sin\ \theta\ \cos\ \varphi,\ br\ \sin\ \theta\ \sin\ \varphi,\ cr\ \cos\ \theta).$

a) $$J = \left(\frac{\partial F}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi}\right) = \begin{pmatrix} a\ \sin\theta\ \cos\varphi & ar\cos\theta\ \cos\varphi & -ar\ \sin\theta\ \sin\varphi \\ b\ \sin\theta\ \sin\varphi & br\cos\theta\ \sin\varphi & br\ \sin\theta\ \cos\varphi \\ c\ \cos\theta & -cr\ \sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \underbrace{\begin{pmatrix} \sin\theta\ \cos\varphi & \cos\theta\ \cos\varphi & -\sin\varphi \\ \sin\theta\ \sin\varphi & \cos\theta\ \sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}}_{\text{orthogonal (wie bei Kugelkoordinaten)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r\sin\theta \end{pmatrix}$$

$\det J = abc\ r^2 \sin\theta = 0 \Leftrightarrow r = 0$ oder $\sin\theta = 0$
$\Rightarrow F$ umkehrbar für $r \notin 0$ und $\theta \notin k\pi, k \in \mathbb{Z}$.

b)
$$\begin{aligned} x &= ar\ \sin\theta\ \cos\varphi & \qquad \tfrac{x}{a} &= r\ \sin\theta\ \cos\varphi \\ y &= br\ \sin\theta\ \sin\varphi & \Rightarrow \quad \tfrac{y}{b} &= r\ \sin\theta\ \sin\varphi \\ z &= cr\ \cos\theta & \tfrac{z}{c} &= r\ \cos\theta \end{aligned}$$

$\Rightarrow$ Wähle wie bei Kugelkoordinaten

$$r = \sqrt{\left(\tfrac{x}{a}\right)^2 + \left(\tfrac{y}{b}\right)^2 + \left(\tfrac{z}{c}\right)^2}$$

$$\theta = \arctan \frac{\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}}{\frac{z}{c}} + \pi$$

$$\varphi = \arctan \tfrac{ya}{xb} + \pi$$

$$g:\ U \subset \mathbb{R}^3 \to \mathbb{R}^3, (x,y,z) \mapsto \begin{pmatrix} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2} \\ \arctan \frac{\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}}{z/c} + \pi \\ \arctan \frac{ya}{xb} + \pi \end{pmatrix}$$

$\Rightarrow\ f\big(g(-a,-b,-c)\big) = f(\sqrt{3}, \pi\text{-}\arctan\sqrt{2}, \tfrac{5}{4}\pi) = (-a,-b,-c)$

3) a) $f(x,y,z) = 2y^3 + x^2 + 3y^2 + 2z^2 - 2yz + x - y$

grad $f = (2x+1, 6y^2 + 6y - 2z - 1, 4z - 2y) = (0,0,0)$

$\Rightarrow\ 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$
$$\left.\begin{aligned} 6y^2 + 6y - 2z - 1 &= 0 \\ 4z - 2y = 0 \Rightarrow y &= 2z \end{aligned}\right\} \Rightarrow\ 6y^2 + 5y - 1 = 0$$

$\Rightarrow\ y_{1,2} = \frac{-5\pm\sqrt{25+24}}{12} = \frac{-5\pm 7}{12} = -1$ oder $\frac{1}{6}$

$\Rightarrow$ Kritische Punkte: $(x_1, y_1, z_1) = \left(-\frac{1}{2}, -1, -\frac{1}{2}\right)$ mit $f(x_1, y_1, z_1) = \frac{5}{4}$

$$(x_1, y_2, z_2) = \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{12}\right) f(x_2, y_2, z_2) = -\frac{109}{216}$$

Hesse-Matrix: $H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12y+6 & -2 \\ 0 & -2 & 4 \end{pmatrix}$

$H(x_1, y_1, z_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & 4 \end{pmatrix}$ indefinit, da $\det \begin{pmatrix} -6 & -2 \\ -2 & 4 \end{pmatrix} = -20 < 0$

d.h. ein Eigenwert < 0 und 2 Eigenwerte > 0

$H(x_2, y_2, z_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 4 \end{pmatrix}$ positiv definit, da $\det \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix} > 0$

und Spur $\begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix} > 0 \Rightarrow$ alle EW > 0

$\Rightarrow$ (x_1, y_1, z_1) Sattelpunkt, (x_2, y_2, z_2) Minimum.

3) b) $f(x, y, z) = x^2 + y^2 + 3z^2 + xy$

$\rightarrow$ Suche kritische Punkte von $g(x, y, z) = f(x, y, z) - \lambda(x^2 + y^2 + z^2 - 1)$

$$0 = \frac{\partial g}{\partial x} = 2x + y - 2\lambda x \qquad (I)$$

$$0 = \frac{\partial g}{\partial y} = 2y + x - 2\lambda y \qquad (II)$$

$$0 = \frac{\partial g}{\partial Z} = 6z - 2\lambda z = (6 - 2\lambda)z \Rightarrow z = 0 \text{ oder } 2\lambda = 6$$

1. Fall: $z = 0$:

$$0 = (I) \cdot y - (II) \cdot x = y^2 - x^2 \Rightarrow x = \pm y$$

2. Fall: $2\lambda = 6$

$$\left.\begin{array}{rl} (I) & 0 = y - 4x \\ (II) & 0 = x - 4y \end{array}\right\} \Rightarrow x = y = 0$$

Zusatzbedingung: $x^2 + y^2 + z^2 = 1$

$$\Rightarrow \quad \text{kritische Punkte:} \quad \underbrace{\begin{matrix}(0, & 0, & 1)\\(0, & 0, & -1)\end{matrix}}_{} \quad \underbrace{\begin{matrix}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\\(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)\end{matrix}}_{} \quad \underbrace{\begin{matrix}(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)\\(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)\end{matrix}}_{}$$

$$\text{Funktionswerte von } f: \quad 3 \qquad \tfrac{3}{2} \qquad \tfrac{1}{2}$$

$\Rightarrow$ Maximalwert von f : 3

Minimalwert von f : $\frac{1}{2}$

4) a)
$$\int_V (x^2+y^2)dxdydz = \int_{-L}^{L} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz (x^2+y^2)\Theta(R^2-y^2-z^2)$$

$$\overset{y=r\cos\varphi}{\underset{z=r\sin\varphi}{=}} \int_{-L}^{L} dx \int_0^R rdr \int_0^{2\pi} d\varphi (x^2+r^2\cos^2\varphi)$$

$$= 2\pi \int_{-L}^{L} x^2 dx \int_0^R rdr + 2L\int_0^R r^3 dr \int_0^{2\pi} \cos^2\varphi d\varphi$$

$$= 2\pi \cdot 2\frac{L^3}{3}\ \frac{1}{2}R^2 + 2L\ \frac{R^4}{4}\left[\frac{\varphi}{2}+\frac{\sin 2\varphi}{4}\right]_0^{2\pi}$$

$$= 2\pi L R^2\left(\frac{L^2}{3}+\frac{R^2}{4}\right).$$

b)
$$\int_V (x^2+y^2)dxdydz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)\Theta(R^2-x^2-y^2-\lambda^2 z^2)dxdydz$$

$$\overset{\zeta=\lambda z}{=} \frac{1}{\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x^2+y^2)\Theta(R^2-x^2-y^2-\zeta^2)dxdyd\zeta$$

Kugelkoordinaten:
$$x = r\sin\theta\ \cos\varphi$$
$$y = r\sin\theta\ \sin\varphi$$
$$\zeta = r\cos\theta$$

$$\Rightarrow \int_V (x^2+y^2)dxdydz = \frac{1}{\lambda}\int_0^R r^2 dr\int_0^{\pi}\sin\theta\ d\theta\int_0^{2\pi} d\varphi\ r^2\ \sin^2\theta$$

$$= \frac{2\pi}{\lambda}\int_0^R r^4 dr \underbrace{\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta}_{\overset{u=\cos\theta}{=}\int_{-1}^{1}(1-u^2)du = 2-\frac{2}{3}=\frac{4}{3}}$$

$$= \frac{8\pi}{15}\ \frac{R^5}{\lambda}$$

$$= \frac{2}{5}R^2\cdot\frac{4\pi}{3}\ \frac{R^3}{\lambda}$$

5) a) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$\det(\lambda E - A) = (\lambda - 2)(\lambda + 2) + 5 = \lambda^2 - 4 + 5 = \lambda^2 + 1$

$\Rightarrow$ Eigenwerte $\lambda_\pm = \pm i$

$A - \lambda_+ E = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \Rightarrow$ Eigenvektor $v_+ = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$A - \lambda_- E = \begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix} \Rightarrow$ Eigenvektor $v_- = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

$B = \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix}$, $B^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -2+i \\ -1 & 2+i \end{pmatrix}$

$e^{At} = B \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} B^{-1} = \frac{1}{2i} \begin{pmatrix} 2+i & 2-i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & -2+i \\ -1 & 2+i \end{pmatrix}$

$= \begin{pmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{pmatrix}$

$\Rightarrow$ Fundamentalsystem: $y(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} y_1^{(0)} + \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix} y_2^{(0)}$

b) $0 = Ay_c + b \Rightarrow y_c = -A^{-1}b = -\begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c) Allgemeine Lösung um $y' = Ay + b$:

$y(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix} y_1^{(0)} + \begin{pmatrix} -5\sin t \\ \cos t - 2\sin t \end{pmatrix} y_2^{(0)} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Anfangsbedingungen: $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = y(0) = \begin{pmatrix} y_1^{(0)} + 2 \\ y_2^{(0)} + 1 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1^{(0)} \\ y_2^{(0)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow y(t) = \begin{pmatrix} 2\cos t - \sin t + 2 \\ \cos t + 1 \end{pmatrix}$.

6) $y'(x) = \frac{xy(x)+1}{1-x^2} = \frac{x}{1-x^2} y(x) + \frac{1}{1-x^2}$.

Lösung der homogenen DGL $y' = \frac{x}{1-x^2} y$:

$\int \frac{dy}{y} = \int \frac{x}{1-x^2} dx$

$\ell n\ y = -\frac{1}{2}\ell n(1-x^2) + c$

$\Rightarrow\ y = \frac{y_0}{\sqrt{1-x^2}}$

Variation der Konstanten: $y_0 = y_0(x)$

$\Rightarrow\ y' = \frac{y_o'}{\sqrt{1-x^2}} + y_0 \frac{d}{dx}\frac{1}{\sqrt{1-x^2}} = \frac{x}{1-x^2}\ \frac{y_0}{\sqrt{1-x^2}} + \frac{1}{1-x^2}$

$\Rightarrow\ y_0'(x) = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow\ y_0(x) = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin\ x + c$

$\Rightarrow$ allgemeine Lösung: $y(x) = \frac{\arcsin\ x + c}{\sqrt{1-x^2}}$

speziell für $y(0) = 0$ folgt: $c = 0 \Rightarrow y = \frac{\arcsin\ x}{\sqrt{1-x^2}}$.

7) $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

Parametrisierung von M :

$$\begin{aligned}
\gamma &: \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto (\gamma_x, \gamma_y, \gamma_z) = (x, y, e^{-x^2-y^2}) \\
\gamma^*\omega &= \gamma_x d\gamma_y \wedge d\gamma_z + \gamma_y d\gamma_z \wedge d\gamma_x + \gamma_z d\gamma_x \wedge d\gamma_y \\
d\gamma_z &= e^{-x^2-y^2}(-2xdx - 2ydy) \\
\Rightarrow \gamma^*\omega &= -2x^2 e^{-x^2-y^2} dy \wedge dx - 2y^2 e^{-x^2-y^2} dy \wedge dx + e^{-x^2-y^2} dx \wedge dy \\
&= (1 + 2x^2 + 2y^2) e^{-x^2-y^2} dx \wedge dy \\
\int_M \omega &= \int_{\mathbb{R}^2} \gamma^*\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + 2x^2 + 2y^2) e^{-x^2-y^2} dxdy \\
&\overset{\substack{x=r\cos\varphi \\ y=r\sin\varphi}}{=} \int_0^{\infty} rdr \int_0^{2\pi} d\varphi (1 + 2r^2) e^{-r^2} dr \\
&\overset{u=r^2}{=} 2\pi \int_0^{\infty} \tfrac{1}{2}(1 + 2u) e^{-u} du = \pi \underbrace{\left[-e^{-u}\right]_0^{\infty}}_{=1} + 2\pi \int_0^{\infty} ue^{-u} du \\
\int_0^{\infty} ue^{-u} du &= \underbrace{\left[-ue^{-u}\right]_0^{\infty}}_{=0} + \int_0^{\infty} e^{-u} du = 1 \\
&\Rightarrow \int_M \omega = 3\pi
\end{aligned}$$

8) Satz von Stokes: $\int\limits_{\partial U} \alpha = \int\limits_{U} d\alpha$

a)

$$\begin{aligned}
\alpha &= [\cos y + xf(x^2+y^2)]dx + [\sin x + yf(x^2+y^2)]dy \\
&\quad \text{mit } f : \mathbb{R} \to \mathbb{R}, r \mapsto r^{101} \\
d\alpha &= [-\sin y dy + f'(x^2+y^2)(2x^2dx + 2xydy)] \wedge dx \\
&+ [\cos x dx + f'(x^2+y^2)(2y^2dy + 2xydx)] \wedge dy \\
&= -\sin y dy \wedge dx + 2xyf'(x^2+y^2)dy \wedge dx \\
&+ \cos x dx \wedge dy + 2xyf'(x^2+y^2)dx \wedge dy \\
&= (\cos x + \sin y)dx \wedge dy \\
\Rightarrow \int\limits_{\partial U} \alpha &= \int\limits_{U} d\alpha = \int\limits_0^{\frac{\pi}{4}}\int\limits_0^{\frac{\pi}{4}} (\cos x + \sin y)dxdy = \tfrac{\pi}{4}[\sin x]_0^{\frac{\pi}{4}} - \tfrac{\pi}{4}[\cos y]_0^{\frac{\pi}{4}} \\
&= \tfrac{\pi}{4} \cdot \left(\tfrac{\sqrt{2}}{2} - 0\right) - \tfrac{\pi}{4}\left(\tfrac{\sqrt{2}}{2} - 1\right) = \tfrac{\pi}{4}
\end{aligned}$$

b)

$$\begin{aligned}
\alpha &= dg \text{ mit } g : \mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \sin[xy \arctan(x-y)] \\
\int\limits_{\partial U} \alpha &= \int\limits_{U} d\alpha = \int\limits_{U} ddg = 0 \text{ da } ddg = d^2g = 0 \\
&\quad \text{für zweimal stetig differenzierbare Funktion } g
\end{aligned}$$