

Universität Regensburg

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NWF I - Mathematik

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Klausur zur Analysis II für Physiker - Musterlösung

1) $\gamma : [0, 1] \rightarrow \mathbb{R}^2, t \mapsto (t^3, 2 - 2t^2).$

a) $\dot{\gamma}(t) = (3t^2, -4t)$

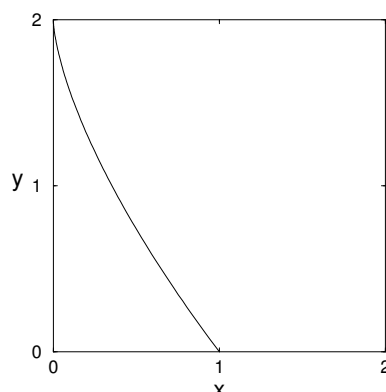
$$\begin{aligned} \text{Länge } L &= \int_0^1 \|\dot{\gamma}(t)\| dt = \int_0^1 \sqrt{9t^4 + 16t^2} dt \\ &= \int_0^1 t \sqrt{9t^2 + 16} dt \quad \begin{array}{l} u=9t^2+16 \\ du=18tdt \end{array} \quad \frac{1}{18} \int_{16}^{25} \sqrt{u} du \\ &= \frac{1}{18} \cdot \frac{2}{3} [u^{3/2}]_{16}^{25} = \frac{1}{27} (125 - 64) = \frac{61}{27} \end{aligned}$$

b) $\dot{\gamma}(t_0) = (3t_0^2, -4t_0) = \alpha(1, -2)$

$$\Rightarrow 3t_0^2 = 2t_0 \Rightarrow t_0 = \frac{2}{3}$$

$$\Rightarrow x_0 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}, \quad y_0 = 2 - 2 \cdot \left(\frac{2}{3}\right)^2 = \frac{10}{9}$$

c)



d) Fläche $= \int_0^1 dx \int_0^{y(x)} dy = \int_0^1 y(x) dx$

$$y(x) = 2 - 2(t(x))^2 \quad \text{mit} \quad x = t^3 \Rightarrow t = x^{1/3}$$

$$= 2 - 2x^{2/3}$$

$$\Rightarrow \text{Fläche} = 2 - 2 \cdot \frac{3}{5} = \frac{4}{5}$$

2) $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3, (r, \theta, \varphi) \mapsto (ar \sin \theta \cos \varphi, br \sin \theta \sin \varphi, cr \cos \theta)$.

$$\begin{aligned} \text{a) } J &= \left(\frac{\partial F}{\partial r}, \frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial \varphi} \right) = \begin{pmatrix} a \sin \theta \cos \varphi & ar \cos \theta \cos \varphi & -ar \sin \theta \sin \varphi \\ b \sin \theta \sin \varphi & br \cos \theta \sin \varphi & br \sin \theta \cos \varphi \\ c \cos \theta & -cr \sin \theta & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \underbrace{\begin{pmatrix} \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi \\ \sin \theta \sin \varphi & \cos \theta \sin \varphi & \cos \varphi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix}}_{\text{orthogonal (wie bei Kugelkoordinaten)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \sin \theta \end{pmatrix} \end{aligned}$$

$$\det J = abc r^2 \sin \theta = 0 \Leftrightarrow r = 0 \text{ oder } \sin \theta = 0$$

$\Rightarrow F$ umkehrbar für $r \notin 0$ und $\theta \notin k\pi, k \in \mathbb{Z}$.

$$\begin{aligned} \text{b) } x &= ar \sin \theta \cos \varphi & \frac{x}{a} &= r \sin \theta \cos \varphi \\ y &= br \sin \theta \sin \varphi & \Rightarrow \frac{y}{b} &= r \sin \theta \sin \varphi \\ z &= cr \cos \theta & \frac{z}{c} &= r \cos \theta \end{aligned}$$

\Rightarrow Wähle wie bei Kugelkoordinaten

$$r = \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2}$$

$$\theta = \arctan \frac{\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}}{\frac{z}{c}} + \pi$$

$$\varphi = \arctan \frac{ya}{xb} + \pi$$

$$g: U \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, (x, y, z) \mapsto \begin{pmatrix} \sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2} \\ \arctan \frac{\sqrt{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2}}{z/c} + \pi \\ \arctan \frac{ya}{xb} + \pi \end{pmatrix}$$

$$\Rightarrow f(g(-a, -b, -c)) = f(\sqrt{3}, \pi - \arctan \sqrt{2}, \frac{5}{4}\pi) = (-a, -b, -c)$$

3) a) $f(x, y, z) = 2y^3 + x^2 + 3y^2 + 2z^2 - 2yz + x - y$

$$\text{grad } f = (2x + 1, 6y^2 + 6y - 2z - 1, 4z - 2y) = (0, 0, 0)$$

$$\Rightarrow \left. \begin{aligned} 2x + 1 &= 0 \Rightarrow x = -\frac{1}{2} \\ 6y^2 + 6y - 2z - 1 &= 0 \\ 4z - 2y &= 0 \Rightarrow y = 2z \end{aligned} \right\} \Rightarrow 6y^2 + 5y - 1 = 0$$

$$\Rightarrow y_{1,2} = \frac{-5 \pm \sqrt{25+24}}{12} = \frac{-5 \pm 7}{12} = -1 \text{ oder } \frac{1}{6}$$

⇒ Kritische Punkte: $(x_1, y_1, z_1) = \left(-\frac{1}{2}, -1, -\frac{1}{2}\right)$ mit $f(x_1, y_1, z_1) = \frac{5}{4}$

$$(x_2, y_2, z_2) = \left(-\frac{1}{2}, \frac{1}{6}, \frac{1}{12}\right) f(x_2, y_2, z_2) = -\frac{109}{216}$$

$$\text{Hesse-Matrix: } H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12y + 6 & -2 \\ 0 & -2 & 4 \end{pmatrix}$$

$$H(x_1, y_1, z_1) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -2 \\ 0 & -2 & 4 \end{pmatrix} \text{ indefinit, da } \det \begin{pmatrix} -6 & -2 \\ -2 & 4 \end{pmatrix} = -20 < 0$$

d.h. ein Eigenwert < 0 und 2 Eigenwerte > 0

$$H(x_2, y_2, z_2) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 4 \end{pmatrix} \text{ positiv definit, da } \det \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix} > 0$$

$$\text{und Spur } \begin{pmatrix} 8 & -2 \\ -2 & 4 \end{pmatrix} > 0 \Rightarrow \text{alle EW} > 0$$

⇒ (x_1, y_1, z_1) Sattelpunkt, (x_2, y_2, z_2) Minimum.

3) b) $f(x, y, z) = x^2 + y^2 + 3z^2 + xy$

→ Suche kritische Punkte von $g(x, y, z) = f(x, y, z) - \lambda(x^2 + y^2 + z^2 - 1)$

$$0 = \frac{\partial g}{\partial x} = 2x + y - 2\lambda x \quad (I)$$

$$0 = \frac{\partial g}{\partial y} = 2y + x - 2\lambda y \quad (II)$$

$$0 = \frac{\partial g}{\partial z} = 6z - 2\lambda z = (6 - 2\lambda)z \Rightarrow z = 0 \text{ oder } 2\lambda = 6$$

1. Fall: $z = 0$:

$$0 = (I) \cdot y - (II) \cdot x = y^2 - x^2 \Rightarrow x = \pm y$$

2. Fall: $2\lambda = 6$

$$\left. \begin{array}{l} (I) \quad 0 = y - 4x \\ (II) \quad 0 = x - 4y \end{array} \right\} \Rightarrow x = y = 0$$

Zusatzbedingung: $x^2 + y^2 + z^2 = 1$

$$\Rightarrow \text{kritische Punkte: } \underbrace{\begin{pmatrix} 0, & 0, & 1 \\ 0, & 0, & -1 \end{pmatrix}}_3 \quad \underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}}, & 0 \\ -\frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}}, & 0 \end{pmatrix}}_{\frac{3}{2}} \quad \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{2}}, & 0 \\ \frac{1}{\sqrt{2}}, & -\frac{1}{\sqrt{2}}, & 0 \end{pmatrix}}_{\frac{1}{2}}$$

Funktionswerte von f :

$$\Rightarrow \text{Maximalwert von } f : \quad 3$$

$$\text{Minimalwert von } f : \quad \frac{1}{2}$$

$$4) \text{ a) } \int_V (x^2 + y^2) dx dy dz = \int_{-L}^L dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz (x^2 + y^2) \Theta(R^2 - y^2 - z^2)$$

$$\stackrel{\substack{y=r \cos \varphi \\ z=r \sin \varphi}}{=} \int_{-L}^L dx \int_0^R r dr \int_0^{2\pi} d\varphi (x^2 + r^2 \cos^2 \varphi)$$

$$= 2\pi \int_{-L}^L x^2 dx \int_0^R r dr + 2L \int_0^R r^3 dr \int_0^{2\pi} \cos^2 \varphi d\varphi$$

$$= 2\pi \cdot 2 \frac{L^3}{3} \cdot \frac{1}{2} R^2 + 2L \frac{R^4}{4} \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4} \right]_0^{2\pi}$$

$$= 2\pi LR^2 \left(\frac{L^2}{3} + \frac{R^2}{4} \right).$$

$$\text{b) } \int_V (x^2 + y^2) dx dy dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2) \Theta(R^2 - x^2 - y^2 - \lambda^2 z^2) dx dy dz$$

$$\stackrel{\zeta=\lambda z}{=} \frac{1}{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 + y^2) \Theta(R^2 - x^2 - y^2 - \zeta^2) dx dy d\zeta$$

$$\text{Kugelkoordinaten: } \begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ \zeta &= r \cos \theta \end{aligned}$$

$$\Rightarrow \int_V (x^2 + y^2) dx dy dz = \frac{1}{\lambda} \int_0^R r^2 dr \int_0^{\pi} \sin \theta d\theta \int_0^{2\pi} d\varphi r^2 \sin^2 \theta$$

$$= \frac{2\pi}{\lambda} \int_0^R r^4 dr \underbrace{\int_0^{\pi} \sin \theta (1 - \cos^2 \theta) d\theta}_{\substack{u=\cos \theta \\ = \int_{-1}^1 (1-u^2) du = 2 - \frac{2}{3} = \frac{4}{3}}}$$

$$= \frac{8\pi}{15} \frac{R^5}{\lambda}$$

$$= \frac{2}{5} R^2 \cdot \frac{4\pi}{3} \frac{R^3}{\lambda}$$

5) a) $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

$$\det(\lambda E - A) = (\lambda - 2)(\lambda + 2) + 5 = \lambda^2 - 4 + 5 = \lambda^2 + 1$$

\Rightarrow Eigenwerte $\lambda_{\pm} = \pm i$

$$A - \lambda_+ E = \begin{pmatrix} 2 - i & -5 \\ 1 & -2 - i \end{pmatrix} \Rightarrow \text{Eigenvektor } v_+ = \begin{pmatrix} 2 + i \\ 1 \end{pmatrix}$$

$$A - \lambda_- E = \begin{pmatrix} 2 + i & -5 \\ 1 & -2 + i \end{pmatrix} \Rightarrow \text{Eigenvektor } v_- = \begin{pmatrix} 2 - i \\ 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 + i & 2 - i \\ 1 & 1 \end{pmatrix}, \quad B^{-1} = \frac{1}{2i} \begin{pmatrix} 1 & -2 + i \\ -1 & 2 + i \end{pmatrix}$$

$$e^{At} = B \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} B^{-1} = \frac{1}{2i} \begin{pmatrix} 2 + i & 2 - i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} 1 & -2 + i \\ -1 & 2 + i \end{pmatrix}$$

$$= \begin{pmatrix} \cos t + 2 \sin t & -5 \sin t \\ \sin t & \cos t - 2 \sin t \end{pmatrix}$$

$$\Rightarrow \text{Fundamentalsystem: } y(t) = \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix} y_1^{(0)} + \begin{pmatrix} -5 \sin t \\ \cos t - 2 \sin t \end{pmatrix} y_2^{(0)}$$

b) $0 = Ay_c + b \Rightarrow y_c = -A^{-1}b = - \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

c) Allgemeine Lösung um $y' = Ay + b$:

$$y(t) = \begin{pmatrix} \cos t + 2 \sin t \\ \sin t \end{pmatrix} y_1^{(0)} + \begin{pmatrix} -5 \sin t \\ \cos t - 2 \sin t \end{pmatrix} y_2^{(0)} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{Anfangsbedingungen: } \begin{pmatrix} 4 \\ 2 \end{pmatrix} = y(0) = \begin{pmatrix} y_1^{(0)} + 2 \\ y_2^{(0)} + 1 \end{pmatrix} \Rightarrow \begin{pmatrix} y_1^{(0)} \\ y_2^{(0)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow y(t) = \begin{pmatrix} 2 \cos t - \sin t + 2 \\ \cos t + 1 \end{pmatrix}.$$

6) $y'(x) = \frac{xy(x)+1}{1-x^2} = \frac{x}{1-x^2}y(x) + \frac{1}{1-x^2}$.

Lösung der homogenen DGL $y' = \frac{x}{1-x^2}y$:

$$\int \frac{dy}{y} = \int \frac{x}{1-x^2} dx$$

$$\ln y = -\frac{1}{2}\ln(1-x^2) + c$$

$$\Rightarrow y = \frac{y_0}{\sqrt{1-x^2}}$$

Variation der Konstanten: $y_0 = y_0(x)$

$$\Rightarrow y' = \frac{y_0'}{\sqrt{1-x^2}} + y_0 \frac{d}{dx} \frac{1}{\sqrt{1-x^2}} = \frac{x}{1-x^2} \frac{y_0}{\sqrt{1-x^2}} + \frac{1}{1-x^2}$$

$$\Rightarrow y_0'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_0(x) = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

$$\Rightarrow \text{allgemeine Lösung: } y(x) = \frac{\arcsin x + c}{\sqrt{1-x^2}}$$

$$\text{speziell f\u00fcr } y(0) = 0 \text{ folgt: } c = 0 \Rightarrow y = \frac{\arcsin x}{\sqrt{1-x^2}}.$$

7) $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

Parametrisierung von M :

$$\gamma \quad : \quad \mathbb{R}^2 \rightarrow \mathbb{R}^3, (x, y) \mapsto (\gamma_x, \gamma_y, \gamma_z) = (x, y, e^{-x^2-y^2})$$

$$\gamma^*\omega = \gamma_x d\gamma_y \wedge d\gamma_z + \gamma_y d\gamma_z \wedge d\gamma_x + \gamma_z d\gamma_x \wedge d\gamma_y$$

$$d\gamma_z = e^{-x^2-y^2}(-2xdx - 2ydy)$$

$$\Rightarrow \gamma^*\omega = -2x^2 e^{-x^2-y^2} dy \wedge dx - 2y^2 e^{-x^2-y^2} dy \wedge dx + e^{-x^2-y^2} dx \wedge dy$$

$$= (1 + 2x^2 + 2y^2)e^{-x^2-y^2} dx \wedge dy$$

$$\int_M \omega = \int_{\mathbb{R}^2} \gamma^*\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 + 2x^2 + 2y^2)e^{-x^2-y^2} dx dy$$

$$\stackrel{\substack{x=r \cos \varphi \\ y=r \sin \varphi}}{=} \int_0^{\infty} r dr \int_0^{2\pi} d\varphi (1 + 2r^2)e^{-r^2} dr$$

$$\stackrel{u=r^2}{=} 2\pi \int_0^{\infty} \frac{1}{2}(1 + 2u)e^{-u} du = \pi \underbrace{[-e^{-u}]_0^{\infty}}_{=1} + 2\pi \int_0^{\infty} ue^{-u} du$$

$$\int_0^{\infty} ue^{-u} du = \underbrace{[-ue^{-u}]_0^{\infty}}_{=0} + \int_0^{\infty} e^{-u} du = 1$$

$$\Rightarrow \int_M \omega = 3\pi$$

8) Satz von Stokes: $\int_{\partial U} \alpha = \int_U d\alpha$

$$\text{a) } \alpha = [\cos y + xf(x^2 + y^2)]dx + [\sin x + yf(x^2 + y^2)]dy$$

$$\text{mit } f : \mathbb{R} \rightarrow \mathbb{R}, r \mapsto r^{101}$$

$$d\alpha = [-\sin y dy + f'(x^2 + y^2)(2x^2 dx + 2xy dy)] \wedge dx$$

$$+ [\cos x dx + f'(x^2 + y^2)(2y^2 dy + 2xy dx)] \wedge dy$$

$$= -\sin y dy \wedge dx + 2xy f'(x^2 + y^2) dy \wedge dx$$

$$+ \cos x dx \wedge dy + 2xy f'(x^2 + y^2) dx \wedge dy$$

$$= (\cos x + \sin y) dx \wedge dy$$

$$\Rightarrow \int_{\partial U} \alpha = \int_U d\alpha = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} (\cos x + \sin y) dx dy = \frac{\pi}{4} [\sin x]_0^{\frac{\pi}{4}} - \frac{\pi}{4} [\cos y]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} \cdot \left(\frac{\sqrt{2}}{2} - 0 \right) - \frac{\pi}{4} \left(\frac{\sqrt{2}}{2} - 1 \right) = \frac{\pi}{4}$$

$$\text{b) } \alpha = dg \text{ mit } g : \mathbb{R}^2 \rightarrow \mathbb{R}, (x, y) \mapsto \sin[xy \arctan(x - y)]$$

$$\int_{\partial U} \alpha = \int_U d\alpha = \int_U ddg = 0 \text{ da } ddg = d^2g = 0$$

für zweimal stetig differenzierbare Funktion g