# Hasse principles for higher-dimensional fields

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to Jürgen Neukirch, in memoriam

**Abstract**

For rather general excellent schemes $X$, K. Kato defined complexes of Gersten-Bloch-Ogus type involving the Galois cohomology groups of all residue fields of $X$. For arithmetically interesting schemes, i.e., schemes over global or local fields, or over their rings of integers, he developed a web of conjectures on some of these complexes, generalizing the fundamental exact sequence of Brauer groups for a global field. He proved these conjectures for low dimensions. We prove Kato's conjecture (with infinite coefficients) over number fields. In particular it gives a Hasse principle for function fields $F$ over a number field $K$, involving the corresponding function fields $F_v$ over the completions $K_v$ of $K$. We get a conditional result over global fields $K$ of positive characteristic, assuming resolution of singularities. This is unconditional for $\dim(X) \leq 3$, due to recent results on resolution. There are also applications to other cases considered by Kato.

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## §0 Introduction

In this paper we prove some conjectures of K. Kato [Ka] which were formulated to generalize the classical exact sequence of Brauer groups for a global field $K$,

$$0 \longrightarrow Br(K) \longrightarrow \bigoplus_v Br(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0\,, \tag{0.1}$$

to function fields $F$ over $K$ and varieties $X$ over $K$. In the above sequence, which is also called the Hasse-Brauer-Noether sequence, the sum runs over all places $v$ of $K$, and $K_v$ is the completion of $K$ with respect to $v$. The injectivity of the restriction map into the sum of local Brauer groups is called the Hasse principle.

Kato's generalization does not concern Brauer groups but rather the following Galois cohomology groups. Let $L$ be any field and let $n > 0$ be an integer. Write $n = mp^r$ with $p = \mathrm{char}(L)$ the characteristic of $L$ and $p \nmid m$ (so that $n = m$ for $\mathrm{char}(L) = 0$). Define the following Galois groups for $i, j \in \mathbb{Z}$

$$H^i(L, \mathbb{Z}/n\mathbb{Z}(j)) := H^i(L, \mu_m^{\otimes j}) \oplus H^{i-j}(L, W_r\Omega^i_{L,log}) \tag{0.2}$$

where $\mu_m$ is the Galois module of $m$-th roots of unity (in the separable closure $L^{sep}$ of $L$) and $W_r\Omega^i_{L,log}$ is the logarithmic part of the de Rham-Witt sheaf $W_r\Omega^i_L$ [Il] I 5.7 (an étale sheaf, regarded as a Galois module). It is a fact that $Br(L)[n] = H^2(L, \mathbb{Z}/n\mathbb{Z}(1))$, where $A[n] = \{ x \in A \mid nx = 0\}$ denotes the $n$-torsion in an abelian group $A$, so the $n$-torsion of the exact sequence can be identified with an exact sequence

$$(0.3) \qquad 0 \longrightarrow H^2(K, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \bigoplus_v H^2(K_v, \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\,.$$

In fact, this sequence is often used for the Galois cohomology of number fields, independently of Brauer groups; it is closely related to class field theory and Tate-Poitou duality.

For the generalization let $F$ be the function field in $d$ variables over a global fields $K$ and assume $F/K$ is primary, i.e., that $K$ is separably closed in $F$. For each place $v$ of $K$, let $F_v$ be the corresponding function field over $K_v$: If $F = K(V)$, the function field of a geometrically integral variety $V$ over $K$, then $F_v = K_v(V \times_K K_v)$. Then Kato conjectured:

**Conjecture 1** The following restriction map is injective:

$$H^{d+2}(F, \mathbb{Z}/n\mathbb{Z}(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/n\mathbb{Z}(d+1))\,.$$

Note that this generalizes the injectivity in (0.3), which is the case $d = 0$ and $F = K$. On the other hand it is known that the corresponding restriction map for Brauer groups is not in general injective for $d \geq 1$: If $X$ is a smooth projective curve over a number field which has a $K$-rational point, then for $F = K(X)$ the kernel of $\mathrm{Br}(F) \to \prod_v \mathrm{Br}(F_v)$ is isomorphic to the Tate-Shafarevich group of the Jacobian $\mathrm{Jac}(X)$. Kato proved conjecture 1 for $d = 1$ [Ka]. Here we prove the following variant (see Theorems 2.7 (a) and 2.8). For a field $L$, a prime $\ell$ and integers $i$ and $j$ we let

$$(0.4) \qquad H^i(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j)) = \varinjlim H^i(L, \mathbb{Z}/\ell^n\mathbb{Z}(j))\,,$$

where the inductive limit is taken via the obvious monomorphisms $\mathbb{Z}/\ell^n\mathbb{Z}(j) \hookrightarrow \mathbb{Z}/\ell^{n+1}(j)$ (for $\ell$ invertible in $L$) and $W_n\Omega_{L,log} \hookrightarrow W_{n+1}\Omega_{L,n+1}$ (if $\ell = \mathrm{char}(L)$, respectively.

**Theorem 0.1** Let $K$ be a number field, and let $\ell$ be any prime, and let $F$ be a function field in $d$ variables over $K$ such that $F/K$ is primary. Then the restriction map

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))\,.$$

is injective.

For $d = 2$ this result was already proved in [Ja4]. Concerning the case of finite coefficients, i.e., the original conjecture 1, we note the following. For any field $L$, any prime $\ell$ and any integer $t \geq 0$, there is a symbol map

$$h^t_{L,\ell} : K^M_t(L)/\ell \longrightarrow H^t(L, \mathbb{Z}/\ell(t)),$$

where $K^M_t(L)$ denotes the $t$-th Milnor $K$-group of $L$ ([Mi1] and [BK] §2). Extending an earlier conjecture of Milnor [Mi1] for $\ell = 2 \neq \mathrm{char}(L)$, it was conjectured by Bloch and Kato that this map is always an isomorphism. In particular we have the following Bloch-Kato conjecture for a field $L$:

$\mathrm{BK}(L, t, \ell)$: For any field $M$ over $L$, $h^t_{M,\ell}$ is surjective.

The status of this conjecture is as follows. The conjecture is known for $\ell = \mathrm{char}(L)$ (Bloch-Gabber-Kato [BK]), and in the following cases for $\ell \neq \mathrm{char}(L)$: for $t = 1$ (classical Kummer theory), for $t = 2$ (Merkurjev-Suslin [MS]) and for $\ell = 2$ (Voevodsky [V1]). It has been announced by Rost and Voevodsky ([SJ] and [V2]) that it holds in general. Then we have (see Theorems 2.7 and 2.9)

**Theorem 0.2** Let $K$ be a number field, let $\ell$ be a prime, and let $F$ be a primary function field in $d$ variables over $K$. If $\mathrm{BK}(K, d+1, \ell)$ holds, then conjecture 1 holds for $F$ and $\mathbb{Z}/\ell^n\mathbb{Z}$, for all $n$, i.e., the map

$$H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/\ell^n\mathbb{Z}(d+1))$$

is injective for all $n \geq 0$.

It should be noted that Kato did in fact use $\mathrm{BK}(K, 2, \ell)$, i.e., the Merkurjev-Suslin theorem, in his proof of conjecture 1 for $d = 1$. From the validity of $\mathrm{BK}(K, m, 2)$ for all $m$, see [V1], we obtain:

**Corollary 0.3** The map

$$H^{d+2}(F, \mathbb{Z}/2\mathbb{Z}) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/2\mathbb{Z})$$

is injective.

As in the classical case and the case of $d = 1$ (see the appendix to [Ka]), and the case of $d = 2$ in [Ja4] this has applications to quadratic forms over $F$, see [CTJ].

Our proof of Theorem 0.1 uses resolution of singularities. The same proof gives the following conditional result in the function field case (see Theorem 2.7).

**Theorem 0.4** Let $K$ be a global function field, i.e., a function field in one variable over a finite field, and let $\ell$ be a prime invertible in $K$. Let $F/K$ be a primary function field in $d$ variables.
(a) Assume the following condition holds, where $K'$ is the perfect hull of $K$ and $F' = FK' = (F \otimes_K K')_{red}$ is the function field associated to $F$ over $K'$:

$\mathrm{RS}(F', K')$: For every smooth variety $V$ over $K'$ with function field $F'$ there exists a cofinal set of smooth open subvarieties $U \subset V$ which admit a good compactification over $K'$, i.e., an open embedding $U \hookrightarrow X$ where $X$ is smooth projective over $K'$ and $Y = X - U$ (with the reduced subscheme structure) is a divisor with normal crossings.
Then the map

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

is injective. In particular this holds for $d \leq 3$.
(b) Assume in addition that $\mathrm{BK}(K, d+1, \ell)$ holds. Then the map

$$H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \longrightarrow \bigoplus_v H^{d+2}(F_v, \mathbb{Z}/\ell^n\mathbb{Z}(d+1))$$

is injective for all $n \geq 0$.

Kato also stated a conjecture on the cokernel of the above restriction map, in the following way. Let $L$ be a global or local field, let $X$ be any variety over $L$, and let $n$ be an integer. Then in [Ka]

Kato defined the following homological complex $C^{2,1}(X,\mathbb{Z}/n\mathbb{Z})$ of Galois cohomology groups:

$$\cdots \bigoplus_{x\in X_a} H^{a+2}(k(x),\mathbb{Z}/n\mathbb{Z}(a+1)) \longrightarrow \bigoplus_{x\in X_{a-1}} H^{a+1}(k(x),\mathbb{Z}/n\mathbb{Z}(a)) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{x\in X_1} H^{3}(k(x),\mathbb{Z}/n\mathbb{Z}(2)) \longrightarrow \bigoplus_{x\in X_0} H^{2}(k(x),\mathbb{Z}/n\mathbb{Z}(1)).$$

where the term involving $X_a$ is placed in degree $a$. Here $X_a$ denotes the set of points $x \in X$ of dimension $a$, and $k(x)$ denotes the residue field of $x$. A complex of the same shape can also be defined via the method of Bloch and Ogus, and it is shown in [JSS] that these two definitions agree up to (well-defined) signs (see also §4 for a discussion of more general complexes $C^{a,b}(X,\mathbb{Z}/n\mathbb{Z})$). Now let $K$ be a global field and let $X$ be a variety over $K$. Then there are obvious maps of complexes $C^{2,1}(X,\mathbb{Z}/n\mathbb{Z}) \to C^{2,1}(X_v,\mathbb{Z}/n\mathbb{Z})$ for each place $v$ of $K$, where $X_v = X \times_K K_v$, and these induce a map of complexes

$$\beta_{X,n} : C^{2,1}(X,\mathbb{Z}/n\mathbb{Z}) \longrightarrow \bigoplus_v C^{2,1}(X_v,\mathbb{Z}/n\mathbb{Z})\,.$$

Then Kato conjectured the following [Ka].

**Conjecture 2** let $X$ be a global field, let $n > 0$ be an integer, and let $X$ be a connected smooth projective variety over $K$. Then the above map induces isomorphisms

$$H_a(C^{2,1}(X,\mathbb{Z}/n\mathbb{Z})) \xrightarrow{\sim} \bigoplus_v H_a(C^{2,1}(X_v,\mathbb{Z}/n\mathbb{Z}))$$

for $a > 0$, and an exact sequence

$$0 \longrightarrow H_0(C^{2,1}(X,\mathbb{Z}/n\mathbb{Z})) \longrightarrow \bigoplus_v H_0(C^{2,1}(X_v,\mathbb{Z}/n\mathbb{Z})) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0\,.$$

Note that we obtain a sequence of the shape (0.3) for $X = \mathrm{Spec}(K)$, where the complexes are concentrated in degree zero. Conjecture 2 is implied by the following two conjectures

(a) The map $\beta_{X,n} : C^{2,1}(X,\mathbb{Z}/n\mathbb{Z}) \longrightarrow \oplus_v C^{2,1}(X_v,\mathbb{Z}/n\mathbb{Z})$ is injective.

(b) Let $C'(X,\mathbb{Z}/n\mathbb{Z})$ be the cokernel of $\beta_{X,n}$. Then one has $H_0(C'(X,\mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$, and $H_a(C'(X,\mathbb{Z}/n\mathbb{Z})) = 0$ for $a > 0$.

Conversely, if one assumes that every function field over the perfect hull $K'$ of $K$ has a smooth projective model (which holds for a number field), then conjecture 2, taken for all smooth projective varieties over all finite extensions of $K$, is equivalent to conditions (i) and (ii), taken for the same varieties.

Kato [Ka] proved conjecture 2 for $d = 1$. Here we prove the following result. For a field $L$, a prime $\ell$ and a variety $X$ over $L$ let

$$C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) = \varinjlim C^{2,1}(X,\mathbb{Z}/\ell^n\mathbb{Z})\,,$$

where the inductive limit is induced by the obvious transition maps (see (0.4)).

**Theorem 0.5** Let $K$ be a number field, let $\ell$ be any prime, and let $X$ be a connected smooth projective variety of dimension $d$ over $K$. Then the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-version of conjecture holds, i.e., we have:

(a) The map $\beta_{X,\ell^\infty} : C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \oplus_v\, C^{2,1}(X_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is injective.

(b) Let $C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be the cokernel of $\beta_{X,\ell^\infty}$. Then one has $H_0(C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, and $H_a(C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $a > 0$.

If $\mathrm{BK}(K, d+1, \ell)$ holds, then conjecture 2 is true for $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficients, for all $n > 0$, i.e., one has the following for all $n > 0$:

(a') The map $\beta_{X,\mathbb{Z}/\ell^n\mathbb{Z}} : C^{2,1}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow \oplus_v\, C^{2,1}(X_v, \mathbb{Z}/\ell^n\mathbb{Z})$ is injective.

(b') For the cokernel $C'(X, \mathbb{Z}/\ell^n\mathbb{Z})$ of $\beta_{X,\mathbb{Z}/\ell^n\mathbb{Z}}$ one has $H_0(C'(X, \mathbb{Z}/\ell^n\mathbb{Z})) = \mathbb{Z}/\ell^n\mathbb{Z}$, and $H_a(C'(X, \mathbb{Z}/\ell^n\mathbb{Z})) = 0$ for $a > 0$.

Here (a) and (a') are easily deduced from Theorems 0.1 and 0.2. For the proof of Theorem 0.4 we use another instance of resolution of singularities, and with the same method we obtain the following result for global function fields, unconditional for $d \le 3$.

**Theorem 0.6** Let $K$ be a global function field with perfect hull $K'$, let $\ell$ be a prime invertible in $K$, and let $X$ be a connected smooth projective variety of dimension $d$ over $K$.

(a) Assume $\mathrm{RS}(F', K')$ (see 0.4) holds for all function fields $F'$ of transcendence degree $\le d$ over $K'$. Then $\beta_{X,\ell^\infty} : C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \oplus_v\, C^{2,1}(X_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is injective. In particular this holds for $d \le 3$.

(b) Assume further that for every integral variety $Z$ of dimension $\le d$ over $K'$ there is a proper birational $K'$-morphism $\pi : \tilde{Z} \to Z$ with a smooth projective $K'$-variety $\tilde{Z}$. Then for the cokernel $C'(X, \mathbb{Z}/\ell^n\mathbb{Z})$ of $\beta_{X,\mathbb{Z}/\ell^n\mathbb{Z}}$ one has $H_0(C'(X, \mathbb{Z}/\ell^n\mathbb{Z})) = \mathbb{Z}/\ell^n\mathbb{Z}$, and $H_a(C'(X, \mathbb{Z}/\ell^n\mathbb{Z})) = 0$ for $a > 0$. In particular this holds for $d \le 3$.

(c) If $\mathrm{BK}(K, d+1, \ell)$ holds, then the above holds with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ replaced by $\mathbb{Z}/\ell^n\mathbb{Z}$.

The claims on the varieties of dimension $\le 3$ hold because both conditions on resolution of singularities are known to hold for $d \le 2$, and they hold by recent results on resolution of singularities in [CP1] and [CP2] and [CJS] for $d = 3$.

Finally our techniques also allow to get results on another conjecture of Kato, over finite fields. For any variety over a finite field $k$ and any natural number $n$, Kato considered a complex $C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})$ which is of the form

$$\cdots \bigoplus_{x \in X_a} H^{a+1}(k(x), \mathbb{Z}/n\mathbb{Z}(a)) \longrightarrow \bigoplus_{x \in X_{a-1}} H^{a}(k(x), \mathbb{Z}/n\mathbb{Z}(a-1)) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{x \in X_1} H^{2}(k(x), \mathbb{Z}/n\mathbb{Z}(1)) \longrightarrow \bigoplus_{x \in X_0} H^{1}(k(x), \mathbb{Z}/n\mathbb{Z})$$

with the term involving $X_a$ placed in (homological) degree $a$ (this is another special case of the general complexes $C^{a,b}(X, \mathbb{Z}/n\mathbb{Z})$). Kato conjectured the following (where the case $a = 0$ is easy):

**Conjecture 3** If $X$ is connected, smooth and projective over a finite field $k$, then one has

$$H_a(C^{1,0}(X, \mathbb{Z}/n\mathbb{Z})) = \begin{cases} 0 & , \quad a \neq 0, \\ \mathbb{Z}/n\mathbb{Z} & , \quad a = 0\,. \end{cases}$$

For $\dim(X) = 1$ this conjecture amounts to (0.3) with $K = k(X)$, and for $\dim(X) = 2$ the conjecture follows from [CTSS] for $n$ invertible in $k$, and from [Gr] and [Ka] if $n$ is a power of $\mathrm{char}(k)$. Saito [Sa] proved that $H_a(C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $\dim(X) = 3$ and $\ell \neq \mathrm{char}(k)$. For $X$ of any dimension Colliot-Thélène [CT] (for $\ell \neq \mathrm{char}(k)$) and Suwa [Su] (for $\ell = \mathrm{char}(k)$ proved that $H_a(C^{1,0}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $0 < a \leq 3$. Here we prove the following conditional result (which gives back the cited results for $\dim(X) \leq 3$):

**Theorem 0.7** Let $X$ be connected, smooth and projective of dimension $d$ over a finite field $k$, and let $\ell$ be a prime. If $\mathrm{RS}(F, k)$ holds for all function fields of transcendence degree $\leq d$ over $k$, and if any integral variety $Z$ of dimension $\leq d$ admits a proper birational morphism $\pi : \tilde{X} \to X$ with smooth projective variety $\tilde{X}$, then one has

$$H_a(C^{1,0}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} 0 & , \quad a \neq 0, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a = 0\,. \end{cases}$$

If in addition $\mathrm{BK}(k, d, \ell)$ holds, then the same is true with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ replaced by $\mathbb{Z}/\ell^n\mathbb{Z}$, for any natural number $n$.

Due to the results on resolution of singularities in [CP1], [CP2] and [CJS], the above gives back the quoted results for $\dim(X) \leq 3$. By another technique, the results in [CJS] are used in [JS2] to prove $H_a(C^{1,0}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $X$ of arbitrary dimension and $0 < a \leq 4$. Finally, Kato also formulated an arithmetic analogue of conjecture 3, for regular flat proper schemes of $\mathrm{Spec}(\mathbb{Z})$, and in [JS1] some results on this are obtained using Theorem 0.5.

This paper had a rather long evolution time. Theorem 0.1 was obtained in 1990, rather short after the proof of Theorem 0.1 and Theorem 0.5 for $d = 2$ in [Ja4]. In 1996, right after the appearance of [GS], it became clear to me how to obtain Theorem 0.5, but a first account was only written in 2004. Meanwhile I had also noticed that these methods allow a proof of Theorem 0.7, i.e., a proof of Kato's conjecture over finite fields, assuming resolution of singularities. Part of the delay was caused by the long time to complete the comparison of Kato's original complexes with the complexes of Gersten-Bloch-Ogus type used here, which was recently accomplished [JSS].

This paper is dedicated to my teacher and friend Jürgen Neukirch, who helped and inspired me in so many ways by his support and enthusiasm. I also thank Jean-Louis Colliot-Thélène for his long lasting interest in this work, for the discussions on the rigidity theorems 2.9 and 4.14, and for the proof of Theorem 2.10. Moreover I thank Wayne Raskind, Florian Pop, Tamás Szamuely and Thomas Geisser for their interest and useful hints and discussions. In establishing the strategy for proving Theorems 0.7 and 0.5 I profited from an incomplete preprint by Michael Spieß. My contact with Shuji Saito started with the subject of this paper, and I thank him for all these years of a wonderful collaboration and the countless inspirations I got from our discussions.

## §1 First reductions and a Hasse principle for global fields

Let $K$ be a global field, and let $F$ be a function field of transcendence degree $d$ over $K$. We assume that $K$ is separably closed in $F$. For every place $v$ of $K$, let $K_v$ be the completion of

$K$ at $v$, and let $F_v$ be the corresponding function field over $K_v$: there exists a geometrically irreducible variety $V$ of dimension $d$ over $K$, such that $F = K(V)$, and then $F_v = K_v(V_v)$, where $V_v = V \times_K K_v$ (this is integral, since $F/K$ is primary and $K_v/K$ is separabel, see [EGA IV],2, (4.3.2) and (4.3.5)). This definition does not depend on the choice of $V$.

Fix a prime $\ell \neq \mathrm{char}(K)$. We want to study the map

$$\mathrm{res}\colon\ H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to \prod_v H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

induced by the restrictions from $F$ to $F_v$. For this it will be useful to first replace the completions $K_v$ by the Henselizations. For each place $v$ of $K$, denote by $K_{(v)}$ the Henselization of $K$ at $v$. It can be regarded as a subfield of a fixed separable closure $\overline{K}$ of $K$, equal to the fixed field of a decomposition group $G_v$ at $v$. For $V$ as above, let $F_{(v)} = K_{(v)}(V \times_K K_{(v)})$ be the corresponding function field over $K_{(v)}$. Since $K_{(v)}$ is separably algebraic over $K$ and linearly disjoint from $F$, $F_{(v)}$ is equal to the composite $FK_{(v)}$ in a fixed separable closure $\overline{F}$ of $F$. We obtain a diagram of fields

(1.1)

$$\begin{array}{ccc} & & \overline{F} \\ & & | \\ \overline{K} & — & F\overline{K} \\ | & & | \\ K_{(v)} & — & F_{(v)} \\ | & & | \\ K & — & F \end{array}$$

which identifies $G_K = \mathrm{Gal}(\overline{K}/K)$ with $\mathrm{Gal}(F\overline{K}/F)$, and $G_{K_{(v)}} = \mathrm{Gal}(\overline{K}/K_{(v)})$ with $\mathrm{Gal}(F\overline{K}/FK_{(v)})$.

**Proposition 1.2** Let $M$ be a discrete $\ell$-primary torsion $G_F$-module. The restriction map

$$H^{d+2}(F, M) \to \prod_v H^{d+2}(F_{(v)}, M)$$

has image in the direct sum $\bigoplus_v H^{d+2}(F_{(v)}, M)$. There is a commutative diagram

$$\begin{array}{cccc} f: & H^{d+2}(F,M) & \longrightarrow & \bigoplus_v H^{d+2}(F_{(v)}, M) \\ & \downarrow & & \downarrow \\ g: & H^2(K, H^d(F\overline{K}, M)) & \longrightarrow & \bigoplus_v H^2(K_{(v)}, H^d(F\overline{K}, M)), \end{array}$$

in which the horizontal maps are induced by the restrictions, and the vertical maps by the Hochschild-Serre spectral sequences. This diagram is functorial in $M$ and induces canonical isomorphisms

$$\ker(f) \xrightarrow{\sim} \ker(g) \qquad \text{and} \qquad \mathrm{coker}(f) \xrightarrow{\sim} \mathrm{coker}(g) \cong H^d(F\overline{K}, M)(-1)_{G_K}\ .$$

Here $N(n)$ denotes the $n$-fold Tate twist of a $\ell$-primary discrete torsion $G_K$-module $N$, and $N_{G_K}$ denotes its cofixed module, i.e., the maximal quotient on which $G_K$ acts trivially.

**Proof** Diagram (1.1) gives Hochschild-Serre spectral sequences

$$\begin{array}{rcl} E_2^{p,q}(K) & = & H^p(K, H^q(F\overline{K}, M)) \Rightarrow H^{p+q}(F, M) \\ E_2^{p,q}(K_{(v)}) & = & H^p(K_{(v)}, H^q(F\overline{K}, M)) \Rightarrow H^{p+q}(F_{(v)}, M) \end{array}$$

Moreover, for each $v$ we obtain a natural map $E(K) \to E(K_{(v)})$ between the above spectral sequences which gives the restriction maps for $K \subset K_{(v)}$ on the $E_2$-terms and the restriction maps for $F \subset F_{(v)}$ on the abutment, respectively. On the other hand, the field $F\overline{K}$ has cohomological dimension $d$, so that $E_2^{p,q}(K) = 0 = E_2^{p,q}(K_{(v)})$ for $q > d$. This gives a commutative diagram

$$\begin{array}{ccc} H^{d+2}(F, M) & \longrightarrow & H^{d+2}(F_{(v)}, M) \\ \downarrow & & \downarrow \\ H^2(K, H^d(F\overline{K}, M)) & \longrightarrow & H^2(K_{(v)}, H^d(F\overline{K}, M)), \end{array}$$

where the vertical maps are edge morphisms of the spectral sequences. If $v$ is not a real archimedean place, or if $\ell \neq 2$, we have $cd_\ell(K_{(v)}) \leq 2$ and, hence, $E_2^{p,q}(K_{(v)}) = 0$ for $p > 2$, and the right vertical edge morphism is an isomorphism. This already shows the first claim of the proposition, since the restriction map

$$H^2(K, N) \to \prod_v H^2(K_{(v)}, N)$$

is known to have image in the direct sum $\bigoplus_v$ for any torsion $G_K$-module $N$. If $K$ has no real archimedean valuations (or if $\ell \neq 2$), then $cd_\ell(K) = 2$, the left hand edge morphism is an isomorphism as well, and the second claim follows.

So let now $K$ be a number field. Here we need the following lemma.

**Lemma 1.3** The above maps between the spectral sequences induce

(a) surjections for all $r \geq 2$ and all $p + q = d + 1$

$$E_r^{p,q}(K) \twoheadrightarrow \bigoplus_{v|\infty} E_r^{p,q}(K_{(v)}) \ ,$$

(b) surjections for all $r \geq 2$

$$E_r^{2,d}(K) \twoheadrightarrow \bigoplus_{v|\infty} E_r^{2,d}(K_{(v)}) \ ,$$

(c) isomorphisms between the kernels and between the cokernels of the maps

$$E_r^{2,d}(K) \to \bigoplus_v E_r^{2,d}(K_{(v)}) \quad \text{and} \quad E_{r+1}^{2,d}(K) \to \bigoplus_v E_{r+1}^{2,d}(K_{(v)}) \ ,$$

for all $r \geq 2$,

(d) isomorphisms

$$E_r^{p,q}(K) \xrightarrow{\sim} \bigoplus_{v|\infty} E_r^{p,q}(K_{(v)})$$

for all $r \geq 2$ and all $(p, q) \neq (2, d)$ with $p + q \geq d + 2$.

**Proof:** By induction on $r$. Recall that $E_r^{p,q}(K) = 0 = E_r^{p,q}(K_{(v)})$ for all $q > d$ and all $r \geq 2$. Hence, for $r = 2$ the claims (a), (b) and (d) follow from the following well-known facts of global Galois cohomology: the maps

$$\begin{array}{rcl} H^1(K,N) & \to & \bigoplus_{v|\infty} H^1(K_{(v)},N) \\ H^2(K,N) & \to & \bigoplus_{v|S} H^2(K_{(v)},N) \end{array}$$

are surjective for any torsion $G_K$-module $N$ and any finite set $S$ of places, and the maps

$$H^i(K,N) \xrightarrow{\sim} \bigoplus_{v|\infty} H^i(K_{(v)},N)$$

are isomorphisms for such $N$ and all $i \geq 3$. Note that here we could replace $K_{(v)}$ by the more common completion $K_v$, since $G_{K_{(v)}} \cong G_v \cong G_{K_v}$.

Now let $r \geq 2$. For (a) look at the commutative diagram

$$\begin{array}{ccccc} E_r^{p-r,q+r-1}(K) & \xrightarrow{d_r} & E_r^{p,q}(K) & \xrightarrow{d_r} & E_r^{p+r,q-r+1}(K) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \bigoplus_{v|\infty} E_r^{p-r,q+r-1}(K_{(v)}) & \xrightarrow{d_r} & \bigoplus_{v|\infty} E_r^{p,q}(K_{(v)}) & \xrightarrow{d_r} & \bigoplus_{v|\infty} E_r^{p+r,q-r+1}(K_{(v)}) \end{array}$$

coming from the map of spectral sequences. We may assume $p \geq 1$(since $E_r^{0,d+1} = 0$), and hence $(p+r, q-r+1) \neq (2,d)$. Then $\beta$ is surjective and $\gamma$ is an isomorphism, by induction assumption (for (a) and (d)). By taking homology of both rows, we obtain a surjection $E_{r+1}^{p,q}(K) \twoheadrightarrow \bigoplus_{v|\infty} E_{r+1}^{p,q}(K_{(v)})$ as wanted for (a).

For (d) we look at the same diagram where now we may assume that $p \geq 2, (p,q) \neq (2,d) \neq (p+r, q-r+1)$, that $\beta$ and $\gamma$ are bijective, and that $\alpha$ is surjective (by induction assumption for (a), (b) and (d)). Hence we get the isomorphism

$$E_{r+1}^{p,q}(K) \xrightarrow{\sim} \bigoplus_{v|\infty} E_{r+1}^{p,q}(K_{(v)}).$$

For (b) and (c) consider the *exact* commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{r+1}^{2,d}(K) & \longrightarrow & E_r^{2,d}(K) & \xrightarrow{d_r} & E_r^{2+r,d-r+1}(K) \\ & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\beta} & & \wr\downarrow{\scriptstyle\gamma} \\ 0 & \longrightarrow & \bigoplus_{v\in S'} E_{r+1}^{2,d}(K_{(v)}) & \longrightarrow & \bigoplus_{v\in S'} E_r^{2,d}(K_{(v)}) & \xrightarrow{\partial} & \bigoplus_{v|\infty} E_r^{2+r,d-r+1}(K_{(v)}) \end{array}$$

for any set of places $S \supset \{v \mid \infty\}$, in which $\partial = \bigoplus_{v \in S'} d_r(K_{(v)})$(note that $E_r^{p,q}(K_{(v)}) = 0$ for $p > 2$ and $v \nmid \infty$). The map $\gamma$ is an isomorphism by induction assumption (for (d)). Hence for $S' = \{v \,|\, \infty\}$ the surjectivity of $\beta$ implies the one for $\beta'$, i.e., we get (b) for $r+1$. If $S'$ is the set of all places, we see that clearly $\ker(\beta') = \ker(\beta)$, and that $\mathrm{coker}(\beta') = \mathrm{coker}(\beta)$, since $\mathrm{im}(\partial \circ \beta) = \mathrm{im}(\partial)$ by induction assumption for (b). Thus we get (c) for $r$ from (b) and (d) for $r$.

We use the lemma to complete the proof of proposition 1.2. From what we have shown, we have $E_\infty^{0,d+2} = E_\infty^{0,d+1} = 0$ for $K$ and all $K_{(v)}$, and isomorphisms

$$E_\infty^{p,q}(K) \xrightarrow{\sim} \bigoplus_v E_\infty^{p,q}(K_{(v)}).$$

for all $(p,q)$ with $p+q=d+2, p\geq 3$ (note that $E_2^{p,q}(K_{(v)}) = 0$ for $p \geq 3$ and $v \nmid \infty$). Hence kernel and cokernel of

$$H^{d+2}(F,M) \to \bigoplus_v H^{d+2}(F_{(v)},M)$$

can be identified with kernel and cokernel of

$$E_\infty^{2,d}(K) \to \bigoplus_v E_\infty^{2,d}(K_{(v)}),$$

respectively. But these coincide with kernel and cokernel of

$$E_2^{2,d}(K) = H^2(K, H^d(F\overline{K},M)) \to \bigoplus_v H^2(K_{(v)}, H^d(F\overline{K},M)) = \bigoplus_v E_2^{2,d}(K_{(v)})$$

respectively, by (c) of the lemma. Finally, for any finite $\ell$-primary $G_K$-module $N$, Poitou-Tate duality gives an exact sequence

$$H^2(K,N) \longrightarrow \bigoplus_v H^2(K_{(v)},N) \longrightarrow H^0(K,N^*)^\vee \longrightarrow 0\,,$$

where $N^*$ denotes the finite $G_K$-module $Hom(N,\mu)$ where $\mu$ is the group of roots of unity in $\overline{K}$ and $M^\vee$ is the Pontrjagin dual of a finite $G_K$-module. But then we have canonical identifications

$$\begin{aligned} & H^0(K, Hom(N,\mu))^\vee = Hom_{G_K}(N, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^\vee \\ \cong\ & Hom_{G_K}(N(-1), \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee = Hom(N(-1)_{G_K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee = N(-1)_{G_K}^{\vee\vee} \cong N(-1)_{G_K}\,. \end{aligned}$$

This shows the last isomorphism of 1.2

**Remarks 1.4** (a) Proposition 1.2 extends to the case where $F$ is a function field over $K$, but $K$ is not necessarily separably closed in $F$, by replacing $F_{(v)}$ with $F\otimes_K K_{(v)}$ and $F\overline{K}$ with $F\otimes_K\overline{K}$. The cohomology groups of these rings have to be interpreted as the étale cohomology groups of the associated affine schemes; with this the proof carries over verbatim. In more down-to-earth (but more tedious) terms, we may note that $(F\otimes_K \overline{K})_{red} \cong \prod_\sigma (F\otimes_{\tilde{K},\sigma} \overline{K})$, where $\tilde{K}$ is the separable closure of $K$ in $F$ and $\sigma$ runs over the $K$-embeddings of $\tilde{K}$ into $\overline{K}$. Similarly, $F\otimes_K K_{(v)} \cong \prod_\sigma(\prod_w F\otimes_{\tilde{K}} \sigma(\tilde{K})_{(\sigma w)})$, where $w$ runs over the places of $\tilde{K}$ above $v$, $\sigma w$ is the corresponding place of $\sigma(\tilde{K})$ above $v$, and $\sigma(\tilde{K})_{(\sigma w)}$ is the Henselization of $\sigma(\tilde{K})$ at $\sigma w$. The étale cohomology groups referred to above can thus be identified with sums of Galois cohomology groups of the fields introduced above, and the claim also follows by applying Proposition 1.2 to $F/K'$.

(b) A consequence of Proposition 1.2 is that the restriction map

$$f' : H^{d+2}(F,M) \to \prod_v H^{d+2}(F_v,M)$$

has image in the direct sum $\bigoplus_v \subset \prod_v$ as well, since it factors through the map $f$ in 1.2. Moreover, as we shall see in §2, the maps $H^{d+2}(F_{(v)},M) \to H^{d+2}(F_v,M)$ are injective, so that $\ker(f') = \ker(f)$. For $d>0$, however, $H^{d+2}(F_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$ is much bigger than $H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, and proposition 1.2 does not extend to the completions. In particular, everywhere in

[Ja 4] the completions $K_v$ should be replaced by the Henselizations $K_{(v)}$ (In loc. cit., the notations $FK_v$ and $F\overline{K}_v$ are problematic; they should be interpreted as $F_v$ and $F_v\overline{K_v}$. Even then $\mathrm{Gal}(\overline{FK_v}/F\overline{K_v}) = \mathrm{Gal}(\overline{F_v}/F_v\overline{K_v})$ is not isomorphic to $\mathrm{Gal}(\overline{F}/F\overline{K})$, but much bigger, as was kindly pointed out to me by J.-L. Colliot-Thélène and J.-P. Serre). The comparison of $\mathrm{coker}(f')$ and $\mathrm{coker}(f)$ is more subtle, see §4.

By proposition 1.2, we may study the restriction map

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

by considering the restriction map

$$\beta_N : H^2(K, N) \to \bigoplus_v H^2(K_{(v)}, N) \cong \bigoplus_v H^2(K_v, N)$$

for the $G_K$-module $N = H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$. Here we have used the isomorphism $G_{K_v} \xrightarrow{\sim} G_{K_{(v)}}$ to rewrite the latter map in terms of the more familiar completions $K_v$. Recall that $F = K(V)$, the function field of a geometrically irreducible variety $V$ of dimension $d$ over $K$. From this we obtain

$$H^d(F\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) = \varinjlim_{U \subset V} H^d_{\text{ét}}(U \times_K \overline{K},\, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

where the limit is over all affine open subvarieties $U$ of $V$. In fact étale cohomology commutes with this limit ([Mi2] III 1.16 ), so that the right hand side is the étale cohomology group $H^d_{\text{ét}}(\mathrm{Spec}(\overline{K}(V)), \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, which can be identified with the Galois cohomology group on the left hand side. Since $H^2(K, \varinjlim N_i) = \varinjlim H^2(K, N_i)$ for a direct limit of $G_K$-modules $N_i$, and since the same holds for the $K_v$, it thus suffices to study the maps

$$\beta_B : H^2(K, B) \to \bigoplus_v H^2(K_v, B)$$

for $B = H^d(U \times_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$, where $U \subseteq V$ runs through all open subvarieties of $V$, or through a cofinal set of them. For this we shall use the following Hasse principle, which generalizes [Ja 3] th. 3.

**Theorem 1.5** Let $K$ be a global field, and let $\ell \neq \mathrm{char}(K)$ be a prime number.

(a) Let $A$ be a discrete $G_K$-module which is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^m$ for some $m$ as an abelian group, and mixed of weights $\neq -2$ as a Galois module. Then the restriction map induces isomorphisms

$$\beta_A : H^2(K, A) \xrightarrow{\sim} \bigoplus_v H^2(K_v, A) = \bigoplus_{v \in S \text{or } v|\ell} H^2(K_v, A),$$

where $S$ is a finite set of bad places for $A$.

(b) Let $T$ be a finitely generated free $\mathbb{Z}_\ell$-module with continuous action of $G_K$ making $T$ mixed of weights $\neq 0$. Then for any finite set $S'$ of places of $K$ the restriction map in continuous cohomology

$$\alpha_T : H^1(K, T) \to \prod_{v \notin S'} H^1(K_v, T)$$

is injective.

Before we prove this, let us explain the notion of a mixed $G_K$-representation and a bad place $v$ for it. A priori, this is defined for a $\mathbb{Q}_\ell$-representation $V$ of $G_K$ (i.e., a finite-dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of $G_K$) - see [De ] 1.2 and 3.4.10, and 1.6 below. We extend it to a module like $A$ above or more generally, to a discrete $\ell$-primary torsion $G_K$-module of cofinite type (resp. to a finitely generated $\mathbb{Z}_\ell$-module $T$ with continuous action of $G_K$), by calling $A$ (resp. $T$) pure of weight $w$ or mixed, if this holds for the $\mathbb{Q}_\ell$-representation $T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$), where $T_\ell A = \varprojlim_n A[\ell^n]$ is the Tate module of $A$. In the same way we define the bad places for $A$ (resp. $T$) to be those of the associated $\mathbb{Q}_\ell$-representations. It remains to recall

**Definition 1.6** (a) A $\mathbb{Q}_\ell$-representation $V$ of $G_K$ is pure of weight $w \in \mathbb{Z}$, if there is a finite set $S \supset \{v \nmid \infty\}$ of places of $K$ such that

- (i) $V$ is unramified outside $S \cup \{v \nmid \ell\}$, i.e., for $v \notin S, v \nmid \ell$, the inertia group $I_v$ at $v$ acts trivially on $V$,
- (ii) for every place $v \notin S, v \nmid \ell$, the eigenvalues $\alpha$ of the geometric Frobenius $Fr_v$ at $v$ acting on $V$ are algebraic numbers with
$$|\iota\alpha| = (Nv)^{\frac{w}{2}}$$
for every embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, where $Nv$ is the cardinality of the finite residue field of $v$.

Every such set $S$ will be called a set of bad places for $V$; the places not in $S$ are called good.

(b) $V$ is called mixed, if it has a filtration $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ by subrepresentations such that every quotient $V_i/V_{i-1}$ is pure of some weight $w_i$. The weights and bad places of $V$ are those present in some non-trivial quotient $V_i/V_{i-1}$.

**Remarks and examples 1.7** (a) If $v$ is a place of $K$, any extension $w$ of $v$ to $\overline{K}$ determines a decomposition group $G_w \subset G_K$ and an inertia group $I_w \subset G_w$. The arithmetic Frobenius $\varphi_w$ is a well-defined element in $G_w/I_w$; under the canonical isomorphism $G_w/I_w \xrightarrow{\sim} \mathrm{Gal}(k(w)/k(v))$ it corresponds to the automorphism $x \mapsto x^{N_v}$ of $k(w)$. The geometric Frobenius $Fr_w$ is the inverse of $\varphi_w$. If $I_w$ acts trivially on $V$, then the action of $Fr_w$ on $V$ is well-defined. If we do not fix a choice of $w$, everything is well-defined up to conjugacy in $G_K$, and we use the notation $G_v, I_v$, and $Fr_v$. Thus "$I_v$ acts trivially" means that one and hence any $I_w$ , for $w|v$, acts trivially, and then the eigenvalues of $Fr_v$ are well-defined, since they depend only on the conjugacy class.

(b) If $V$ is pure of weight $w$, then the same holds for every $\mathbb{Q}_\ell$-$G_K$-subquotient. If $V'$ is pure of weight $w'$, then $V \otimes_{\mathbb{Q}_\ell} V'$ is pure of weight $w + w'$.

(c) The representation $\mathbb{Q}_\ell(1)$ is unramified outside $S = \{v \mid \infty \cdot \ell\}$, and for $v \notin S, \varphi_v$ acts on $\mathbb{Q}_\ell(1)$ by multiplication with $N_v$. Therefore $\mathbb{Q}_\ell(1)$ is pure of weight $-2$, and $\mathbb{Q}_\ell(i)$ is pure of weight $-2i$.

(d) If $X$ is a smooth and proper variety over $K$, then its $i$-th étale cohomology group $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell), \overline{X} = X \times_K \overline{K}$, is pure of weight $i$ by the smooth and proper base change theorems and by Deligne's proof of the Weil conjectures over finite fields (cf., e.g., [Ja 2] proof of lemma 3). The set $S$ can be taken to be the set of places where $X$ has bad reduction, i.e., such that for $v \notin S, X$ has good reduction at $v$, viz., a smooth projective model $\mathcal{X}_v$ over $\mathcal{O}_v$, the ring of integers in $K_v$, with $\mathcal{X}_v \times_{\mathcal{O}_v} K_v = X_v$.

**Proof of theorem 1.5:** Part (a) is implied by (b). In fact, $A$ is mixed of weights $\neq -2$ if and only if its Kummer dual $T = \mathrm{Hom}(A, \mu)$ (where $\mu$ is the Galois module of roots of unity in $\overline{K}$) is mixed of weights $\neq 0$, and the kernels of $\beta_A$ and $\alpha_T$ for $S' = \emptyset$ are dual to each other by the theorem of Tate-Poitou (and passing to the limits over the finite modules $A[\ell^n]$ and $T/\ell^n T = \mathrm{Hom}(A[\ell^n], \mu)$, respectively). Moreover, by Tate-Poitou the cokernel of $\alpha_A$ is isomorphic to $H^0(K,T)^\vee \cong A(-1)_{G_k}$, and this is zero by the hypothesis on the weights. Finally, by local Tate duality, $H^2(K_v, A)$ is dual to $H^0(K_v, T)$, and for good places $v \nmid \ell$ this is zero if $T$ is mixed of weights $\neq 0$

Part (b) generalizes [Ja 3] Theorem 3a), which covers the case of a pure $T$. The generalization follows by induction: let

$$0 \to T' \to T \to T'' \to 0$$

be an exact sequence of $\mathbb{Z}_\ell$-$G_K$-modules as in (b), and let $S'$ be a finite set of primes. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\prod\limits_{v\notin S'} H^0(K_v,T'') & \longrightarrow & \prod\limits_{v\notin S'} H^1(K_v,T') & \longrightarrow & \prod\limits_{v\notin S'} H^1(K_v,T) & \longrightarrow & \prod\limits_{v\notin S'} H^1(K_v,T'') \\
\uparrow & & \uparrow{\scriptstyle \beta_{T'}} & & \uparrow{\scriptstyle \beta_T} & & \uparrow{\scriptstyle \beta_{T''}} \\
H^0(K,T'') & \longrightarrow & H^1(K,T') & \longrightarrow & H^1(K,T) & \longrightarrow & H^1(K,T'')
\end{array}$$

If $\beta_{T''}$ is injective and $H^0(K_v, T'') = 0$ for all $v \notin S'$( which is the case for $T''$ pure of weight $\neq 0$ and $S'$ containing all bad places for $T''$ and all $v \,|\, \ell$, by loc. cit.), then $\beta_T$ is injective if and only if $\beta_T$ is. Since we may always enlarge the set $S'$, the proof proceeds by induction on the length of a filtration with pure quotients, which exists on $T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, by definition, and hence on $T$ by pull-back.

## §2 Injectivity of the global-local map

Let $K$ be a global field, let $\ell \neq \mathrm{char}(K)$ be a prime, and let $U$ be a smooth, quasi-projective, geometrically irreducible variety of dimension $d$ over $K$. Following the strategy of section 1, we study the $G_K$-module $H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. Assume the following condition, which holds for number fields by Hironaka's resolution of singularities in characteristic zero [Hi].

> $\mathrm{RS}_{gc}(U)$: There is a *good compactification* for $U$, i.e., a smooth projective variety $X$ over $K$ containing $U$ as an open subvariety such that $Y = X \smallsetminus U$, with its reduced closed subscheme structure, is a divisor with simple normal crossings.

Recall that $Y$ is said to have simple normal crossings if its irreducible components $Y_1, \ldots, Y_N$ are smooth projective subvarieties $Y_i \subset X$ such that for all $1 \leq i_1 < \ldots < i_\nu \leq N$, the $\nu$-fold intersection $Y_{i_1,\ldots,i_\nu} := Y_{i_1} \cap \ldots \cap Y_{i_\nu}$ is empty or smooth projective of pure dimension $d - \nu$, so the same is true for the disjoint union

$$Y^{[\nu]} := \coprod_{1 \leq i_1 < \ldots < i_\nu \leq N} Y_{i_1,\ldots,i_\nu} \qquad (1 \leq \nu \leq d),$$

and for $Y^{[0]} := Y_\emptyset := X$.

This geometric situation gives rise to a spectral sequence

$$E_2^{p,q} = H^p(\overline{Y^{[q]}}, \mathbb{Q}_\ell(-q)) \Rightarrow H^{p+q}(\overline{U}, \mathbb{Q}_\ell)\,, \tag{2.1}$$

see, e.g., [Ja 1] 3.20. It is called the weight spectral sequence, because it induces the weight filtration on the $\ell$-adic representation $H^n(\overline{U}, \mathbb{Q}_\ell)$. In fact, $E_2^{p,q}$ is pure of weight $p+2q$. Therefore the same is true for the $E_\infty^{p,q}$-terms, and if $\tilde{W}_q$ denotes the canonical ascending filtration on the limit term $H^n(\overline{U}, \mathbb{Q}_\ell)$ for which $\tilde{W}_q/\tilde{W}_{q-1} = E_\infty^{n-q,q}$, then its $n$-fold shift $W. := \tilde{W}.[-n]$ (i.e., $W_i = \tilde{W}_{i-n}$) is the unique weight filtration : $W_i/W_{i-1} \cong E_\infty^{2n-i,i-n}$ is pure of weight $i$. Moreover, for $r > 3$ the differentials

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

are morphisms between Galois $\mathbb{Q}_\ell$-representation of different weights (viz., $p+2q$ and $p+2q-r+2$) and hence vanish, so that $E_\infty^{p,q} = E_3^{p,q}$.

Note that $E_2^{p,q} = 0$ for $p < 0$ or $q < 0$. Hence the weights occurring in $H^n(\overline{U}, \mathbb{Q}_\ell)$ lie in $\{n, \ldots, 2n\}, W_{2n-1}$ is mixed of weights $w \leq 2n-1$, and

$$\begin{aligned} H^n(\overline{U}, \mathbb{Q}_\ell)/W_{2n-1} &= W_{2n}/W_{2n-1} = E_3^{0,n} \\ &= \ker(H^0(\overline{Y^{[n]}}, \mathbb{Q}_\ell(-n)) \xrightarrow{d_2^{0,n}} H^2(\overline{Y^{[n-1]}},\ \mathbb{Q}_\ell(-n+1)). \end{aligned}$$

In particular, the Galois action on $(W_{2n}/W_{2n-1})(n)$ factors through a finite quotient, since this is the case for $H^0(\overline{Y^{[n]}}, \mathbb{Q}_\ell)$.

We want to say something similar for $H^n(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, at least for $n = d$ ($= \dim U$). If $U$ is affine, then we have an exact sequence

$$\ldots \to H^d(\overline{U}, \mathbb{Z}_\ell) \to H^d(\overline{U}, \mathbb{Q}_\ell) \to H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0,$$

since $H^{d+1}(\overline{U}, \mathbb{Z}_\ell) = 0$ by weak Lefschetz [Mi2] VI 7.2. Thus $B_1 = H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible, and there is an exact sequence

$$0 \longrightarrow A_1 \longrightarrow B_1 \longrightarrow C_1 \longrightarrow 0 \tag{2.2}$$

in which $A_1 = \mathrm{im}(W_{2d-1}H^d(\overline{U}, \mathbb{Q}_\ell) \longrightarrow H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell))$ is divisible and of weights $w \in \{d, \ldots, 2d-1\}$, and in which $C_1$ is a quotient of $H^d(\overline{U}, \mathbb{Q}_\ell)/W_{2d-1}$, divisible and pure of weight $2d$. We need to know $C_1$ precisely, not just up to isogeny, and this requires more arguments - note that in the $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-analogue of (2.1) the differentials $d_1^{p,q}$ will not in general vanish for $r \geq 3$.

For a better control of this spectral sequence we replace $U$ by a smaller variety, as follows. By the Bertini theorem, there is a hyperplane $H$ in the ambient projective space whose intersection with $X$ and all $Y_{i_1,\ldots,i_\nu}$ is transversal, i.e., gives smooth divisors in these (in particular, the intersection with $Y_{i_1,\ldots,i_d}$ is empty for all $d$-tupels $(i_1, \ldots, i_d)$). This means that $\tilde{Y} = \bigcup_{i=1}^{N+1} Y_i$, with $Y_{N+1} := H \cap X$, is again a divisor with strict normal crossings on $X$. As explained in section 1, it is possible for our purposes to replace $U$ by the open subscheme $U^0 = X \smallsetminus \tilde{Y} = U \smallsetminus (H \cap U)$, because such subschemes form a cofinal subset in the set of all opens $U \subseteq V$, $F = K(V)$. Now we have the following description for $B_0 := H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$.

**Proposition 2.2** There is an exact sequence

$$0 \to A_0 \to H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to C_0 \to 0$$

in which $A_0$ is divisible and mixed of weights in $\{d, \ldots, 2d-1\}$, and

$$C_0 = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d)$$

for a finitely generated free $\mathbb{Z}$-module $I$ with discrete action of $G_K$. Moreover, there is an exact sequence

$$0 \to I' \to I \to I'' \to 0$$

of $G_K$-modules with

$$I'' = \mathbb{Z}[\pi_0(\overline{Y^{[d]}})]$$

$$I' = \ker(\mathbb{Z}[\pi_0(\overline{Y^{[d-1]} \cap H})] \overset{\beta}{\twoheadrightarrow} \mathbb{Z}[\pi_0(\overline{Y^{[d-1]}})])$$

where $Y^{[d-1]} \cap H := \coprod_{1 \le i_1 < \ldots < i_{d-1} \le N} Y_{i_1,\ldots,i_{d-1}} \cap H$, and where $\beta$ is induced by the inclusions $Y_{i_1,\ldots,i_\nu} \cap H \hookrightarrow Y_{i_1,\ldots,i_\nu}$.

**Proof** For $1 \le i_1 < \ldots < i_\nu \le N$ define

$$Y^0_{i_1,\ldots,i_\nu} := Y_{i_1,\ldots,i_\nu} \smallsetminus (Y_{i_1,\ldots,i_\nu} \cap H)$$

by removing the smooth hyperplane section with $H$, and let $Y^{0[\nu]} \subseteq Y^{[\nu]}$ be the disjoint union of these open subvarieties for fixed $\nu$ (with $Y^{0[0]} := X^0 := X \smallsetminus (X \cap H)$). Then $Y^0 = \bigcup_{i=1}^{N} Y^0_i$ is a divisor with (strict) normal crossing on $X^0$ with $U^0 = X^0 \smallsetminus Y^0$, and hence there is a spectral sequence

$$E_2^{p,q} = H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-q)) \Rightarrow H^{p+q}(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{2.3}$$

by the same arguments as for (2.1) (the properness is not needed in the proof).

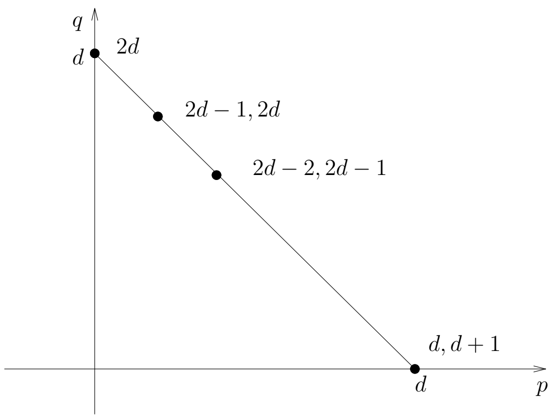
But now the $Y^0_{i_1,\ldots,i_q}$ are affine varieties, as complements of hyperplane sections, and of dimension $d-q$, so that

$$H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-q)) = \begin{cases} 0 & \text{for } p > d-q, \\ \text{divisible} & \text{for } p = d-q\,, \end{cases}$$

by weak Lefschetz. Moreover, by the Gysin sequences

$$\ldots \to H^p(\overline{Y_{\underline{i}}}, \mathbb{Q}_\ell) \to H^p(\overline{Y^0_{\underline{i}}}, \mathbb{Q}_\ell) \to H^{p-1}(\overline{Y_{\underline{i}} \cap H}, \mathbb{Q}_\ell(-1)) \to \ldots\ ,$$

$H^p(\overline{Y^{0[q]}}, \mathbb{Q}_\ell)$ is mixed with weights $p$ and $p+1$, since $H^p(\overline{Y_{\underline{i}}}, \mathbb{Q}_\ell)$ and $H^{p-1}(\overline{Y_{\underline{i}} \cap H}, \mathbb{Q}_\ell(-1))$ are pure of weights $p$ and $p+1$, respectively. Hence the spectral sequence (2.3) is much simpler than (2.1) and has the following $E_2$-layer:




The terms vanish for $p+q > d$, and on the line $p+q = d$ the $E_2^{p,q}$-terms - and hence also the $E_\infty^{p,q}$-terms which are quotients - are divisible and mixed of the indicated weights. Note that $H^0(\overline{Y^{0[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d))$ is pure of weight $2d$.

Let $F^{\cdot}$ be the descending filtration on $B_0 = H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for which $F^\nu/F^{\nu-1} = E_\infty^{\nu,d-\nu}$. Then we see that $F^2$ is divisible and mixed of weights $\leq 2d-1$. Next,

$$F^1/F^2 \cong E_2^{1,d-1} = H^1(\overline{Y^{0[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d+1))$$

is the cohomology of a (usually non-connected) smooth affine curve, and by the Gysin sequence

$$\begin{array}{rl} 0 & \to \; H^1(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^1(\overline{Y^{0[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ & \to \; H^0(\overline{Y^{[d-1]} \cap H}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) \to H^2(\overline{Y^{0[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0 \end{array}$$

there is an exact sequence

$$0 \to A' \to F^1/F^2 \to C' \to 0$$

where $A'$ is divisible of weight $2d-1$ and where $C' = I' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, with $I'$ defined by the exact sequence

$$0 \to I' \to \mathbb{Z}[\pi_0(\overline{Y^{[d-1]} \cap H})] \to \mathbb{Z}[\pi_0(\overline{Y^{[d-1]}})] \to 0.$$

Finally,

$$C'' := F^0/F^1 \cong H^0(\overline{Y^{0[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(-d))$$

is the cohomology of $Y^{0(d)} = Y^{(d)}$ which is a union of points, and $C'' = I'' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ for

$$I'' = \mathbb{Z}[\pi_0(\overline{Y^{(d)}})].$$

Let $A_0$ be the preimage of $A'$ in $F^1$, and $C_0 = B_0/A_0$. Then we have exact sequences

$$\begin{array}{ccccccccc} 0 & \to & A_0 & \to & B_0 & \to & C_0 & \to & 0 \\ 0 & \to & F^2 & \to & A_0 & \to & A' & \to & 0 \\ 0 & \to & C' & \to & C_0 & \to & C'' & \to & 0 \end{array}$$

Hence $A_0$ is divisible and mixed of weights $\leq 2d-1$, and $C_0$ is divisible of weight $2d$. This determines $A_0$ and $C_0$ uniquely (there is no non-trivial $G_K$-morphism between such modules), and so the spectral sequence (2.1) for $\overset{0}{U} = X \smallsetminus \tilde{Y}$ instead of $U = X \smallsetminus Y$ shows that $C_0$ is a quotient of

$$\ker(H^0(\overline{\tilde{Y}^{(d)}}, \mathbb{Q}_\ell(-d)) \to H^2(\overline{\tilde{Y}^{(d-1)}}, \mathbb{Q}_\ell(-d+1))).$$

Hence the action of $G_K$ on $C_0(d)$ factors through a finite quotient $G$. This in turn shows that the extension

$$0 \to C' \to C_0 \to C'' \to 0$$

comes from an extension

$$0 \to I' \to I \to I'' \to 0$$

of $G$-modules by tensoring with $\mathbb{Q}_\ell/\mathbb{Z}_\ell(d)$. In fact, applying a Tate twist is an exact functor on $\mathbb{Z}_\ell$-$G_k$-modules, and one has isomorphisms (where the tensor products are over $\mathbb{Z}$)

$$\begin{array}{rcl} \operatorname{Ext}^1_G(I'', I') \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & \xrightarrow{\sim} & \operatorname{Ext}^1_{\mathbb{Z}_\ell[G]}(I'' \otimes \mathbb{Z}_\ell, I' \otimes \mathbb{Z}_\ell) \\ & \xrightarrow{\sim} & \operatorname{Ext}^1_G(I'' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell, I' \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell), \end{array}$$

since $\mathbb{Z}_\ell$ is flat over $\mathbb{Z}$, and since the functor $T \mapsto T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell$ is an equivalence between $\mathbb{Z}_\ell$-lattices and divisible $\ell$-torsion modules of cofinite type (with action of $G$) preserving exact sequences. Finally,

$$\operatorname{Ext}^1_G(I'', I') \to \operatorname{Ext}^1_G(I'', I') \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_\ell$$

is surjective for a finite group $G$.

We are now ready to prove

**Theorem 2.4** The restriction map

$$\beta_B : H^2(K, B) \to \bigoplus_v H^2(K_v, B)$$

is injective for $B = H^d(\overline{U^0}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$.

**Proof** We follow the method of [Ja 4]. By applying the $(d+1)$-fold Tate twist to the sequence $0 \to A_0 \to B_0 \to C_0 \to 0$ of Proposition 2.2, we get an exact sequence

$$0 \to A \to B \to C \to 0\,.$$

It induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \bigoplus\limits_{v\in S} H^1(K_v, C) & \longrightarrow & \bigoplus\limits_{v} H^2(K_v, A) & \longrightarrow & \bigoplus\limits_{v} H^2(K_v, B) \\
 & \uparrow \alpha_{C,S} & (*) & \wr\uparrow \beta_A & & \uparrow \beta_B \\
\cdots \longrightarrow & H^1(K, C) & \longrightarrow & H^2(K, A) & \longrightarrow & H^2(K, B)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & \bigoplus\limits_{v} H^2(K_v, C) & \longrightarrow & \bigoplus\limits_{v\infty} H^3(K_v, A) & \longrightarrow \cdots \\
 & \uparrow \beta_C & & \wr\uparrow \gamma A & \\
\longrightarrow & H^2(K, C) & \longrightarrow & H^3(K, A) & \longrightarrow \cdots
\end{array}$$

for a suitable finite set $S$ of places of $K$. In fact, if $S_{bad}$ is a set of bad places for $A$, then for any $S \supset S_{bad} \cup \{v \mid \ell\}$, $H^2(K_v, A) = 0$ for $v \notin S$, and thus $(*)$ is commutative. By Theorem 1.5, $\beta_A$ is an isomorphism, since $A$ is divisible and mixed of weights $\leq -3$, and by Tate duality, $\gamma_A$ is an isomorphism (for all torsion modules $A$). To show the injectivity of $\beta_B$ by the 5-lemma, it therefore suffices to show that $C$ satisfies

$$\begin{array}{lllcll}
(H) & (i) & \alpha_{C,S} : H^1(K, C) & \to & \bigoplus\limits_{v\in S} H^1(K_v, C) & \text{is surjective for all finite } S. \\
 & (ii) & \beta_C : H^2(K, C) & \to & \bigoplus\limits_{v} H^2(K_v, C) & \text{is injective.}
\end{array}$$

Let $I, I'$ and $I''$ be as in theorem 2.2, so that $C = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$. We have exact sequences

$$\begin{array}{ccccccccc}
0 & \to & I' & \to & I & \to & I'' & \to & 0 \\
0 & \to & I' & \to & I_2 & \to & I_3 & \to & 0\,,
\end{array}$$

in which $I''$, $I_2$ and $I_3$ are permutation modules, i.e., of the form $\mathbb{Z}[M]$ for a $G_K$-set $M$. Thus $(H)$ holds for $C$ by repeated application (first to $I'', I_2$ and $I_3$, then to $I'$, and finally to $I$) of the following result

**Theorem 2.5** Let $I_1$, $I_2$ and $I_3$ be finitely generated free $\mathbb{Z}$-modules with discrete $G_K$-action, and let $C_i = I_i \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$ for $i = 1, 2, 3$. Assume that $I_3$ is a permutation module.

(a) Property $(H)$ holds for $C_3$.

(b) If $0 \to I_1 \to I_2 \to I_3 \to 0$ is an exact sequence, then $(H)$ holds for $C_1$ if and only if it holds for $C_2$.

The following observation will help to prove part (b).

**Lemma 2.6** Let $I$ be a finitely generated free $\mathbb{Z}$-module with discrete $G_K$-action, and let $T$ be the torus over $K$ with cocharacter module $X_*(T) = I$. Then property $(H)(i)$ (resp. $(H)(ii)$) holds for $C = I \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)$ if and only if $T$ satisfies

$$\begin{array}{rll} (H'_\ell)(i): & \alpha_{T,S,\ell}: H^1(K,T)\{\ell\} & \to \bigoplus_{v\in S} H^1(K_v,T)\{\ell\} \text{ is surjective for all finite } S. \\ \text{(resp. } (H'_\ell)(ii): & \beta_{T,\ell}: H^2(K,T)\{\ell\} & \to \bigoplus_{v} H^2(K_v,T)\{\ell\} \text{ is injective.)} \end{array}$$

**Proof** Recall that $T(\overline{K}) = I \otimes_{\mathbb{Z}} \overline{K^\times}$ and $H^i(K,T) = H^i(K,T(\overline{K}))$ by definition. Since $\ell \neq \mathrm{char}(K)$, $T(\overline{K})$ is $\ell$-divisible, and the Kummer sequences

$$0 \to I \otimes_{\mathbb{Z}} \mu_{\ell^n} \to T(\overline{K}) \xrightarrow{\ell^n} T(\overline{K}) \to 0 \tag{2.7}$$

identify $C$ with $T(\overline{K})\{\ell\}$, the $\ell$-primary torsion subgroup of $T(\overline{K})$. Similar results hold for the fields $K_v$, and the cohomology sequences associated to (2.7) for all $n$ give rise to a commutative diagram with exact rows

$$\begin{array}{ccccccc} \bigoplus_{v\in S} T(K_v)\otimes \mathbb{Q}_\ell/Z_\ell & \longrightarrow & \bigoplus_{v\in S} H^1(K_v,C) & \longrightarrow & \bigoplus_{v\in S} H^1(K_v,T)\{\ell\} & \longrightarrow & 0 \\ \uparrow \omega_{T,S} & & \uparrow \alpha_{C,S} & & \uparrow \alpha_{T,S,\ell} & & \\ T(K)\otimes \mathbb{Q}_\ell/Z_\ell & \longrightarrow & H^1(K,C) & \longrightarrow & H^1(K,T)\{\ell\} & \longrightarrow & 0 \end{array}$$

and to a commutative diagram with horizontal isomorphisms

$$\begin{array}{ccc} \bigoplus_{v} H^2(K_v,C) & \xrightarrow{\sim} & \bigoplus_{v} H^2(K_v,T)\{\ell\} \\ \uparrow \beta_C & & \uparrow \beta_{T,\ell} \\ H^2(K,C) & \xrightarrow{\sim} & H^2(K,T)\{\ell\}\ . \end{array}$$

Here the vertical maps are induced by the various restriction maps, and we used that $T(K) = H^0(K,T(\overline{K}))$ and $T(K_v) = H^0(K_v,T(\overline{K}_v))$ for a separable closure $\overline{K_v}$ of $K_v$. Note that $H^i(K,T)$ and $H^i(K_v,T)$ are torsion groups for $i \geq 1$.

Now the map $\omega_{T,S}$ is surjective for any torus $T$ and any finite set of places $S$ ([Ja 4] lemma 2). This proves the lemma.

**Proof of theorem 2.5** Let $I_3$ be a permutation module. Then $I_3$ is a direct sum of modules of the form $I_0 = \mathrm{Ind}_{K'}^K(\mathbb{Z}) = \mathbb{Z}[G_K/G_{K'}]$ for some finite separable extension $K'$ of $K$. Let $T_0$ be the torus with cocharacter module $I_0$. Then

$$H^i(K,T_0) \cong H^i(K',\mathbb{G}_m) = H^i(K',\overline{K}^\times) \tag{2.8}$$

by Shapiro's lemma, and similarly

$$H^i(K_v,T_0) \cong \bigoplus_{w|v} H^i(K'_\omega,\mathbb{G}_m)\,; \tag{2.9}$$

where $w$ runs through the places of $K'$ above $v$. Thus

$$H^1(K,T_0) = 0 = H^1(K_v,T_0)$$

by Hilbert's theorem 90, and $\beta_{T_0} \to \bigoplus_v H^2(K_v, T_0)$ is injective by the classical theorem of Brauer-Hasse-Noether for $K'$. This shows property $(H'_\ell)$ for the torus $T_3$ with cocharacter module $I_3$, for all primes $\ell$, and hence part (a) of theorem 2.5.

For part (b), let $T_i$ be the torus with cocharacter module $I_i$ $(i = 1, 2, 3)$. Then we have an exact sequence

$$0 \to T_1 \to T_2 \to T_3 \to 0$$

with $H^1(K, T_3) = 0 = H^1(K_v, T_3)$ by assumption and the above. This gives exact commutative diagrams

$$\begin{array}{ccccccc}
\bigoplus_{v\in S} T_3(K_v) & \xrightarrow{\delta} & \bigoplus_{v\in S} H^1(K_v, T_1) & \longrightarrow & \bigoplus_{v\in S} H^1(K_v, T_2) & \longrightarrow & 0 \\
\uparrow{\scriptstyle \omega} & & \uparrow{\scriptstyle \alpha_{T_1,S}} & & \uparrow{\scriptstyle \alpha_{T_2,S}} & & \\
T_3(K) & \longrightarrow & H^1(K, T_1) & \longrightarrow & H^1(K, T_2) & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_v H^2(K_v, T_1) & \longrightarrow & \bigoplus_v H^2(K_v, T_2) & \longrightarrow & \bigoplus_v H^3(K_v, T_3) \\
 & & \uparrow{\scriptstyle \beta_{T_1}} & & \uparrow{\scriptstyle \beta_{T_2}} & & \uparrow{\scriptstyle \beta_{T_3}} \\
0 & \longrightarrow & H^2(K, T_1) & \longrightarrow & H^2(K_v, T_2) & \longrightarrow & H^3(K_v, T_3)
\end{array} .$$

Since $\beta_{T_3}$ is injective by assumption, one has an isomorphism $\ker\beta_{T_1} \xrightarrow{\sim} \ker\beta_{T_2}$. On the other hand, the groups $H^1(K_v, T_1)$ have finite exponent $n$ (by Hilbert's theorem 90 we can take $n = [K' : K]$, if $K'/K$ is a finite Galois extension splitting $T_1$). Hence $\delta$ factors through $\bigoplus_{v\in S} T_3(K_v)/n$. But $\omega \otimes \mathbb{Z}/n\mathbb{Z}$ is surjective for every $n$: Indeed,

$$K^\times / K^{\times^n} \to \bigoplus_{v\in S} K_v^\times / (K_v^\times)^n$$

is surjective for all $n$ by weak approximation for $K$, and the same for all finite extensions $K'$ of $K$ gives the result for $T_3$ (cf. (2.8) and (2.9) for $i = 0$). This gives an isomorphism coker $\alpha_{T_1,S} \xrightarrow{\sim}$ coker $\alpha_{T_2,S}$ and hence (b).

This completes the proof of theorem 2.4, and we can now show:

**Theorem 2.7** Let $F$ be a function field in $d$ variables over $K$, such that $K$ is separably closed in $F$, and let $\ell$ be a prime invertible in $K$.
(a) If $K$ is a number field, then the restriction map

$$H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \to \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$$

is injective.
(b) Let $K$ have positive characteristic and perfect hull $K'$, let $F = K(V)$ for an integral variety $V$ over $K$, and let $F' = (F \otimes_K K')_{red} = K'((V \times_K K')_{red})$ be the corresponding function field over $K'$. Then the same result holds under the following condition:

> RS$(F', K')$: Condition RS$_{gc}(U)$ (see the beginning of this section) holds for a cofinal set of open subsets $U \subset (V \times_K K')_{red}$.

Note that this condition in fact only depends on $F'$ and $K'$.
(c) Assume that BK$(K, d+1, \ell)$ (the Bloch-Kato conjecture, see the introduction) holds. Then the same results hold after replacing $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ with $\mathbb{Z}/\ell^n\mathbb{Z}$, for each $n > 0$.

**Proof** (a) follows immediately from theorem 2.4 and the remarks preceding theorem 1.5. For (b) one can replace $K$ by $K'$, because the considered groups do not change for $\ell \neq \mathrm{char}(K)$, and then the argument is the same. Note that there might not exist a smooth model $V$ of $F$ over $K$, not to mention a smooth proper model, even for curves (where one always has a unique regular compactification, which might be non-smooth). Now assume $\mathrm{BK}(K, d+1, \ell)$. It is well-known that this implies that the Galois symbol maps $K_{d+1}^M(L)/\ell^n \to H^{d+1}(L, \mathbb{Z}/\ell^n\mathbb{Z})$ are surjective for all fields $L/K$ and all $n > 0$. Therefore we obtain a surjective map

$$K_{d+1}^M(L) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^{d+1}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

and thus the divisibility of the group on the right. Hence the long exact cohomology sequence associated to the exact sequence

$$0 \to \mathbb{Z}/\ell^n\mathbb{Z}(d+1) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1) \xrightarrow{\ell^n} \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1) \to 0$$

gives an exact sequence

$$0 \to H^{d+2}(L, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \xrightarrow{i} H^{d+2}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \xrightarrow{\ell^n} H^{d+2}(L, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))\,, \tag{2.8}$$

i.e., the injectivity of the map $i$. Now the commutative diagram

$$\begin{array}{ccc} H^{d+2}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) & \longrightarrow & \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ H^{d+2}(F, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) & \longrightarrow & \bigoplus_v H^{d+2}(F_{(v)}, \mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \end{array}$$

implies the claim in (c) (the injectivity of the map $i$ on the right is not needed).

To have the same result with $F_v$ in place of $F_{(v)}$, it suffices to show:

**Theorem 2.9** For any $n \in \mathbb{N}$ and all $i, j \in \mathbb{Z}$, the restriction map

$$H^i(F_{(v)}, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i(F_v, \mathbb{Z}/n\mathbb{Z}(j))$$

is injective.

This is related to a more precise rigidity result (for $n$ invertible in $K$) on the Kato complexes recalled in the next section which we shall also need in the following sections. However, as was pointed out to me by J.-L. Colliot-Thélène, the injectivity above follows by a simple argument, and in the following general version:

**Theorem 2.10** Let $K/k$ be a field extension satisfying the following property

(SD) If a variety $Y$ over $k$ has a $K$-rational point, then it also has a $k$-rational point.

Let $F$ be a set-valued contravariant functor on the category of all $k$-schemes such that

(FP) $\varinjlim_i F(A_i) \cong F(A)$ via the natural map, for any inductive system $(A_i)$ of $k$-algebras and $A = \varinjlim_i A_i$. (Here we write $F(B) := F(\mathrm{Spec}\, B)$ for a $k$-algebra $B$.)

Let $V$ be a geometrically integral variety over $k$, and write $k(V)$ (resp. $K(V)$) for the function field of $V$ (resp. $V \times_k K$). Then the map

$$F(k(V)) \to F(K(V))$$

is injective.

**Proof of Theorem 2.9** We may apply 2.10 to the extension $K_v/K_{(v)}$ and the functor $F(X) = H^i_{\text{ét}}(X, M)$ for any fixed discrete $G_{K_{(v)}}$-module $M$ (regarded as étale sheaf by pull-back) to get the injectivity of

$$H^i(F_{(v)}, M) \to H^i(F_v, M) \ .$$

In fact, property (SD) (for "strongly dense") is known to hold in this case (cf. [Gr] Theorem 1), and the commuting with limits as in (FP) (for "finitely presented") is a standard property of étale cohomology (cf. [Mi2] III 1.16).

**Proof of theorem 2.10** The field $K$ can be written as the union of its subfields $K_i$ which are finitely generated (as fields) over $k$. Every $K_i$ can of course be written as the fraction field of a finitely generated $k$-algebra $A_i$.

Now let $\alpha \in F(k(V))$, and assume that $\alpha$ vanishes in $F(K(V))$. By (FP), there is an $i$ such that $\alpha$ already vanishes in $F(K_i(V))$. Moreover, there is a non-empty affine open $V' \subseteq V$ and a $\beta \in F(V')$ mapping to $\alpha$ in $F(k(V))$. Finally there is a non-empty affine open $U \subseteq Z_i \times_k V'$, where $Z_i =$ Spec $A_i$, such that $\beta$ vanishes under the composite map $F(V') \to F(Z_i \times_k V') \to F(U)$.

Now it follows from Chevalley's theorem that the image of $U$ under the projection $p : Z_i \times_k V' \to Z_i$ contains a non-empty affine open $U'$ ($p$ maps constructible set to constructible sets, and is dominant). Now $U'$ has a $K$-point $\mathrm{Spec}(K) \to \mathrm{Spec}(K_i) \hookrightarrow U'$. Hence, by property (SD), $U'$ has a $k$-rational point $Q$. Then $W = p^{-1}(Q) \cap U$ is open and non-empty in $p^{-1}(Q) = Q \times_k V' \cong V'$. By functoriality, $\beta$ vanishes in $F(W)$, and thus $\alpha$ in $F(k(V))$.

## §3 A Hasse principle for unramified cohomology

To investigate the cokernel of $\beta_B$ (notations as in section 2), we could follow the method of [Ja 4] and show that it is isomorphic to $\mathrm{coker}(\beta_C)$. By describing the edge morphisms in the spectral sequence (2.3) we could prove theorem 3.1 below. Instead, we prefer to argue more directly.

We shall make repeated use of the following. Let $i : Y \hookrightarrow X$ be a closed immersion of smooth varieties over a field $K$, of pure codimension $c$. Then, for every integer $n$ invertible in $K$ and every integer $r$, one has a long exact Gysin sequence

$$\ldots \to H^{\nu-1}(U, \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{\delta} H^{\nu-2c}(Y, \mathbb{Z}/n\mathbb{Z}(r-c)) \xrightarrow{i_*} H^{\nu}(X, \mathbb{Z}/n\mathbb{Z}(r)) \xrightarrow{j^*} H^{\nu}(U, \mathbb{Z}/n\mathbb{Z}(r)) \to \ldots$$

where $U = X \smallsetminus Y$ is the open complement of $Y$ and $j : U \hookrightarrow X$ is the open immersion. We call $i_*$ and $\delta$ the Gysin map and the residue map for $i : Y \hookrightarrow X$, respectively. If $i' : Y' \hookrightarrow Y$ is another closed immersion, with $Y'$ smooth and of pure codimension $d$ in $Y$, then the diagram of Gysin sequences

$$\begin{array}{ccccccc}
H^{\nu-1}(U,\mathbb{Z}/n\mathbb{Z}(r)) & \xrightarrow{\delta} & H^{\nu-2c}(Y,\mathbb{Z}/n\mathbb{Z}(r-c)) & \xrightarrow{i_*} & H^{\nu}(X,\mathbb{Z}/n\mathbb{Z}(r)) & \xrightarrow{j^*} \\
\uparrow{\scriptstyle j'^*} & & \uparrow{\scriptstyle i'_*} & & \| & \\
H^{\nu-1}(U',\mathbb{Z}/n\mathbb{Z}(r)) & \xrightarrow{\delta} & H^{\nu-2(c+d)}(Y',\mathbb{Z}/n\mathbb{Z}(r-c-d)) & \xrightarrow{(i\circ i')_*} & H^{\nu}(X,\mathbb{Z}/n\mathbb{Z}(r)) & \xrightarrow{(j\circ j')^*}
\end{array}$$

is commutative, where $j' : U \hookrightarrow U'$ is the open immersion.

**Theorem 3.1** Let $X$ be a smooth, proper, irreducible variety of dimension $d$ over a finitely generated field $K$. Let $Y = \bigcup_{i=1}^{N} Y_i$ be a union of smooth irreducible divisors on $X$ intersecting transversally such that $X \smallsetminus Y_1$ is affine (this holds, e.g., if $Y_1$ is a smooth hyperplane section), and let $U = X \smallsetminus Y$. Then, for any prime $\ell$ invertible in $K$, and with the notations of the beginning of section 2, the sequence

$$0 \to H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \xrightarrow{e} H^0(\overline{Y^{[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K} \xrightarrow{d_2} H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{G_K}$$

is exact, where $e$ and $d_2$ are defined as follows. The specialization map $e$ is induced by the compositions

$$(3.2)\qquad \begin{aligned} & H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) \xrightarrow{\delta} H^{d-1}(\overline{Y_{i_d} \smallsetminus (\bigcup_{i\neq i_d} Y_i)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)) \\ & \xrightarrow{\delta} H^{d-2}(\overline{Y_{i_{d-1},i_d} \smallsetminus (\bigcup_{i\neq i_{d-1},i_d} Y_i)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-2)) \longrightarrow \dots \\ & \dots \xrightarrow{\delta} H^1(\overline{Y_{i_2,\dots,i_d} \smallsetminus (\bigcup_{i\neq i_2,\dots,i_d} Y_i)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \xrightarrow{\delta} H^0(\overline{Y}_{i_1,\dots,i_d}, \mathbb{Q}_\ell/\mathbb{Z}_\ell), \end{aligned}$$

where each $\delta$ is the connecting morphism in the obvious Gysin sequence. On the other hand $d_2 = \sum_{\mu=1}^{d} (-1)^\mu \, \delta_\mu$, where $\delta_\mu$ is induced by the Gysin map associated to the inclusions

$$Y_{i_1,\dots,i_d} \hookrightarrow Y_{i_1,\dots,\hat{i}_\nu,\dots,i_d}$$

(and $\hat{i}_\nu$ means omission of $i_\nu$, as usual).

**Proof** Write $H^i(\overline{Z}, j)$ instead of $H^i(\overline{Z}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$, for short. We proceed by induction on $r$, the number of components of $Y$. If $r = 1$, then the Gysin sequence

$$\dots \to H^d(\overline{X}, d) \to H^d(\overline{U}, d) \to H^{d-1}(\overline{Y}, d-1) \to \dots$$

shows that $H^d(\overline{U}, d)$ is mixed with weights $-d$ and $-d+1$. Hence we can only have $H^d(\overline{U}, d)_{G_K} \neq 0$ and $Y^{[d]} \neq \emptyset$ for $d = 1$. In this case we have an exact sequence

$$0 \to H^1(\overline{X}, 1) \to H^1(\overline{U}, 1) \xrightarrow{\delta} H^0(\overline{Y}, 0) \to H^2(\overline{X}, 1) \to 0$$

Without loss of generality, we may assume that $X$ is geometrically irreducible over $K$ (otherwise this is the case over a finite extension $K'$ of $K$, and everything reduces to this situation, since we have induced modules). Letting $C = \mathrm{im}(\delta)$, we have

$$H^1(\overline{U}, 1)_{G_K} \xrightarrow{\sim} C_{G_K}$$

since $H^1(\overline{X},1)_{G_K} = 0$ ($H^1(\overline{X},1)$ is divisible and of weight $-1$), and there is an exact sequence

$$0 \to C \to \mathrm{Ind}_{K(x)}^{K}(\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \mathbb{Q}_\ell/\mathbb{Z}_\ell \to 0$$

where $K(x)$ is the residue field of the unique point $x \in Y$. But this sequence stays exact after taking cofixed modules: the action of $G_K$ factors through a finite quotient $G$, and $H_1(G, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ (this group is dual to $H^1(G, \mathbb{Z}_\ell) = 0$). Putting things together, we have an exact sequence

$$0 \to H^1(\overline{U},1)_{G_K} \xrightarrow{e} H^0(\overline{Y},0)_{G_K} \xrightarrow{d_2} H^2(\overline{X},1)_{G_K} \to 0$$

as wanted.

Now let $r > 1$. Note that $U$ is affine, since $U \hookrightarrow X$ is known to be an affine morphism [ ], and $X \smallsetminus Y_1$ is affine. Now $Z = \bigcup_{i=1}^{r-1} Y_i$ is a divisor with normal crossings on $X$ that fulfills all the assumptions of the theorem, and the same is true for $Y_r \cap Z = \bigcup_{i=1}^{r-1} (Y_r \cap Y_i)$ on $Y_1$.

We claim that we obtain a commutative diagram

(3.3)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H^d(\overline{X\smallsetminus Z},d)_{G_K} & \longrightarrow & H^d(\overline{U},d)_{G_K} & \xrightarrow{\delta} & H^{d-1}(Y_r\smallsetminus(Y_r\cap Z),d-1)_{G_K} & \longrightarrow & 0 \\
 & & \downarrow e & (1) & \downarrow e & (2) & \downarrow e & & \\
0 & \longrightarrow & H^0(\overline{Z^{[d]}},0)_{G_K} & \longrightarrow & H^0(\overline{Y^{[d]}},0)_{G_K} & \longrightarrow & H^0(\overline{(Y_r\cap Z)^{[d-1]}},0)_{G_K} & \longrightarrow & 0 \\
 & & \downarrow d_2 & (3) & \downarrow d_2 & (4) & \downarrow d_2 & & \\
0 & \longrightarrow & H^2(\overline{Z^{[d-1]}},1)_{G_K} & \longrightarrow & H^2(\overline{Y^{[d-1]}},1)_{G_K} & \longrightarrow & H^2(\overline{(Y_r\cap Z)^{[d-2]}},1)_{G_K} & \longrightarrow & 0
\end{array}$$

with exact rows: The first row comes from the Gysin sequence for $(X \smallsetminus Z, Y_r \smallsetminus Z_r)$

$$\ldots \to H^d(\overline{X\smallsetminus Z},d) \to H^d(\overline{U},d) \xrightarrow{\delta} H^{d-1}(\overline{Y_r \smallsetminus Z_r}, d-1) \to 0$$

in which $H^{d+1}(\overline{X\smallsetminus Z},d) = 0$ by weak Lefschetz. Next note that

$$\begin{array}{rcl}
Y^{[d]} & = & \coprod\limits_{1\le i_1<\ldots<i_\nu\le r} Y_{i_1,\ldots,i_\nu} \\
Z^{[d]} & = & \coprod\limits_{1\le i_1<\ldots<i_\nu\le r-1} Y_{i_1,\ldots,i_\nu} \\
(Y_r\cap Z)^{[d-1]} & = & \coprod\limits_{1\le i_1<\ldots<i_{\nu-1}\le r-1} Y_r\cap Y_{i_1,\ldots,i_{\nu-1}} \\
 & = & \coprod\limits_{1\le i_1<\ldots<i_\nu = r} Y_{i_1,\ldots,i_\nu}
\end{array}$$

Hence one has commutative diagrams

(3.4)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i(\overline{Z^{[d]}},j) & \longrightarrow & H^i(\overline{Y^{[d]}},j) & \longrightarrow & H^i(\overline{(Y_r\cap Z)^{[d-1]}},j) & \longrightarrow & 0 \\
 & & \downarrow d_2 & (3') & \downarrow d_2 & (4') & \downarrow d_2 & & \\
0 & \longrightarrow & H^{i+2}(Z^{[d-1]},j+1) & \longrightarrow & H^{i+2}(Y^{[d-1]},j+1) & \longrightarrow & H^{i+2}((Y_r\cap Z)^{[d-2]},j+1) & \longrightarrow & 0
\end{array}$$

with split exact rows, where the two left maps $d_2$ are $d_2 = \sum\limits_{\mu=1}^{\nu} (-1)^{\mu}\delta_{\mu}$, with $\delta_{\mu}$ being induced by the inclusions

$$Y_{i_1,\ldots,i_\nu} \hookrightarrow Y_{i_1,\ldots,\widehat{i_\mu},\ldots,i_\nu}$$

and where the right hand $d_2$ is similarly defined as $d_2 = \sum\limits_{\mu=1}^{\nu-1} (-1)^{\mu}\delta_{\mu}$, with $\delta_{\mu}$ being induced by the inclusions

$$Y_{i_1,\ldots,i_{\nu-1}} \cap Y_r \hookrightarrow Y_{i_1,\ldots,\widehat{i_\mu},\ldots,i_{\nu-1}} \cap Y_r\,.$$

In fact, the commuting of (3') is trivial, and the square (4') commutes since it commutes with $\delta_\mu, 1 \le \mu \le \nu - 1$ in place of $d_2$, whereas $\delta_\nu$ vanishes after projection onto $(Y_r \cap Z)^{(\nu-1)}$ (the last component of $(i_1 \ldots, \hat{i}_\nu)$ cannot be $r$). This implies the commutativity of (3) and (4), and the exactness of the two involved rows.

The commutativity of (2) is clear: For $1 \le i_1 < \ldots < i_d = r$ the specialization map (3.2) is the composition

$$H^d(\overline{U}, d) \xrightarrow{\delta} H^{d-1}(\overline{Y_r \setminus Z}, d-1) \xrightarrow{\delta} H^{d-2}(\overline{Y_{i_{\nu-1}} \cap Y_r \setminus (\bigcup_{i \neq i_{\nu-1},r} (Y_i \cap Y_r))}, d-2)$$

$$\ldots \xrightarrow{\delta} H^1(\overline{Y_{i_2}, \ldots, r \setminus (\bigcup_{i \neq i_2,\ldots,r} (Y_i \cap Y_r)}, 1) \xrightarrow{\delta} H^0(\overline{Y_{i_1,\ldots,r}}, 0).$$

The commutativity of (1) is implied by the commutativity of

$$\begin{array}{lccccccc}
e: & H^d(\overline{U},d) & \xrightarrow{\delta} & H^{d-1}(\overline{Y_{i_d} \setminus \bigcup_{i\neq i_d} Y_i}, d-1) & \longrightarrow \cdots & \cdots & \longrightarrow & H^0(\overline{Y_{i_1,\ldots,i_d}}, 0) \\
 & \uparrow & & \uparrow & & \uparrow & & \| \\
e: & H^d(\overline{X-Z},d) & \xrightarrow{\delta} & H^{d-1}(\overline{Y_{i_d} \setminus \bigcup_{i\neq i_{d,r}} Y_i}, d-1) & \longrightarrow \cdots & \cdots & \longrightarrow & H^0(\overline{Y_{i_1,\ldots,i_d}}, 0)
\end{array}$$

for $1 \le i_1 < \ldots < i_d < r$, where the vertical maps are the restriction maps for the open immersions obtained by deleting $Y_r$ everywhere (note that $Y_{i_1,\ldots,i_d} \cap Y_r = \emptyset$ for $i_d \neq r$). This commutativity follows from the compatibility of the corresponding Gysin sequences with restriction to open subschemes.

Given the diagram (3.3), we can carry out the induction step: by induction the left and right column are exact, hence so is the middle one by a straightforward diagram chase.

We now pass to function fields. Recall the following definition [CT].

**Definition 3.5** Let $k$ be a field and let $F$ be a function field over $k$. For an integer $n$ invertible in $k$, the unramified cohomology $H^i_{nr}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) \subseteq H^i(F, \mathbb{Z}/n\mathbb{Z}(j))$ is defined as the subset of elements lying in the image of

$$H^i_{\text{ét}}(\operatorname{Spec} A, \mathbb{Z}/n\mathbb{Z}\mathbb{Z}(j)) \to H^i(F, \mathbb{Z}/n\mathbb{Z}(j))$$

for all discrete valuation rings $A \subseteq F$ containing $k$.

If $\lambda$ is a discrete valuation of $F$ which is trivial on $k$, and if $A_\lambda$ and $k(\lambda)$ are the associated valuation ring and residue field, respectively, then one has an exact Gysin sequence

$$\ldots H^i_{\text{ét}}(\operatorname{Spec} A_\lambda, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i(F, \mathbb{Z}/n\mathbb{Z}(j)) \xrightarrow{\delta_\lambda} H^{i-1}(k(\lambda), \mathbb{Z}/n\mathbb{Z}(j-1)) \to \ldots\ldots,$$

since purity is known to hold in this situation. We call the map $\delta_\lambda$ the residue map for $\lambda$. This shows:

**Lemma 3.6** One has

$$H^i_{nr}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) = \ker(H^i(F, \mathbb{Z}/n\mathbb{Z}(j)) \to \prod_\lambda H^{i-1}(k(\lambda), \mathbb{Z}/n\mathbb{Z}(j-1)))\,,$$

where the sum is over all discrete valuations $\lambda$ of $F/k$, and the components of the map are the residue maps $\delta_\lambda$.

We will need the following fact (cf. [CT]).

**Proposition 3.7** Let $X$ be a smooth proper variety over $k$, and let $F = k(X)$ be its function field. Then

$$H^i_{nr}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) = \ker(H^i(F, \mathbb{Z}/n\mathbb{Z}\mathbb{Z}(j) \stackrel{\delta_X}{\to} \bigoplus_{v\in X^1} H^{i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(j-1)))$$

where $X^i = \{x \in X \,|\, \dim\mathcal{O}_{X,x} = i\}$ for $i \geq 0$, $k(x)$ is the residue field of $x \in X$, and $\delta$ is the map from the Bloch-Ogus complexes for étale cohomology [BO]. In particular,

$$H^i_{nr}(F/k, \mathbb{Z}/n\mathbb{Z}(j)) \cong H^0_{\mathrm{Zar}}(X, \mathcal{H}^i_n(j))$$

where $\mathcal{H}^i_n(j)$ is the Zariski sheaf on $X$ associated to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mathbb{Z}/n\mathbb{Z}(j))$.

We note that, by definition of the Bloch-Ogus sequence, the components of $\delta_X$ are the residue maps $\delta_{X,x} := \delta_{\lambda(x)}$, where $\lambda(x)$ is the discrete valuation associated to $x$ (so that $A_{\lambda(x)} = \mathcal{O}_{X,x}$ and $k(\lambda(x)) = k(x)$. The main result of the present section is now:

**Theorem 3.8** Let $K$ be a global field, let $\ell$ be a prime invertible in $K$, and let $F$ be a function field in $d$ variables over $K$, $d > 0$, such that $K$ is separably closed in $F$. For every place $v$ of $K$ let $K_{(v)}$ be the Henselization of $K$ at $v$, and let $F_{(v)} = FK_{(v)}$ be the corresponding function field over $K_{(v)}$.
(a) If $K$ is a number field, then the restriction maps induce an isomorphism

$$H^{d+2}_{nr}(F, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1)) \stackrel{\sim}{\longrightarrow} \bigoplus_v H^{d+2}_{nr}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))\,.$$

(b) The same holds for a function field $K$ with perfect hull $K'$, if condition $\mathrm{RS}(F', K')$ holds for the function field $F' = (F \otimes_K K')_{red}$ (see Theorem 2.7).
(c) If $\mathrm{BK}(K, d+1, \ell)$ holds (the Bloch-Kato conjecture, see the introduction), then the same results hold after replacing $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ by $\mathbb{Z}/\ell^n\mathbb{Z}$, for all $n > 0$.

**Proof** Let $a = (a_v) \in \oplus_v H^{d+2}_{nr}(F_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$. Because only finitely many of the $a_v$ are non-zero, and $K_{(v)}$ is algebraic over $K$, there exists a geometrically irreducible variety $U$ over $K$ with function field $F$ such that $a$ is represented by an element $a_U \in \oplus_v H^{d+2}(U \times_K K_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d+1))$. Moreover, by property $\mathrm{RS}(F', K')$ (which holds if $\mathrm{char}(K) = 0$), after possibly shrinking $U$ and passing to a finite inseparable extension $L$ of $K$ and the scheme $(U \times_K L)_{red}$ (which does not change the considered groups), we may assume that $U$ is smooth and can be embedded in a smooth projective variety $X$ over $K$ as an open subvariety such that the complement $Y = X \smallsetminus U$ is a divisor with simple normal crossings, say with components $Y_i$ $(i = 1, \ldots, r)$. By possibly applying Bertini's theorem as in section 2 (before Proposition 2.2) and removing a suitable hyperplane section (which does not matter for our purposes), we may assume that $X \smallsetminus Y_1$ is affine, i.e., that $U \subseteq X \supseteq Y$ satisfies the assumptions of theorem 3.1.

Abbreviating $H^i(?, j)$ for $H^i(?, \mathbb{Q}_\ell/\mathbb{Z}_\ell(j))$, we get a commutative diagram with exact rows

(3.9)

$$\begin{array}{ccccccc}
0 \longrightarrow & \bigoplus\limits_v H^{d+2}_{nr}(F_{(v)}/K_{(v)}, d+1) & \longrightarrow & \bigoplus\limits_v H^{d+2}(F_{(v)}, d+1) & \xrightarrow{\bigoplus\limits_v \delta_{X_{(v)}}} & \bigoplus\limits_v \bigoplus\limits_{y \in X^1_{(v)}} H^{d+1}(K_{(v)}(y), d) \\
& \uparrow \beta' & & \uparrow \beta(F/K) & & \uparrow \beta'' \\
0 \longrightarrow & H^{d+2}_{nr}(F/K, d+1) & \longrightarrow & H^{d+2}(F, d+1) & \xrightarrow{\delta_X} & \bigoplus\limits_{x \in X^1} H^{d+1}(K(x), d)
\end{array}$$

where $X_{(v)} = X \times_K K_{(v)}$ and in which $\beta(F/K)$ is the restriction map, $\beta''$ is induced by the restrictions for the field extensions $K_{(v)}/K(x)$ for $y$ lying above $x$, and $\beta'$ is the induced map. Note that $F_{(v)} \cong F \otimes_K K_{(v)}$ is the function field of $X_{(v)}$ over $K_{(v)}$. The commutativity of the right square is easily checked (contravariance of Gysin sequences for pro-étale maps). Now, for $x \in X^1$ and a place $v$ of $K$, every $y \in X_{(v)}$ lying above $x$ is again of codimension 1, since $X_{(v)} \to X$ is integral. Hence

(3.10)
$$\coprod_{\substack{y \in X^1_{(v)} \\ y|x}} \mathrm{Spec}(K_{(v)}(y)) = \coprod_{\substack{y \in X_{(v)} \\ y|x}} \mathrm{Spec}(K_{(v)}(y)) = X_{(v)} \times_X K(x)\ ,$$

the fibre of the pro-étale morphism $X_{(v)} \to X$ over $x$. This is again isomorphic to

(3.11)
$$\begin{array}{rcl}
(X \times_K K_{(v)}) \times_X K(x) & \cong & \mathrm{Spec}(K(x) \otimes_K K_{(v)}) \\
& \cong & \mathrm{Spec}(K(x) \otimes_{K\{x\}} (K\{x\} \otimes_K K_{(v)})) \\
& \cong & \coprod\limits_{w|v} \mathrm{Spec}(K(x) \otimes_{K\{x\}} K\{x\}_{(w)})
\end{array}$$

where $K\{x\}$ is the separable closure of $K$ in $K(x)$ (which is a finite extension of $K$) and where $w$ runs over the places $w$ of $K\{x\}$ above $v$. This shows that $\beta''$ can be identified with the map

(3.12)
$$\bigoplus_{x \in X^1} \beta(K(x)/K\{x\}) : \bigoplus_{x \in X^1} H^{d+1}(K(x), d) \longrightarrow \bigoplus_{x \in X^1} \bigoplus_{w \in P(K\{x\})} H^{d+1}(K(x)_{(w)}, d)$$

where $P(K\{x\})$ is the set of places of the global field $K\{x\}$. Hence $\beta''$ is injective as well as $\beta(F/K)$, by Theorem 2.7. (Note that $K(x)$, for $x \in X^1$, is a function field in $d-1$ variables over $K\{x\}$). A diagram chase then shows that it suffices to show that the image $\overline{a}$ of $a$ in coker$\beta(F/k)$ is zero. By assumption, $\overline{a}$ maps to zero under

(3.13)
$$\text{coker } \beta(F/K) \longrightarrow \text{coker } \beta'' = \bigoplus_{x \in X^1} \text{coker } \beta(K(x)/K\{x\})\,.$$

**Lemma 3.14** The map (3.13) can be identified with the map of cofixed modules

$$H^d(F\overline{K}, d)_{G_K} \longrightarrow \bigoplus_{x \in X^1} H^{d-1}(K(x) \otimes_K \overline{K}, d-1)_{G_K} \cong (\bigoplus_{y \in \overline{X}^1} H^{d-1}(\overline{K}(y), d-1)\,)_{G_K}$$

induced by the residue map $\delta_{\overline{X}}$ for $\overline{X} = X \times_K \overline{K}$.

**Proof** By (3.10) and (3.11), the map $\beta''$ can also be identified with the map

$$\bigoplus_{x\in X^1} (\, \beta(K(x)/K) : \, H^{d+1}(K(x), d) \longrightarrow \bigoplus_{v \in P(K)} H^{d+1}(K(x) \otimes_K K_{(v)}, d)\,)\,.$$

Therefore, the map (3.13) can be identified with the map coker $\beta_1 \longrightarrow$ coker $\beta_2$ induced the commutative diagram

$$\begin{array}{ccc} \bigoplus_v H^2(K_v, H^d(F\overline{K}, d+1)) & \longrightarrow & \bigoplus_{x\in X^1} \bigoplus_v H^2(K_v, H^{d-1}(K(x)\otimes_K \overline{K}, d)) \\ \uparrow{\scriptstyle \beta_1} & & \uparrow{\scriptstyle \beta_2} \\ H^2(K, H^d(F\overline{K}, d+1)) & \longrightarrow & \bigoplus_{x\in X^1} H^2(K, H^{d-1}(K(x)\otimes_K \overline{K}, d)) \end{array},$$

where the vertical maps are induced by the residue maps

$$H^d(F\overline{K}, d+1) \longrightarrow \bigoplus_{x\in X^1} H^{d-1}(K(x)\otimes_K \overline{K}, d) \cong \bigoplus_{y\in \overline{X}^1} H^{d-1}(\overline{K}(y), d)$$

for $\overline{X}$. This follows from Proposition 1.2, Remark 1.4 (a) and the fact that the Hochschild-Serre spectral sequence is compatible with the connecting morphisms for Gysin sequences. The latter statement follows from the fact that the Hochschild-Serre spectral sequence for étale (hyper)cohomology of complexes is functorial with respect to morphisms in the derived category, and that the Gysin isomorphisms are compatible with proétale base change.

Finally, as seen in the proof of Proposition 1.2, for any discrete torsion $\mathbb{Z}_\ell - G_K$-modules $M$ there is a canonical isomorphism

$$\text{(3.15)} \qquad \operatorname{coker}(\beta_M : \, H^2(K,M) \to \bigoplus_v H^2(K_v, M)) \xrightarrow{\sim} M(-1)_{G_K}\,,$$

which is functorial in $M$. This proves Lemma 3.14.

We are now ready to prove Theorem 3.8. Recall that our element $a$ was the image of an element $a_U \in \oplus_v H^{d+2}(U \times_K K_{(v)}, d+1)$. On the other hand, we have a commutative diagram of edge morphisms for the Hochschild-Serre spectral sequence

$$\begin{array}{ccc} H^{d+2}(F_{(v)}, d+1) & \xrightarrow{e_{F,v}} & H^2(K_{(v)}, H^d(F\overline{K}, d+1)) \\ \uparrow & & \uparrow \\ H^{d+2}(U\times_K K_{(v)}, d+1) & \xrightarrow{e_{U,v}} & H^2(K_{(v)}, H^d(U\times_K K_{(v)}, d+1)) \end{array}.$$

Here the lower edge morphism exists, because $U$ is affine by our assumptions and hence $U \times_K K_{(v)}$ has cohomological dimension $d$. This diagram shows that the image $\overline{a}$ of $a$ in coker $\beta(F/K) = H^d(F\overline{K}, d)_{G_K}$ is the image of an element $\overline{a_U} \in H^d(U\times_K \overline{K}, d)_{G_K}$ (to wit, the image of $(\oplus_v e_{U,v})(b)$ in coker $\beta_{M(U)} = M(U)(-1)_{G_K}$ for $M(U) := H^d(U \times_K \overline{K}, d+1)$). Now we claim that $\overline{a_U}$ lies in the kernel of the map

$$e : H^d(\overline{U}, d)_{G_K} \to H^0(\overline{Y^{[d]}}, 0)_{G_K}$$

introduced in Theorem 3.1. Since the assumptions of 3.1 are fulfilled for $U$, we then conclude that $\overline{a_U}$ is zero and hence $\overline{a}$ is zero as wanted. With the notations of (3.2), the claimed vanishing of $e(\overline{a_U})$ follows from the following commutative diagram, for each $(i_1, \ldots, i_d)$ and each $y \in Y_{i_1,\ldots,i_d}$

(3.16)

$$\begin{array}{ccccc}
H^d(\overline{U},d)_{G_K} & \longrightarrow & H^{d-1}(\overline{Y_{i_d}\smallsetminus(\bigcup_{i\neq i_d} Y_i)},d-1)_{G_K} & \longrightarrow & H^{d-2}(\overline{Y_{i_{d-1},i_d}\smallsetminus(\bigcup_{i\neq i_{d-1},i_d} Y_i)},d-2)_{G_K} \\
\downarrow & & \downarrow & & \downarrow \\
H^d(F\overline{K},d)_{G_K} & \longrightarrow & H^{d-1}(K(y_{i_d})\otimes_K \overline{K},d-1)_{G_K} & \longrightarrow & H^{d-2}(K(y_{i_{d-1},i_d})\otimes_K \overline{K},d-2)_{G_K}
\end{array}$$

$$\begin{array}{ccc}
\cdots & \longrightarrow & H^0(\overline{\{y\}},0)_{G_K} \\
 & & \Vert \\
\cdots & \longrightarrow & H^0(K(y_{i_1,\dots,i_d})\otimes_K \overline{K},0)_{G_K}
\end{array}$$

in which $y_{\underline{i}}$ is the generic point of the component $Y_{\underline{i}}^y$ of $Y_{\underline{i}}$ in which $y$ lies, for any $\underline{i}=(i_r,\dots,i_d)$, so that $K(y_{\underline{i}})$ is the function field of $Y_{\underline{i}}^y$. In fact, the maps in the bottom line are all induced by the residue maps, by definition, and the image of $\overline{a_U}$ under the left vertical map is $\overline{a}$. As we have noted, the image of $\overline{a}$ in $H^{d-1}(K(y_{i_{\underline{d}}})\otimes_K\overline{K},d-1)_{G_K}$ vanishes, for every choice of $(i_1,\dots,i_d)$, $1\leq i_1<\dots<i_d\leq r$ (note that $y_{i_d}\in X^1$). Therefore the image of $\overline{a_U}$ in $H^0(\overline{\{y\}},0)_{G_K}$ vanishes for every $y\in\overline{Y^{[d]}}$ as claimed.

## §4 A Hasse principle for Bloch-Ogus-Kato complexes

Let $X$ be an excellent scheme, let $n\geq 1$ be an integer, and let $r,s\in\mathbb{Z}$. Under some conditions on $X$, $n$ and $(r,s)$, there are homological complexes of Gersten-Bloch-Ogus-Kato type

$C^{r,s}(X,\mathbb{Z}/n\mathbb{Z})$ :

$$\dots\to\bigoplus_{x\in X_i}H^{r+i}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i))\xrightarrow{\partial}\bigoplus_{x\in X_{i-1}}H^{r+i-1}(k(x),\mathbb{Z}/n\mathbb{Z}(s+i-1))\to\dots$$
$$\dots\to\bigoplus_{x\in X_0}H^{r}(k(x),\mathbb{Z}/n\mathbb{Z}(s))$$

where the term for $X_i=\{x\in X\mid \dim(x)=i\}$ is placed in degree $i$.

If $X$ is separated of finite type over a field $L$, $n$ is invertible in $L$ and $(r,s)$ are arbitrary, these complexes where defined by Bloch and Ogus [BO], by using the étale homology for such schemes, defined by

$$H_a(X,\mathbb{Z}/n\mathbb{Z}(b)):=H^{-a}(X,Rf^!\mathbb{Z}/n\mathbb{Z}(-b)), \tag{4.1}$$

where $Rf^!$ is the functor on constructible étale $\mathbb{Z}/n\mathbb{Z}$-schemes defined in [SGA 4] XVIII, for the structural morphism $f:X\to\operatorname{Spec}L$. In fact, Bloch and Ogus constructed a niveau spectral sequence

$$E^1_{p,q}(X,\mathbb{Z}/n\mathbb{Z}(b))=\bigoplus_{x\in X_p}H_{p+q}(k(x),\mathbb{Z}/n\mathbb{Z}(b))\Rightarrow H_{p+q}(X,\mathbb{Z}/n\mathbb{Z}(b))$$

where, by definition, $H_a(k(x),\mathbb{Z}/n\mathbb{Z}(b))=\varinjlim H_a(U,\mathbb{Z}/n\mathbb{Z}(b))$ for $x\in X$, where the limit is over all open subschemes $U\subseteq\overline{\{x\}}$ of the Zariski closure of $x$. By purity, there is an isomorphism

$H_a(U, \mathbb{Z}/n\mathbb{Z}(b)) \cong H^{2p-a}(U, \mathbb{Z}/n\mathbb{Z}(p-b))$ for $U$ irreducible and smooth of dimension $p$ over $L$. Thus one has a canonical isomorphism

$$E^1_{p,q}(X, \mathbb{Z}/n\mathbb{Z}(b)) \cong \bigoplus_{x \in X_p} H^{p-q}(k(x), \mathbb{Z}/n\mathbb{Z}(p-b))\,. \tag{4.2}$$

This is clear for a perfect field $L$, because then $\overline{\{x\}}$ is generically smooth. So the limit can be carried out over the smooth $U \subseteq \overline{\{x\}}$, and

$$\varinjlim_{U \subseteq \overline{\{x\}}} H^{2p-a}(U, \mathbb{Z}/n\mathbb{Z}(p-b)) = H^{2p-a}(k(x), \mathbb{Z}/n\mathbb{Z}(p-b)),$$

since $\varprojlim U = \operatorname{Spec} k(x)$, and since étale cohomology commutes with this limit. For a general field $L$, we may pass to the separable hull, because of invariance of étale cohomology with respect to base change with radical morphisms.

Using the identification (4.2), one may define

$$C^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) = E^1_{\cdot,-r}(X, \mathbb{Z}/n\mathbb{Z}(-s))\,.$$

With this definition, one obtains the following description of the differential

$$\partial : \bigoplus_{x \in X_i} H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{x \in X_{i-1}} H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1))\,.$$

We may assume that $L$ is perfect. For $y \in X_i$ and $x \in X_{i-1}$ let $\partial_{y,x} = \partial^X_{y,x}$ be the $(y,x)$-component of $\partial$. If $x \notin \overline{\{y\}}$, then $\partial_{y,x} = 0$. If $x$ is a smooth point of $\overline{\{y\}}$, then there is an open smooth neighbourhood $x \in U \subseteq \overline{\{y\}}$. Moreover, any $\alpha \in H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i))$ lies in the image of $H^{r+i}(V, \mathbb{Z}/n\mathbb{Z}(s+i)) \to H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i))$ for some open $V \subseteq U$. Moreover, by making $U$ (and $V$) smaller we may assume that $Z = U \setminus V$ is irreducible and smooth as well, and that $x$ is the generic point of $Z$. Then one has a commutative diagram

$$\begin{array}{ccc} H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i) & \xrightarrow{\partial_{y,x}} & H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \\ \uparrow & & \uparrow \\ H^{r+i}(V, \mathbb{Z}/n\mathbb{Z}(s+i) & \xrightarrow{\partial} & H^{r+i-1}(Z, \mathbb{Z}/n\mathbb{Z}(s+i-1)) \end{array} \tag{4.3}$$

where the vertical maps come from passing to the generic points, and $\partial$ is the connecting morphism for the Gysin sequence for $(U, Z)$. This determines $\partial_{y,x}(\alpha)$. If $x \in \overline{\{y\}}$, but is not necessarily a smooth point of $Y = \overline{\{y\}}$, let $\tilde{Y} \to Y$ be the normalization of $Y$. Any point $x' \in \tilde{Y}$ above $x$ has codimension 1 and thus is a regular point in $\tilde{Y}$. Since the niveau spectral sequence is covariant with respect to proper morphisms, there is a commutative diagram

$$\begin{array}{ccc} H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i)) & \xrightarrow{\bigoplus_{x'|x} \partial^{\tilde{Y}}_{y,x'}} & \bigoplus_{x'|x} H^{r+i-1}(k(x'), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \\ \| & & \downarrow \pi_* \\ H^{r+i}(k(y), \mathbb{Z}/n\mathbb{Z}(s+i)) & \xrightarrow{\partial_{y,x}} & H^{r+i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i-1)) \end{array} \tag{4.4}$$

where $\pi_*$ is induced by $\pi : \tilde{Y} \to Y \hookrightarrow X$. One can check that $\pi_*((\alpha_{x'})) = \sum_{x'|x} \operatorname{Cor}_{x'|x}(\alpha_{x'})$, where $\operatorname{Cor}_{x'|x} : H^{\mu}(k(x'), \mathbb{Z}/n\mathbb{Z}(\nu)) \to H^{\mu}(k(x), \mathbb{Z}/n\mathbb{Z}(\nu))$ is the corestriction for the finite extension

$k(x')/k(x)$ (this also makes sense if this extension has some inseparable part). Since $\partial^{\tilde{Y}}_{y,x'}$ can be treated as before, this determines $\partial_{y,x}$.

For a separable function field $L$ of transcendence degree $d$ over a perfect field $k$ of characteristic $p > 0$, a separated scheme $X$ of finite type over $L$ and $n$ a power of $p$ it was shown in [JS1] and [JSS] 2.11.3 that the theory of Bloch and Ogus can be literally extended to this situation for the case $b = -d$, where the cohomology groups $H^i(X, \mathbb{Z}/n\mathbb{Z}(j))$ and $H^i(k(x), \mathbb{Z}/n\mathbb{Z}(j))$ are defined as in (0.2).

For a general excellent scheme $X$, and arbitrary $n$, the complexes $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ were defined by Kato (and named $C_n^{r,s}(X)$, cf. [Ka]), in a more direct way, by using the Galois cohomology of discrete valuation fields, assuming the following condition:

(∗) If $r = s + 1$ and $p$ is a prime dividing $n$, then for any $x \in X_0$ with $\mathrm{char}(k(x)) = p$ one has $[k(x) : k(x)^p] \leq s$.

It is shown in [JSS] that both definitions agree (up to well-defined signs) for varieties over fields in the cases discussed above.

Now let $K$ be a global field, and let $X$ be a variety over $K$. For every place $v$ of $K$ let $X_v = X \times_K K_v$. Then condition (∗) holds for $X$ and the $X_v$ for $(r, s) = (2, 1)$ and arbitrary $n$. Moreover, one has natural restriction maps $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(X_v, \mathbb{Z}/n\mathbb{Z})$, and Kato stated the following conjecture ([Ka], Conjecture 0.4).

**Conjecture 4.5** Let $X$ be connected, smooth and proper. Then one has isomorphisms

$$H_a(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \xrightarrow{\sim} \bigoplus_v H_a(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})) \text{ for all } a \neq 0,$$

and an exact sequence

$$0 \to H_0(C^{2,1}(X, \mathbb{Z}/n\mathbb{Z})) \to \bigoplus_v H_0(C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})) \to \mathbb{Z}/n\mathbb{Z} \to 0\,.$$

**Remark 4.6** (a) For $X = \mathrm{Spec}\ (L)$, $L$ any finite extension of $K$, the cohomology groups vanish for $a \neq 0$, and the sequence for $a = 0$ becomes the exact sequence

$$0 \to H^2(L, \mathbb{Z}/n\mathbb{Z}(1)) \to \bigoplus_{w \in P(L)} H^2(L_w, \mathbb{Z}/n\mathbb{Z}(1)) \to \mathbb{Z}/n\mathbb{Z} \to 0\,,$$

which is the $n$-torsion of the classical exact sequence

$$0 \to Br(L) \to \bigoplus_{w \in p(L)} Br(L_w) \xrightarrow{\sum_w \mathrm{inv}_w} \mathbb{Q}/\mathbb{Z} \to 0$$

for the Brauer groups (where $\mathrm{inv}_w : Br(L_w) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$ is the 'invariant' map). Thus Kato's conjecture is a generalization of this famous sequence to higher dimensional varieties.
(b) Let $C'(X, \mathbb{Z}/n\mathbb{Z})$ be the cokernel of the homomorphism

$$\beta_{X,n} : C^{2,1}(X, \mathbb{Z}/n\mathbb{Z}) \to \oplus_v C^{2,1}(X_v, \mathbb{Z}/n\mathbb{Z})\,.$$

Then conjecture 4.5 is implied to the following two statements:

(i) $\beta_{X,n}$ is injective.
(ii) $H_0(C'(X,\mathbb{Z}/n\mathbb{Z})) = \mathbb{Z}/n\mathbb{Z}$, and $H_a(C'(X,\mathbb{Z}/n\mathbb{Z})) = 0$ for $a > 0$.
Conversely conjecture 4.5 implies (i) and (ii) by the known case (a) and induction on dimension, provided the occurring function fields have smooth and proper models over the perfect hull of $K$ (which holds over number fields).

We prove the following $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-version of Conjecture 4.5. For a prime $\ell$ invertible in $K$ define $C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ as the inductive limit over $n$ of the complexes $C^{2,1}(X,\mathbb{Z}/\ell^n\mathbb{Z})$; similarly for $X_v$ and $X_{(v)}$. The restriction maps

$$C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to C^{2,1}(X_v,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{4.7}$$

induce a homomorphism of complexes

$$\beta_{X,\ell^\infty} : C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \bigoplus_v C^{2,1}(X_v,\mathbb{Q}_\ell/\mathbb{Z}_\ell)\,,$$

and we write $C'(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for its cokernel.

**Theorem 4.8** Let $K$ be a number field, let $\ell$ be a prime, and let $X$ be a connected smooth projective variety over $K$.
(a) The map $\beta_{X,\ell^\infty}$ is injective.
(b) One has

$$H_a(C'(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} 0 & , \quad a \neq 0, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a = 0\,. \end{cases}$$

For global function fields we obtain a conditional result. For an arbitrary field $k$ with perfect hull $k'$ consider the following conditions on resolution of singularities, where $d$ is a non-negative integer.

$\mathrm{RS}_{gc}(k',d)$: For any smooth variety $V'$ of dimension $\leq d$ over $k'$ there is a cofinal set of open subvarieties $U' \subset V'$ which have a *good compactification* (i.e., fulfill condition $\mathrm{RS}_{gc}(U')$).

$\mathrm{RS}_{pb}(k',d)$: For any proper reduced variety $Z$ of dimension $\leq d$ over $k'$ there exists a *proper birational* morphism $\pi : \tilde{Z} \to Z$ such that $\tilde{Z}$ is smooth and projective.

**Theorem 4.9** Let $K$ be a global field of positive characteristic, let $\ell$ be a prime invertible in $K$, and let $X$ be a connected smooth projective variety of dimension $d$ over $K$.
(a) If $\mathrm{RS}_{gc}(K',d)$ holds, then $\beta_{X,\ell^\infty}$ is injective.
(b) If $\mathrm{RS}_{gc}(K',d)$ and $\mathrm{RS}_{pb}(K,d)$ hold then

$$H_a(C'(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} 0 & , \quad a \neq 0, \\ \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a = 0\,. \end{cases}$$

In particular, the statements in (a) and (b) hold for $d \leq 3$.

**Remark 4.10** (a) As in Remark 4.6 it follows that Theorem 4.8 implies conjecture 4.5 for number fields and $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients, and is in fact equivalent to it. The same is true for the statements of Theorem 4.9.
(b) It will follow from the proof that, for a fixed $X$, it suffices to replace condition $\mathrm{RS}_{gc}(K',d)$ in

Theorem 4.9 by the condition that $\mathrm{RS}_{gc}(K(x)', K\{x\}')$ (see Theorem 2.7) holds for all $x \in X$, where $K\{x\}$ is the separable closure of $K$ in $K(x)$. Similarly, it suffices to have the condition written in $\mathrm{RS}_{pb}(K', d)$ only for all $Z' = (Z \times_K K')_{rd}$ where $Z \subset X$ is a reduced closed subscheme.
(c) Theorems 4.8 and 4.9 have the same proof. The only difference is that the cited conditions on resolution of singularities hold for fields of characteristic zero by [Hi].
(d) It is well-known that the conditions $\mathrm{RS}_{gc}(K', d)$ and $\mathrm{RS}_{pb}(K', d)$ hold for all fields $K$ for $d \leq 2$. But by recent results in [CP1], [CP2] and [CJS] they also holds for finitely generated fields $K$ and $d = 3$. This shows the last claim in 4.9.

As for conjecture 4.5 with finite coefficients, we note the following.

**Lemma 4.11** Let $K$ be a global field, let $\ell$ be a prime, and let $n$ be a natural number. Assume condition $(\mathrm{BK})_{d+1,\ell}$ in the introduction (the Milnor-Bloch-Kato conjecture on the Galois symbol in degree $d+1$ with $\mathbb{Z}/\ell$-coefficients). Then for all smooth projective varieties $X$ over $K$ of dimension $\leq d$, conjecture 4.5 for $X$ and $\mathbb{Q}_\ell/\mathbb{Z}_\ell$-coefficients implies conjecture 4.5 for $X$ and $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficients.

We will prove this below. We will first turn to the

**Proof of Theorem 4.7/4.8 (a)** Assume that $K$ is a number field or that $\mathrm{RS}_{gc}(K, d)$ holds. First note that the restriction map (4.7) factors as

$$C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to C^{2,1}(X_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

where $X_{(v)} = X \times_K K_{(v)}$. These maps of complexes have components

$$\bigoplus_{x \in X_i} H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)) \to \bigoplus_{x \in (X_{(v)})_i} H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)) \to \bigoplus_{x \in (X_v)_i} H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

which in turn can be written as the sum, over all $x \in X_i$, of maps

$$H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1) \to \bigoplus_{\substack{x' \in (X_{(v)})_i \\ x'|x}} H^{i+2}(k(x'), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1) \to \bigoplus_{\substack{x'' \in (X_v)_i \\ x''|x}} H^{i+2}(k(x''), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)$$

By the same reasoning as in the proof of Theorem 3.7, the first map can be identified with

$$H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)) \to \bigoplus_{w|v} H^{i+2}(k(x)K\{x\}_{(w)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1))$$

where $K\{w\}$ is the separable closure of $K$ in $k(x)$, which is a finite extension of $K$, and where $w$ runs over all places of $K\{x\}$ above $v$. Note that $k(x)$ is a function field of transcendence degree $i$ over $K\{x\}$. Therefore the restriction maps above induce an injective map into the direct sum

$$H^{i+2}(k(x), \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1)) \to \bigoplus_{w \in P(K\{x\})} H^{i+2}(k(x)_{(w)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i+1))$$

by Theorem 2.7. This shows that we get maps

$$C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_v C^{2,1}(X_v, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{4.12}$$

of which the first one is injective. The claim of Theorem 4.8/4.9 (a) therefore follows from the next claim.

**Proposition 4.13** Let $K$ be a global field, and let $v$ be a place of $K$. For every variety $V$ over $K_{(v)}$, every integer $n$ invertible in $K$ and all $r, s \in \mathbb{Z}$, the natural map

$$C^{r,s}(V, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(V \times_{K_{(v)}} K_v, \mathbb{Z}/n\mathbb{Z})$$

is injective.

**Proof** In degree $i$, this map is the sum over all $x \in V_i$ of restriction maps

$$H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to \bigoplus_{\substack{x' \in \tilde{V}_i \\ x'|x}} H^{r+i}(k(x'), \mathbb{Z}/n\mathbb{Z}(s+i))$$

where $\tilde{V} = V \times_{K_{(v)}} K_v$. For $x \in V_i$, $k(x)$ is the function field of the integral subscheme (of dimension $i$) $Z = \overline{\{x\}} \subseteq V$. Because $K_v/K_{(v)}$ is separable, and $K_{(v)}$ is algebraically closed in $K_v$, $\tilde{Z} = Z \times_{K_{(v)}} K_v \hookrightarrow \tilde{V}$ is a closed integral subscheme whose generic point $\tilde{x}$ is in $\tilde{V}_i$ and lies above $x$. Let $L$ be the algebraic closure of $K_{(v)}$ in $k(x)$. Then $\tilde{L} = L \otimes_{K_{(v)}} K_v$ is a field, $Z$ is geometrically integral over $L$ with function field $L(Z) = k(x)$, and $\tilde{Z} = Z \times_L \tilde{L}$ with function field $\tilde{L}(\tilde{Z}) = k(\tilde{x})$. Moreover, $L$ is henselian, with completion $\tilde{L}$. Thus it follows from Theorem 2.10 that the natural map

$$H^{r+i}(k(x), \mathbb{Z}/n\mathbb{Z}(s+i)) \to H^{r+i}(k(\tilde{x}), \mathbb{Z}/n\mathbb{Z}(s+i))$$

is injective for all $r, s, i \in \mathbb{Z}$ and all $n \in \mathbb{N}$ invertible in $K_{(v)}$. This proves Proposition 4.13 and thus 4.8/4.9 (a).

Now we start the proof of Theorem 4.8/4.9 (b). The following rigidity result is shown in [Ja 6].

**Theorem 4.14** Let $K$ be a global field, and let $v$ be a place of $K$. For every variety $V$ over $K_{(v)}$, every integer $n$ invertible in $K$ and all $r, s \in \mathbb{Z}$, the natural morphism of complexes

$$C^{r,s}(V, \mathbb{Z}/n\mathbb{Z}) \to C^{r,s}(V \times_{K_{(v)}} K_v, \mathbb{Z}/n\mathbb{Z})$$

is a quasi-isomorphism, i.e., induces isomorphisms in the homology.

In view of this result, and of the factorization (4.12), it suffices to prove Theorem 4.8/4.9 (b) after replacing $X_v$ by $X_{(v)}$ for each $v$. In fact, we have a quasi-isomorphism

(4.15) $$C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\sim} \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z})\,,$$

where the complex $\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z})$ is defined by the exact sequence

(4.16) $$C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0\,.$$

So our task is to show $H_0(\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$, and $H_a(\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) = 0$ for $a \neq 0$, if $X$ is connected, smooth, projective over $K$. Note that all complexes in (4.15) are concentrated in degrees $0, \ldots, d := \dim(X)$.

Next we note the following.

**Lemma 4.17** The complex $\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ can be canonically identified with the complex

$$\ldots \to \bigoplus_{x\in X_r} H^r(K(x)\otimes_K \overline{K},\mathbb{Q}_\ell/\mathbb{Z}_\ell(r))_{G_K} \to \ldots \quad \ldots \to \bigoplus_{x\in X_0} H^0(K(x)\otimes_K \overline{K},\mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_{G_K}\,,$$

i.e., the complex $C^{0,0}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K}$. Similarly, for a natural number $n$, the complex

$$\overline{C}(X,\mathbb{Z}/\ell^n\mathbb{Z}) := \mathrm{coker}[C^{2,1}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \to \oplus_v\, C^{2,1}(X_{(v)},\mathbb{Z}/\ell^n\mathbb{Z})]$$

can be identified with the complex $C^{0,0}(\overline{X},\mathbb{Z}/\ell^n\mathbb{Z})_{G_K}$.

**Proof** This follows easily via the arguments used in the proof of 3.14, together with the explicit description of the differentials in this complex in (4.5) and the covariance of the Hochschild-Serre spectral sequence for corestrictions.

By lemma 4.17, theorem 4.8/4.9 (b) is implied by the following more general theorem.

**Theorem 4.18** Let $K$ be a finitely generated field, let $\ell$ be a prime invertible in $K$ and let $X$ be an irreducible smooth projective variety of dimension $d$ over $K$. If $K$ has positive characteristic, assume that the properties $\mathrm{RS}_{gc}(K',d)$ and $\mathrm{RS}_{pb}(K',d)$ hold where $K'$ is the perfect hull of $K'$ Then for the Kato complex $\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) := C^{0,0}(\overline{X},\mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K}$ one has

$$H_a(\overline{C}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a=0 \\ 0 & , \quad a\neq 0\,. \end{cases}$$

The proof of this will be given in the next section. But with the definitions above we can now give the

**Proof of Lemma 4.11** Let $F$ be a function field in $d$ variables over $K$. Then condition $\mathrm{BK}(K,d+1,\ell)$ implies that the sequence

$$0 \to H^{d+2}(F,\mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \to H^{d+2}(F,\mathbb{Q}/\mathbb{Z}_\ell(d+1)) \xrightarrow{\ell^n} H^{d+2}(F,\mathbb{Q}/\mathbb{Z}_\ell(d+1))$$

is exact (see (2.8)), and the same holds for all $F_v$. On the other hand it is known that $\mathrm{BK}(K,d+1,\ell)$ implies $\mathrm{BK}(K,i,\ell)$ for all $i\leq d+1$. From this we get an exact sequence

$$0 \to C^{2,1}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \to C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C^{2,1}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)\,,$$

and we see that the injectivity of $\beta_{X,\ell^\infty}$ implies the injectivity of $\beta_{X,\ell^n}$ (see the proof of Theorem 2.7 (c)). Now we consider the cokernel of $\beta$.

First assume that $K$ is a global function field. Then we claim that we even have an exact sequence

$$0 \to H^{d+2}(F,\mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \to H^{d+2}(F,\mathbb{Q}/\mathbb{Z}_\ell(d+1)) \xrightarrow{\ell^n} H^{d+2}(F,\mathbb{Q}/\mathbb{Z}_\ell(d+1)) \to 0\,,$$

similarly for all $F_v$. In fact, we have $H^{d+3}(F,\mathbb{Z}/\ell^n\mathbb{Z}(d+1))=0$: If $\ell\neq \mathrm{char}(F)$, then $F$ has $\ell$-cohomological dimension $d+2$, and if $\ell=p=\mathrm{char}(F)$, then we have $H^{d+3}(F,\mathbb{Z}/\ell^n\mathbb{Z}(d+1)) = H^2(F,W_n\Omega_{F,log})$, but $F$ has $p$-cohomological dimension 1. Exactly the same reasoning works for $F_v$. Writing, for $n$ a positive integer or $n=\infty$,

$$C_n := \mathrm{coker}[H^{d+2}(F,\mathbb{Z}/\ell^n\mathbb{Z}(d+1)) \longrightarrow \oplus_v\, H^{d+2}(F_v,\mathbb{Z}/\ell^n\mathbb{Z}(d+1))]\,,$$

where we set $\mathbb{Z}/\ell^\infty\mathbb{Z} := \mathbb{Q}_\ell/\mathbb{Z}_\ell$, we obtain an exact sequence

$$0 \to C_n \to C_\infty \to C_\infty \to 0 .$$

Applied to the morphisms $\beta_{X,\ell^n}$ for $n \in \mathbb{N} \cup \{\infty\}$, we get an exact sequence

$$0 \to C'(X, \mathbb{Z}/\ell^n\mathbb{Z}) \to C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C'(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0 ,$$

and the claim of 4.11 follows.

Now let $K$ be a number field. If $\ell \neq 2$ or if $K$ has no real places, then $F$ has $\ell$-cohomological dimension $d+2$, and we can argue in the same way. In general, we can argue in the following way. In any case, a function field $F$ of transcendence degree $d$ over an algebraically closed field has $\ell$-cohomological dimension $d$ for $\ell$ invertible in $F$. It follows that for any variety $X$ over $K$ the sequence

$$C^{0,0}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \to C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0$$

is exact, where $\overline{X} = X \times_K \overline{K}$ for an algebraic closure $\overline{K}$ of $K$. Obviously this complex stays exact if we pass to the co-invariants under $G_K$, the absolute Galois group of $K$.

Now assume that $\beta_{X,\ell^\infty}$ is injective. By the above we get the following commmutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \xrightarrow{\beta_{X,\ell^\infty}} & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \xrightarrow{\beta_{X,\ell^\infty}} & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle i} & & \\
 & & C^{2,1}(X, \mathbb{Z}/\ell^n\mathbb{Z}) & \xrightarrow{\beta_{X,\ell^n}} & \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Z}/\ell^n\mathbb{Z}) & \longrightarrow & \overline{C}(X, \mathbb{Z}/\ell^n\mathbb{Z}) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

A simple diagram chase now shows that $\beta_{X,\ell^n}$ and $i$ are injective, which gives an exact sequence

$$0 \longrightarrow \overline{C}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \longrightarrow \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0 .$$

This implies conjecture 4.5 for $\mathbb{Z}/\ell^n\mathbb{Z}$-coefficients by 4.15 and 4.6 (b), and concludes the proof of Lemma 4.11.

The idea for proving Theorem 4.18 is to 'localize' the question; but for this we will have to leave the realm of smooth projective varieties. First recall that the complexes $C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})$ exist for arbitrary varieties $X$ over a field $L$, under the conditions on $X$, $n$ and $(r,s)$ stated at the beginning of this section. If $K$ is a global field, the restriction map $C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \Pi_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ still has image in the direct sum and is injective (by 1.2 and the same argument as for 4.8/4.9 (a)), and we may define $\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for arbitrary varieties $X$ over $K$ by exactness of the sequence

$$0 \to C^{2,1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_v C^{2,1}(X_{(v)}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0 . \tag{4.19}$$

At the same time, by the same arguments as in Lemma 4.17, we have an isomorphism

$$\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := C^{0,0}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{G_K}$$

for $\ell$ invertible in $K$.

**Definition 4.20** Let $L$ be a field, and let $\mathcal{C}$ be a category of schemes of finite type over $L$ such that for each scheme $X$ in $\mathcal{C}$ also every closed immersion $i : Y \hookrightarrow X$ and every open immersion $j : U \hookrightarrow X$ is in $\mathcal{C}$.
(a) Let $\mathcal{C}_*$ be the category with the same objects as $\mathcal{C}$, but where morphisms are just the proper maps in $\mathcal{C}$. A homology theory on $\mathcal{C}$ is a sequence of covariant functors

$$H_a(-) : \mathcal{C}_* \to \text{(abelian groups)} \qquad (a \in \mathbb{Z})$$

satisfying the following conditions:

(i) For each open immersion $j : V \hookrightarrow X$ in $\mathcal{C}$, there is a map $j^* : H_a(X) \to H_a(V)$, associated to $j$ in a functorial way.

(ii) If $i : Y \hookrightarrow X$ is a closed immersion in $\mathcal{C}$, with open complement $j : V \hookrightarrow X$, there is a long exact sequence (called localization sequence)

$$\ldots \xrightarrow{\delta} H_a(Y) \xrightarrow{i_*} H_a(X) \xrightarrow{j^*} H_a(V) \xrightarrow{\delta} H_{a-1}(Y) \longrightarrow \ldots$$

(The maps $\delta$ are called the connecting morphisms.) This sequence is functorial with respect to proper maps or open immersions, in an obvious way.

(b) A morphism between homology theories $H$ and $H'$ is a morphism $\phi : H \to H'$ of functors on $\mathcal{C}_*$, which is compatible with the long exact sequences from (ii).

**Lemma 4.21** (a) Let $L$ be a field, and let $r, s$, and $n \geq 1$ be fixed integers with $n$ invertible in $L$, or $r \neq s + 1$, or $p = \mathrm{char}(L) \mid n$ and $r = s + 1$ and $[L : L^p] \leq p^s$. There is a natural way to extend the assignments

$$X \quad \leadsto \quad H_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := H_a(C^{r,s}(X, \mathbb{Z}/n\mathbb{Z})) \qquad (a \in \mathbb{Z})$$

to a homology theory on the category of all varieties over $L$.

(b) The same holds for the assignment

$$X \quad \leadsto \quad \overline{H}_a^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := H_a(\overline{C}^{r,s}(X, \mathbb{Z}/n\mathbb{Z})) \qquad (a \in \mathbb{Z}),$$

where $\overline{C}^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) := C^{r,s}(\overline{X}, \mathbb{Z}/n\mathbb{Z})_{G_L}$, with $\overline{X} = X \times_L \overline{L}$ for a separable closure of $L$.

**Proof** (a): The Bloch-Ogus-Kato complexes are covariant with respect to proper morphisms and contravariant with respect to open immersions. The localization sequence for a closed immersion $i : Y \hookrightarrow X$ with open complement $j : U = X \smallsetminus Y \hookrightarrow X$ is obtained by the short exact sequence of complexes

$$0 \to C^{r,s}(Y, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{i_*} C^{r,s}(X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{j^*} C^{r,s}(U, \mathbb{Z}/n\mathbb{Z}) \to 0$$

(cf. also [JS] Corollary 2.10).

(b): This follows from (a).

The mentioned localization is now obtained by the following observation.

**Lemma 4.22** Let $L$ be a perfect field, and let $\varphi : H \to \widetilde{H}$ be a morphism of homology theories on the category $(\mathcal{V}_L)_*$ of all varieties over $L$ (with proper morphisms. For every integral variety $Z$ over $L$ let $L(Z)$ be its function field. Define

$$H_a(L(Z)) := \varinjlim H_a(U)\,,$$

where the limit is over all non-empty open subvarieties $U$ of $Z$, and define $\widetilde{H}_a(L(Z))$ similarly. Suppose the following holds for every integral variety $Z$ of dimension $d$ over $L$.

(i) $H_a(L(Z)) = 0$ for $a \neq d$,

(ii) $\widetilde{H}_a(L(Z)) = 0$ for $a \neq d$, and

(iii) the map $\varphi : H_d(L(Z)) \to \widetilde{H}_d(L(Z))$ induced by $\varphi$ is an isomorphism.

Then $\varphi$ is an isomorphism of homology theories.

Before we give a proof for this, we note the following.

**Remark 4.23** The homology theory of 4.21 (a) clearly satisfies condition 4.22 (i), because $C_a^{r,s}(L(X), \mathbb{Z}/n\mathbb{Z}) = 0$ for $a \neq \dim X$, if $X$ is integral. Hence 4.22 (i) also hold for the homology theories from 4.19 (b). The proof of Theorem 4.18 will then achieved as follows. In the next section we will define a homology theory $H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, over any field $K$ of characteristic 0, or over a perfect field of positive characteristic assuming suitable resolution of singularities, which a priori satisfies

$$H_a^W(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a = 0 \\ 0 & , \quad a \neq 0\,. \end{cases} \tag{4.24}$$

if $X$ is smooth, projective and irreducible. Moreover we will show that 4.22 (ii) holds for $H^W$. Still under the same assumptions we will construct a morphism

$$\varphi : \overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) := \overline{H}_*^{0,0}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

of homology theories which satisfies 4.22 (iii) if $K$ is finitely generated . Thus, by Lemma 4.22, $\varphi$ is an isomorphism, and hence (4.25) also holds for $\overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, which proves 4.18.

**Proof of Lemma 4.22** For every homology theory $H$ over $L$ there is a strongly converging niveau spectral sequence

$$E^1_{p,q}(X) = \bigoplus_{x \in X_p} H_{p+q}(k(x)) \Rightarrow H_{p+q}(X) \tag{4.25}$$

for every $X$, cf. [BO], and also [JS]. If $\widetilde{E}^1_{p,q} \Rightarrow \widetilde{H}_{p+q}$ is associated to another homology theory $\widetilde{H}$, then every morphism $\varphi : H \to \widetilde{H}$ induces a morphism $E \to \widetilde{E}$ of these spectral sequences, compatible with $\varphi$ on the $E^1$-terms and limit terms. In the situation of 4.22, the conditions (i), (ii) and (iii) imply that $\varphi$ induces isomorphisms on the $E^1$-terms, and hence $\varphi$ also gives an isomorphism between the limit terms, i.e., between $H$ and $\widetilde{H}$.

## §5 Weight complexes

Let $k$ be a field. Let $X$ be a smooth, projective variety of dimension $d$ over $k$, and let $Y = \bigcup_{i=1}^{r} Y_i$ be a divisor with simple normal crossings in $X$ – with a fixed ordering of the smooth components as indicated.

**Definition 5.1** Let $F$ be a covariant functor on the category $\mathcal{SP}_k$ of smooth projective varieties with values in an abelian category $\mathcal{A}$ which is additive in the sense that the natural arrow

$$F(X_1) \oplus F(X_2) \to F(X_1 \amalg X_2)$$

is an isomorphism in $\mathcal{A}$, where $X_1 \amalg X_2$ is the sum (disjoint union) of two varieties $X_1, X_2$ in $\mathcal{SP}_k$. Then define $L^i F(X,Y)$ as the $i$-th homology of the complex

$$C.F(X,Y): \qquad 0 \to F(Y^{[d]}) \to F(Y^{[d-1]}) \to \ldots \to F(Y^{[1]}) \to F(X) \to 0\,.$$

Here $F(Y^{[j]})$ is placed in degree $j$, and the differential $\partial : F(Y^{[j]}) \to F(Y^{[j-1]})$ is $\sum_{\nu=1}^{j} (-1)^{\nu} \delta_{\nu}$ where $\delta_\nu$ is induced by the inclusions

$$Y_{i_1,\ldots,i_j} \hookrightarrow Y_{i_1,\ldots,\hat{i}_\nu,\ldots,i_j}$$

for $1 \leq i_1 < \ldots < i_j \leq r$ (and where $Y^{[0]} = X$, as usual).

**Remark 5.2.** There is the dual notion of an additive, contravariant functor $G$ from $\mathcal{SP}_k$ to $\mathcal{A}$, and here we define $R^i G(X,Y)$ to be the $i$-th cohomology of the complex

$$C^{\cdot}G(X,Y): \qquad G(X) \to G(Y^{[1]}) \to \ldots \to G(Y^{[d-1]}) \to G(Y^{[d]})\,,$$

with $G(Y^{[j]})$ is placed in degree $j$.

We may apply this to the following functors. Let $Ab$ be the category of abelian groups.

**Definition 5.3** For any abelian group $A$ define the covariant functor $H_0(-,A) : \mathcal{SP}_k \to Ab$ and the covariant functor $H^0(-,A) : \mathcal{SP}_k \to Ab$ by

$$\begin{aligned} H_0(X,A) &= \bigoplus_{\alpha \in \pi_0(X)} A = A \otimes_{\mathbb{Z}} \mathbb{Z}[\pi_0(X)] \\ H^0(X,A) &= A^{\pi_0(X)} = \text{ Map } (\pi_0(X), A)\,. \end{aligned}$$

where $\mathbb{Z}[M]$ is the free abelian group on a set $M$, and $\text{Map}(M,N)$ is the set of maps between two sets $M$ and $N$. (Hence if $A$ happens to be a ring, then $H_0(X,A)$ is the free $A$-module on $\pi_0(X)$, and $H_0(X,A) = \text{Hom}_A(H_0(X,A),A)$ is its $A$-dual.) We write $C^W_{\cdot}(X,Y;A)$ for $C.H_0(-,A)(X,Y)$ and call

$$H_i^W(X,Y;A) := L^i H_0(-,A)(X,Y) = H_i(C.(X,Y;A))$$

the weight homology of $(X,Y)$. Similarly define $C^{\cdot}_W(X,Y;A) = C^{\cdot}H_0(-,A)(X,Y)$ and call $H^i_W(X,Y;A) = H^i(C^{\cdot}_W(X,Y;A))$ the weight cohomology of $(X,Y)$.

**Proposition 5.4** Let $Y_{r+1}$ be a smooth divisor on $X$ such that the intersections with the connected components of $Y^{[j]}$ are transversal for all $j$ and connected for all $j \leq d-2$. (Note: If $k$ is infinite, then by the Bertini theorems such a $Y_{r+1}$ exists by taking a suitable hyperplane section, since $\dim Y^{[j]} = d - j \geq 2$ for $j \leq d-2$.) Let $Z = \bigcup_{i=1}^{r+1} Y_i$ (which, by the assumption, is again a divisor with normal crossings on $X$). Then, for any abelian group $A$

$$H_i^W(X, Z; A) = 0 = H_W^i(X, Z; A) \text{ for } i \leq d-1\,.$$

**Proof** Fix $A$ and omit it in the notations. We get a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & H_0(Y^{[d-1]} \cap Y_{r+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \psi_{d-2} & & \\
0 & \longrightarrow & H_0(Y^{[d]}) & \longrightarrow & H_0(Y^{[d-1]}) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H_0(Y^{[1]} \cap Y_{r+1}) & \longrightarrow & H_0(Y_{r+1}) & \longrightarrow & 0 \\
 & & \wr\downarrow \psi_1 & & \wr\downarrow \psi_0 & & \\
\cdots & \longrightarrow & H_0(Y^{[1]}) & \longrightarrow & H_0(X) & \longrightarrow & 0
\end{array}$$

where the bottom line is the complex $C_\cdot^W(X, Y)$, the top line is $C_\cdot^W(Y_{r+1}, Y \cap Y_{r+1})$ (note that $Y \cap Y_{r+1} = \bigcup_{i=1}^{r} (Y_i \cap Y_{r+1})$ is a divisor with strict normal crossings on the smooth, projective variety $Y_{r+1}$), and where $\psi_\nu$ is induced by the inclusions $Y^{[\nu]} \cap Y_{r+1} \hookrightarrow Y^{[\nu]}$.

By the assumption, $\psi_\nu$ is an isomorphism for $\nu \leq d-2$, and a (non-canonically) split surjection for $\nu = d-1$. Hence we have isomorphisms

$$H_i^W(Y_{r+1}, Y \cap Y_{r+1}) \xrightarrow{\sim} H_i^W(X, Y) \quad \text{ for } i \leq d-2\,.$$

Moreover, let $C_{\cdot\cdot}$ be the associated double complex, with $H_0(X)$ placed in degree $(0,0)$ and $\psi_\nu$ being replaced by $(-1)^{\nu+1}\psi_\nu$. Then the associated total complex $s(C_{\cdot\cdot})$ is just the complex $C_\cdot^W(X, Z)$. Hence the result follows, and we have exact sequences

$$0 \to H_d^W(X, Z) \to H_{d-1}^W(Y_{r+1}, Y \cap Y_{r+1}) \to H_{d-1}^W(X, Y) \to 0\,,$$

$$0 \to \ker(\psi)_{d-1} \to H_d^W(X, Z) \to H_0^W(Y^{[d]}) \to 0\,.$$

The proof for $H_W^i(X, Z)$ is dual. (Note, however, that in general $H_\cdot^W(X, Y)$ and $H_W^\cdot(X, Y)$ are related by a coefficient theorem in an non-trivial way).

**Corollary 5.5** $H_d^W(X, Z; \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module, and $H_d^W(X, Z; A) = H_d(X, Z; \mathbb{Z}) \otimes_{\mathbb{Z}} A$. The same holds for $H_W^d(X, Z)$.

**Proof** The first statement follows since $\ker(\psi)_{d-1}$ and $H^0(Y^{[d]})$ have this property for $A = \mathbb{Z}$, and the second claim follows from 5.4 and the universal coefficient theorem. Similarly for $H_W^d(X, Z)$.

**Corollary 5.6** $H_d^W(X, Z; \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible.

Now let $U = X \smallsetminus Y$.

**Proposition 5.7** There are canonical homomorphisms

$$H^d_{\text{ét}}(\overline{U}, \mathbb{Z}/n\mathbb{Z}(d)) \xrightarrow{e} H^W_d(X, Y; \mathbb{Z}/n\mathbb{Z})$$

for $n \in \mathbb{N}$ invertible. If $k$ is finitely generated, and if $X \smallsetminus Y_\nu$ is affine for one $\nu \in \{1, \dots, r\}$, then these induces isomorphisms

$$H^d_{\text{ét}}(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_k} \xrightarrow{\sim} H^W_d(X, Y; \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for all primes $\ell \neq$ char $k$.

This is just a reformulation of theorem 3.1, in which the construction of $e$ does not depend on the assumption that some $X \smallsetminus Y_\nu$ is affine.

**Remark 5.8** In particular, with the notations and assumptions of 5.4, this applies to $(X, Z)$ and $U = X \smallsetminus Z$.

We want to have these results in a more functorial setting. This is possible if resolution of singularities holds, in a suitable form. For any field $k$ let $\mathcal{V}_{k,d}$ be the category of varieties of dimension $\leq d$ over $k$. Recall the following condition, where $k$ is a perfect field.

> $\text{RS}_{pb}(k, d)$ For any variety $Z$ in $\mathcal{V}_{k,d}$, there exists a proper surjective morphism $\pi : \tilde{Z} \to Z$ and an open dense subscheme $U \subseteq Z$ such that $\tilde{Z}$ is smooth quasi-projective over $k$, and $\pi$ induces an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} U$.

By Hironaka's fundamental results [Hi], $\text{RS}_{pb}(k, d)$ holds if $k$ has characteristic zero.

**Theorem 5.9** Let $k$ be a field with perfect hull $k'$, let $A$ be an abelian group, and assume that $\text{RS}_{pb}(k', d)$ holds. There exists a homology theory (in the sense of definition 4.12) $(H^W_a(-, A), a \in \mathbb{Z})$ on the category $(\mathcal{V}_{k,d})_*$ of all varieties of dimension $\leq d$ over $k$ with proper morphisms such that for any smooth, projective and connected variety $X$ of dimension $\leq d$ over $k$ one has

$$H^W_a(X, A) = \begin{cases} 0 & a \neq 0 \\ A & a = 0 \,. \end{cases}$$

**Proof** : First assume that $k$ is perfect. We want to show that the covariant functor (cf. 5.3)

$$F : \; X \;\; \leadsto \;\; H_0(X, A) = \bigoplus_{\pi_0(X)} A$$

on the category $\mathcal{SP}_{k,d}$ of smooth projective varieties of dimension $\leq d$ over $k$ extends to a homology theory on all of $\mathcal{V}_{k,d}$. By analysing the methods of Gillet and Soulé ([GS] proof of 3.1.1) this holds if $\text{RS}_{pb}(k, d)$ holds and if $F$ extends to a contravariant functor on Chow motives over $k$, i.e., admits an action of algebraic correspondences modulo rational equivalence. But the latter is clear – in fact, one has $F(X) = \text{Hom}(CH^0(X), A)$, where $CH^j(X)$ is the Chow group of algebraic cycles of codimension $j$ on $X$, modulo rational equivalence. If $k$ is general, we just define

$$H^W_a(Z, A) := H^W_a(Z \times_k k')$$

where the theory on the right is the one existing over $k'$ by our assumptions and the case of a perfect field. Not that $Z \times_k k'$ is again connected for connected $Z$.

For further properties of this homology theory we need a further instance of resolution of singularities, which again holds for $k$ of characteristic zero by Hironaka [Hi]. Recall the following condition for a perfect field $k$.

> $\mathrm{RS}_{gc}(k,d)$ For any smooth variety $V$ of dimension $\leq d$ over $k$ there exists an open subvariety $U \subset V$ and an open embedding $U \hookrightarrow X$ into a smooth projective variety $X$ over $k$ such that the complement $Y = X \setminus U$ is a divisor with simple normal crossings.

This is obviously equivalent to the formulation before 4.9.

**Theorem 5.10** Let $k$ be a field with separable hull $k'$. Assume that $\mathrm{RS}_{gc}(k',d)$ and $\mathrm{RS}_{pb}(k',d)$ hold. Then the homology theory $H_*^W(-,A)$ of Theorem 5.9 has the property 4.22 (ii).

**Proof** by construction we may assume $k$ is perfect. For every integral variety $Z$ over $k$, we have to show

$$H_a^W(k(Z),A) := \varinjlim H_a^W(V,A) = 0 \text{ for } a \neq \dim(Z)\,, \tag{5.11}$$

where the inductive limit is over all non-empty open subvarieties $V \subset Z$. By $\mathrm{RS}_{gc}(k,d)$ and perfectness of $k$, for every non-empty open subvariety $V \subset Z$, there is an non-empty smooth open subvariety $U \subset V$ and an open embedding $U \hookrightarrow X$ into a smooth projective variety $X$ such that the complement $Y = X \setminus U$ is a divisor with strict normal crossings. Then, by [GS] Prop. 3, $H_a^W(U,A)$ is computed as the homology of the complex

$$\bigoplus_{\pi_0(Y^{[d]})} A \to \bigoplus_{\pi_0(Y^{[d-1]})} A \to \ldots \to \bigoplus_{\pi_0(Y^{[1]})} A \to A,$$

for $d = \dim(Z) = \dim(X)$, where the notations and differentials are as in Definition 5.1. In other words, with the definition in 5.3 we have

$$H_a^W(U,A) \cong H_a^W(X,Y;A)\ .$$

If $k$ is infinite, then, by Bertini's theorem, there exists a smooth hyperplane section $H$ of $X$ whose intersection with all connected components of $Y^{[i]}$ is smooth, and connected for $i \leq d-2$. Writing $Y' = Y \cup H$ (which a divisor with strict normal crossings on $X$) and $U' = X \setminus Y' \subset U \subset V$, we get $H_a^W(U',A) = H_a^W(X,Y';A) = 0$ for $a \neq d$ by Proposition 5.4. Since $V$ was arbitrary, we get property (5.11).

If $k$ is finite, we use the usual norm argument. By what has been shown, for each prime $p$ we find such a hyperplane section after base change to an extension $k'/k$ of degree $[k':k] = p^r$, a power of $p$ (the maximal pro-$p$-extension of $k$ is an infinite field). Then the map

$$H_a^W(V_{k'},A) \to H_a^W(k'(Z_{k'}),A)$$

is zero.

Now we note that there is a homology theory $H^W(-,A;k')$ on all varieties over $k$, defined by $H_a^W(Z,A;k') = H_a^W(Z_{k'},A)$ and the induced structure maps. This is also the homology theory which is obtained by the method of Theorem 5.9, by extending the covariant functor

$$F' : \mathcal{SP}_k \longrightarrow Ab\ ,\ X \rightsquigarrow \bigoplus_{\pi_0(X_{k'})} A$$

to a homology theory on all varieties. There is a morphism of functors $Tr : F' \to F$ (trace, or norm), induced by the natural maps $\pi_0(X_{k'}) \to \pi_0(X)$. By the construction of Gillet and Soulé, it extends to a morphism of homology theories $Tr : H^W(-, A; k') \to H^W(-, A)$, and one checks that the induced maps $H_a^W(Z_{k'}, A) \to H_a^W(Z, A)$ are just the maps obtained from functoriality for proper morphisms. On the other hand, there is also a morphism of functors $Res : F \to F'$ (restriction), induced by the maps

$$\bigoplus_{x\in\pi_0(X)} A \to \bigoplus_{y\in\pi_0(X_{k'})} A \ , \ a_x \mapsto \sum_{y/x} a_x \, .$$

It extends to a morphism of homology theories $Res : H(-, A) \to H^W(-, A; k')$. One has $TrRes = [k' : k] = p^r$, because this holds for the restriction to the functors $F$ and $F'$. The outcomes is that the kernel of $Res$ is killed by $p^r$. From the commutative diagram

$$\begin{array}{ccc} H_a^W(V_{k'}, A) & \to & H_a^W(k'(Z_{k'}), A) \\ \uparrow & & \uparrow \\ H_a^W(V, A) & \to & H_a^W(k(Z), A) \, , \end{array}$$

we then get that the image of the lower horizontal map is killed by $p^r$ for $a \neq d$, because then the upper horizontal map is zero. Since this holds for all $V$ (with varying power of $p$), we conclude that every element in the group $H_a^W(k(Z), A)$ is killed by a power of $p$. Since this also holds for any second prime $q \neq p$, we conclude the vanishing of $H_a^W(k(Z), A)$.

**Theorem 5.12** Let $K$ be a finitely generated field such that the conditions $\mathrm{RS}_{gc}(K', d)$ and $\mathrm{RS}_{pb}(K', d)$ hold for the perfect hull $K'$ of $K$, and let $\ell$ be a prime number invertible in $K$. Let $\overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ and $H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ be the homology theories on $\mathcal{V}_{K,d}$ defined in 4.21 (b) and 5.9, respectively. There exists a morphism

$$\varphi : \overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

of homology theories such that the properties (i) - (iii) of Lemma 4.22 are fulfilled for $\overline{H}$, $H^W$ and $\varphi$.

Evidently this theorem implies Theorem 4.18, in view of 4.22 and 4.23.

Since $H_*^W(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is defined via the method of Gillet and Soulé in [GS] 3.1.1, we need to analyze the constructions in [GS] more closely. There functors on Chow motives with values in abelian categories are extended to homology theories on all varieties. We take the opportunity to give a slightly general version for complexes, in the following form.

**Theorem 5.13** Let $k$ be a perfect field satisfying $\mathrm{RS}_{pb}(k, d)$. Let $\mathcal{C}_{\geq 0}(\mathcal{A})$ be the category of non-negative homological complexes $\ldots \to C_2 \to C_1 \to C_0$ in an abelian category $\mathcal{A}$, and let

$$C : \mathcal{SP}_{k,d} \to \mathcal{C}_{\geq 0}(\mathcal{A})$$

be a covariant functor on the category $\mathcal{SP}_k$ of all smooth projective varieties over $k$. Assume that the associated functors

$$H_a^C : \mathcal{SP}_k \to \mathcal{A} \ , \ X \leadsto H_a(C(X))$$

extend to contravariant functors on the category $\mathcal{CHM}^{eff}(k)$ of effective Chow motives (i.e., motives modulo rational equivalence) over $k$. Then there is a natural way to extend the above functors $H_a^C$ to a homology theory on the category $\mathcal{V}_{k,d}$ of all varieties of dimension $\leq d$ over $k$.

**Proof** (cf. the reasoning in [GS] 5.3) In this proof, we will only consider varieties in $\mathcal{V}_{k,d}$ and call them simply varieties. By $\mathrm{RS}_{pb}(k,d)$ and [GS] Lemma 2, any proper variety $Z$ (recall: in $\mathcal{V}_{k,d}$) has a smooth projective hyperenvelope, i.e., a hyperenvelope (cf. [GS] 1.4.1, or section 7 below) $h : \widetilde{Z}. \to Z$ where the components $\widetilde{Z}_r$ of the simplicial scheme $\tilde{Z}.$ are smooth and projective varieties, and every morphism $f : Y \to X$ of proper varieties has a smooth projective hyperenvelope, i.e., there is a commutative square

$$\begin{array}{ccc} \widetilde{Y}. & \xrightarrow{\widetilde{f}} & \widetilde{X}. \\ {\scriptstyle h_Y}\downarrow & & \downarrow{\scriptstyle h_X} \\ Y & \xrightarrow[f]{} & X \end{array}$$

in which $h_Y$ and $h_X$ are smooth projective hyperenvelopes. If $Z$ is an arbitrary reduced variety over $k$, there is an open embedding $j : Z \subset \overline{Z}$ into a proper variety. Fixing such an embedding $j_Z$ for every reduced variety $Z$, and a smooth projective hypercovering $\widetilde{f}$ for every morphism $f$ of proper varieties, we can proceed as follows. For every smooth projective simplicial variety $Z.$ define the complex $tC(Z.)$ in $\mathcal{C}_{\geq 0}(\mathcal{A})$ as the total complex associated to the bi-complex

$$C(Z.) : \qquad \ldots \to C(Z_r) \stackrel{\sum_{i=0}^{r}(-1)^i (d_i)_*}{\longrightarrow} C(Z_{r-1}) \longrightarrow \ldots \longrightarrow C(Z_0)\,,$$

where the $d_i$ are the face morphisms of $Z.$ , and define $H_a(Z.) := H_a(tC(Z.))$. For every morphism $g : W. \to Z.$ of smooth projective simplicial schemes define

$$C(g) := Cone(tC(W.) \stackrel{g_*}{\longrightarrow} tC(Z.))$$

and $H_a(g) = H_a(C(g))$. Finally, for a variety $Z$ over $k$ define

$$C(Z) := C(\widetilde{i_Z}) \quad \text{and} \quad H_a(Z) := H_a(C(Z))\,,$$

where $i_Z : Y \hookrightarrow \overline{Z}$ is the embedding of the closed complement of $j_Z : Z \hookrightarrow \overline{Z}$. Then, for any smooth projective simplicial variety $Z.$ we have a convergent spectral sequence

$$(5.14) \qquad E^1_{p,q}(Z.) = H_p(H_q(C(Z.))) = H_p(H^C_q(Z.)) \Rightarrow H_{p+q}(Z.) = H_{p+q}(tC(Z.))\,.$$

Here $p$ is the simplicial degree and $q$ is the complex degree, so that $E^1_{p,q}$ is the $p$-th homology of the complex

$$H^C_q(Z.) : \qquad \ldots \longrightarrow H^C_q(Z_r) \stackrel{\sum_{i=0}^{r}(-1)^i (d_i)_*}{\longrightarrow} H^C_q(Z_{r-1}) \longrightarrow \ldots \longrightarrow H^C_q(Z_0)\,.$$

By the assumption on the functors $H^C_q$, the latter complex is obtained by applying the extension of $H^C_q$ to Chow motives to the complex of motives

$$M(Z.) : \qquad M(Z_0) \longrightarrow \ldots \longrightarrow M(Z_{r-1}) \stackrel{\sum_{i=0}^{r}(-1)^i (d_i)^*}{\longrightarrow} M(Z_r) \longrightarrow \ldots\,,$$

$M(X)$ denoting the Chow motive of a smooth projective variety $X$. For every morphism $g : W. \to Z.$ of smooth projective simplicial schemes we have functorially associated morphisms $g_* : H_a(W.) \to H_a(Z.)$ extending to a morphism $E(W.) \to E(Z.)$ of the spectral sequences (5.14). If $g$ is a hyperenvelope of simplicial schemes, then the fundamental result [GS] Proposition 2 asserts that the induced morphism

$$g^* : M(Z.) \longrightarrow M(W.)$$

is a homotopy equivalence. Hence $g$ induces an isomorphism on the $E^1$-terms of the above spectral sequences, and thus an isomorphism $g_* : H_a(W_.) \to H_a(Z_.)$ for all $a$.

Using this, we get the functoriality of our homology theory on $\mathcal{V}_k$ following the reasoning in [GS] 2.2 and 2.3: Let $Ar(\mathcal{P}_k)$ be the category of morphisms in the category $\mathcal{P}_k$ of proper varieties over $k$. For every morphism $g : f_1 \to f_2$ in $Ar(\mathcal{P}_k)$, i.e., any commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{f_1} & X_1 \\ {\scriptstyle g_Y}\downarrow & & \downarrow{\scriptstyle g_X} \\ Y_2 & \xrightarrow{f_2} & X_2 \end{array}$$

there exists a commutative diagram

$$\begin{array}{ccc} \tilde{f} & \xrightarrow{b} & \tilde{f}_2 \\ {\scriptstyle a}\downarrow & & \\ \tilde{f}_1 & & \downarrow{\scriptstyle h_2} \\ {\scriptstyle h_1}\downarrow & & \\ f_1 & \xrightarrow{g} & f_2\,. \end{array}$$

in which $h_1, h_2$ and $a$ are hyperenvelopes. Thus we get morphisms

$$g_* = (a_*)^{-1} b_* : H_a(C(\tilde{f}_1)) \longrightarrow H_a(C(\tilde{f}_2))$$

for all $a \in \mathbb{Z}$. It follows as in [GS] 2.2 that these maps do not depend on the choices and are functorial.

If now $f : Z_1 \to Z_2$ is a proper morphism of varieties, and the embeddings $j_{Z_\nu} : Z_\nu \hookrightarrow \overline{Z}_\nu$ into proper varieties are as above, as well as $i_\nu = i_{Z_\nu} : Y_\nu = \overline{Z}_\nu \setminus Z_\nu \hookrightarrow \overline{Z_\nu}$ (for $\nu = 1, 2$), let $Z_f \subset Z_1 \times Z_2$ be the graph of $f$ and $\overline{Z_f}$ be its closure. Let $i_f : Y_f = \overline{Z_f} - Z_f \hookrightarrow \overline{Z_f}$ be the closed immersion. Then the projections $p_\nu : \overline{X_f} \to \overline{X}_\nu$ map $Y_f$ to $Y_\nu$ (for $\nu = 2$ one needs the properness of $f$), and $\pi_1$ induces an isomorphism $Z_f = \overline{Z_f} - Y_f \to \overline{X_1} - Y_1 = X_1$. This gives a canonical diagram in the category $Ar(\mathcal{P}_k)$ of morphisms in the category $\mathcal{P}_k$ of proper varieties over $k$

$$\begin{array}{ccc} i_f & \xrightarrow{\pi_2} & i_2 \\ {\scriptstyle \pi_1}\downarrow & & \\ i_1 & & \end{array}$$

in which $\pi_1$ is Gersten acyclic (loc. cit.) and hence induces a quasi-isomorphism $C(\widetilde{i_f}) \to C(\widetilde{i_1})$ (for this one reasons via the spectral sequences (5.14)). We obtain

$$f_* := H_a(f) := H_a((\pi_2)_*) H_a((\pi_1)_*)^{-1} : H_a(Z_1) = H_a(C(\tilde{i_1})) \to H_a(C(\tilde{i_2})) = H_a(Z_2)\,.$$

This is functorial with the same argument as in [GS] 2.3.

Moreover, we remark that for another choice of compactifications $j_Z$ and hence maps $i_Z$, say $j'_Z$ and $i'_Z$, the morphisms $H_a(id_Z)$ give canonical isomorphisms between the different constructions of $H_a$. In particular, for a smooth projective variety $X$ we get $H_a(X) = H_a^C(X)$. Also if we take another choice for the hyperenvelopes $\tilde{f}$, then the reasoning in [GS] 2.2 shows, that the resulting homology theory is canonically isomorphic to the one for the first choice. In this sense, the homology theory is canonical.

To obtain the properties 4.20 (a) (i) and (ii) for our homology theory, i.e., contravariance for open immersions and the exact localization sequences, we proceed as in [GS] 2.4: For a variety $Z$ and a closed subvariety $Z' \subset Z$ with open complement $U = Z \setminus Z'$ choose a compactification $Z \hookrightarrow \overline{Z}$, and let $Y = \overline{Z} \setminus Z$ and $Y' = \overline{Z} \setminus U$, so that $Z' = Y' \setminus Y$. Then we choose smooth projective hyperenvelopes $\widetilde{\overline{Z}} \to \overline{Z}$, $\widetilde{Y} \to Y$ and $\widetilde{Y'} \to Y'$ such that one has morphisms

$$\widetilde{i} : \widetilde{Y} \xrightarrow{\widetilde{k}} \widetilde{Y'} \xrightarrow{\widetilde{i'}} \widetilde{\overline{Z}}$$

lifting the closed immersions $i : Y \overset{k}{\hookrightarrow} Y' \overset{i'}{\hookrightarrow} \overline{Z}$. We obtain a triangle of mapping cones

$$C(\widetilde{k}) \longrightarrow C(\widetilde{i}) \longrightarrow C(\widetilde{i'}) \longrightarrow C(\widetilde{k})[-1]$$

which gives rise to the desired exact localization sequence

$$\ldots \to H_a(Z') \to H_a(Z) \to H_a(U) \to H_{a-1}(Z') \to \ldots$$

and thereby also to the pull-back $j^*$ for the open immersion $j : U \to Z$. The functorial properties are easily checked.

**Remark 5.15** Let $C' : \mathcal{SP}_k \to \mathcal{C}_{\geq 0}(\mathcal{A})$ be another functor with the same properties, and let $\varphi : C \to C'$ be a morphism of functors. Then it is clear from the construction, that there is a canonical induced morphism $\varphi : H \to H'$ of the associated homology theories on $\mathcal{V}_k$.

**Proposition 5.16** Let $k$ be a perfect field satisfying $\mathrm{RS}_{pb}(k,d)$, and let $(\mathcal{V}_{k,d})_*$ be the category of all varieties of dimension $\leq d$ over $k$ with all proper morphisms between them. Let

$$C' : (\mathcal{V}_{k,d})_* \longrightarrow \mathcal{C}_{\geq 0}(\mathcal{A})$$

be a covariant functor which is equipped with the following additional data:

(i) For every open immersion $j : U \hookrightarrow Z$ in $(\mathcal{V}_{k,d})_*$ there is a morphism $j^* : C'(Z) \to C'(U)$, associated to $j$ in a functorial way.

(ii) If $i : Y \hookrightarrow Z$ is a closed immersion in $(\mathcal{V}_{k,d})_*$, with open complement $j : U \hookrightarrow Z$, then there is a short exact sequence of complexes

$$0 \to C'(Y) \xrightarrow{i_*} C'(X) \xrightarrow{j^*} C'(U) \to 0\,.$$

This sequence is functorial with respect to proper morphisms and open immersions, in an obvious way.

Let $H'$ be the obvious homology theory on $(\mathcal{V}_{k,d})_*$ deduced from $C'$, with $H'_a(Z) = H_a(C'(Z))$, and the localization sequences (see Definition 4.12 ii)) induced by the exact sequences in (ii) above. Moreover let $C : \mathcal{SP}_k \longrightarrow \mathcal{C}_{\geq 0}(\mathcal{A})$ be a covariant functor satisfying the assumption of Theorem 5.13, and let $H$ be the homology theory on $(\mathcal{V}_{k,d})_*$ derived from $C$ via Theorem 5.13. Then any morphism of functors

$$\phi : C'_{|\mathcal{SP}_{k,d}} \longrightarrow C$$

induces a canonical morphism of homology theories $\varphi : H' \to H$ on $(\mathcal{V}_{k,d})_*$ whose restriction to $\mathcal{SP}_{k,d}$ is given by the maps $H_a(C'(X)) \to H_a(C(X))$ induced by $\phi$ for smooth projective $X$. In particular, $\varphi$ is an isomorphism if the latter maps are isomorphisms for all $a$ and all $X$ in $\mathcal{SP}_{k,d}$.

**Proof** This will follow from the descent lemma 5.17 below (which we only need for the case that $X.$ is proper and $Z.$ is a smooth projective hyperenvelope). We will only consider varieties of dimension $\leq d$ and will omit the $d$ in the arguments and notations. Otherwise let the notations be as in the proof of Theorem 5.13, including the choice of a hyperenvelope $h_f : \tilde{f} \to f$ for each morphism $f$ in $\mathcal{P}_k$ and a compactification $j_Z : Z \hookrightarrow X_Z := \overline{Z}$ with complement $i_Z : Y_Z := \overline{Z} - Z \hookrightarrow X_Z$. The crucial observation is that the constructions in the proof of 5.13 can also be applied to the complexes $C'(-)$. First of all we can define $C'(g) := Cone(tC'(Y.) \xrightarrow{g_*} tC'(X.))$ for any morphism $g : Y. \to X.$ of smooth projective simplicial varieties, where $tC'(Y.)$ is the total complex associated to the bi-complex $C'(Y.)$. However we can also define $C'(f) := Cone(C'(Y) \xrightarrow{f_*} C'(X))$ for any proper morphism $f : Y \to X$ of varieties, and property (ii) above gives a canonical morphism

$$\beta_Z : C'(i_Z) = Cone(C'(Y_Z) \xrightarrow{(i_Z)_*} C'(X_Z)) \longrightarrow C'(Z)\,,$$

which is a quasimorphism. Thus the descent lemma below gives a canonical morphism

$$\alpha_Z : C'(\tilde{i_Z}) \xrightarrow{(h_{i_Z})_*} C'(i_Z) \to C'(Z)$$

which is a quasi-isomorphism. Since $\phi$ induces an obvious morphism $\pi_* : C'(\tilde{i_Z}) \to C(\tilde{i_Z})$ of complexes, we get maps

$$\varphi_a : H'_a(Z) = H_a(C'(Z)) \xrightarrow{(\alpha_Z)^{-1}} H_a(C'(\tilde{i_Z})) \xrightarrow{\phi_*} H_a(C(\tilde{i_Z})) = H_a(Z)\,,$$

where the equalities holds by definition. Now we consider the functoriality of these maps. To carry out the corresponding constructions in the proof of Theorem 5.13 for $C'(-)$, it suffices to note that two types of morphisms $g_1 \to g_2$ between morphisms $g_i$ of smooth projective simplicial schemes induces quasi-isomorphisms $C'(g_1) \to C'(g_2)$. The first is the morphism $a$, constructed for a morphism $f_1 \to f_2$ in $Ar(\mathcal{P}_k)$. But $h_1$ and $ah_1$ are hyperenvelopes and so induce quasi-isomorphisms $C'(\tilde{f_1}) \to C'(f_1)$ and $C'(\tilde{f}) \to C'(f_1)$ by the descent lemma below; so $a$ induces a quasi-isomorphism $C'(\tilde{f}) \to C'(\tilde{f_1})$. The second is the morphism $\pi_1$, constructed for a proper morphism of varieties $f : Z_1 \to Z_2$. Here we get a commutative diagram

$$\begin{array}{ccc}
\tilde{f} & \xrightarrow{b} & \tilde{i}_2 \\
{\scriptstyle a}\downarrow & & \\
\tilde{i}_f & & \downarrow{\scriptstyle h_1} \\
{\scriptstyle h}\downarrow & & \\
i_f & \xrightarrow{\pi_1} & i_1\,.
\end{array}$$

where $h, h_1$ and $a$ are hyperenvelopes. It induces a commutative diagram

$$\begin{array}{ccc}
C'(\tilde{f}) & \xrightarrow{b_*} & C'(\tilde{i_1}) \\
{\scriptstyle (ha)_*}\downarrow & & \downarrow{\scriptstyle (h_1)_*} \\
C'(i_f) & \xrightarrow{(\pi_1)_*} & C'(i_1) \\
{\scriptstyle (\beta_{Z_f})_*}\downarrow & & \downarrow{\scriptstyle (\beta_{Z_1})_*} \\
C'(Z_f) & \xrightarrow{(\pi_1)_*} & C'(Z_1)\,.
\end{array}$$

Here $(ha)_*$ and $(h_1)_*$ are isomorphisms by the descent lemma below, and the lower vertical maps are isomorphisms by the property (ii) for $C'$. Moreover the lower horizontal map is an isomorphism, because $\pi_1$ induces an isomorphism $Z_f \to Z_1$. This shows that the other two horizontal maps are isomorphisms as well. Finally the composition $(\pi_2)_*((\pi_1)_*)^{-1} : H_a(Z_1) \to H_a(Z_2)$ coincides with the morphism $f_*$. It follows from this that one gets commutative diagrams

$$\begin{array}{ccc} H_a(C'(\tilde{i_1})) & \xrightarrow{b_*(a_*)^{-1}} & H_a(C'(\tilde{i_2})) \\ {\scriptstyle (h_1)_*}\downarrow & & \downarrow{\scriptstyle (h_2)_*} \\ H'_a(Z_1) & \xrightarrow{f_*} & H'_a(Z_2) \end{array}$$

in which the bottom map is the one following from the functoriality of $C'(-)$. It is now clear that the constructed maps $H'_a(Z) \to H_a(Z)$ are functorial. The functoriality with respect to the long exact homology sequences follows in a similar way.

**Lemma 5.17** Let $C'(-)$ be as in Proposition 5.16. If $Z_. \to X$ is a hyperenvelope of a variety in $\mathcal{V}_{k,d}$, then the canonical morphism

$$tC'(Z_.) \longrightarrow C'(X)$$

(induced by the morphism $\tilde{Z}_0 \to X$) is a quasi-isomorphism. Here $tC'(Z_.)$ is defined as in the proof of Theorem 5.13.

**Proof** This follows exactly as in the descent theorem [Gi] Theorem 4.1. Let me very briefly recall the three steps: All considered varieties will be in $\mathcal{V}_{k,d}$.

(I) If $Z_. = cosk_0^X(Z) \to X$ is the natural morphism for an envelope $Z \to X$, and $Z \to X$ has a section, then $Z_.$ is homotopy equivalent to the constant simplicial variety $X$, and the claim follows via the convergent spectral sequence

$$E^1_{p,q}(Z_.) = H_p(H_q(C'(Z_.))) \Rightarrow H_{p+q}(tC'(Z_.))\,, \tag{5.18}$$

whose existence follows with the same argument as for (5.13) (it is the spectral sequence for the filtration with respect to the 'simplicial' degree of the bi-complex $C'(Z_.)$).

(II) If we consider $Z_. = cosk_0^X(Z) \to X$ for an arbitrary envelope $Z \to X$, then by the definition of envelopes and by localization in $X$, i.e., by the exact sequence 5.16. (ii) and the induced one for $Z.$, and by noetherian induction, we may reduce to the case that $Z \to X$ has a section.

(III) Then, to extend this to the general case it suffices to show that the morphism of simplicial schemes

$$f : Z[n+1]. := cosk_{n+1}^X sk_{n+1}(Z_.) \longrightarrow cosk_n^X sk_n(Z_.) =: Z[n]. \tag{5.19}$$

induces a quasi-isomorphism $tC'(Z[n+1].) \to C'(Z[n].)$ for all $n \geq 0$, because $Z[n]_j = Z_j$ for $j \leq n$ and hence $H_j(tC'(Z_.)) \xrightarrow{\sim} H_j(Z[n].)$ for $j < n$ by the spectral sequence (5.18). To show that the morphism (5.19), abbreviated $f : X'_. \to X_.$, induces a quasi-isomorphism one then follows the proof of [SGA 4, Vbis] (3.3.3.2). In fact, as noted in [Gi], the reasoning of loc. cit. (3.3.3.3) shows that all morphisms $F_i : X'_i = Z[n+1]_i \to X_i = Z[n]_i$ are envelopes. Note that the diagram in loc. cit. 3.3.3.3 should read

$$\begin{array}{ccccc} K'_\iota & \longrightarrow & \prod X'_r & \overset{pr_i}{\underset{X'(\iota)pr_j}{\rightrightarrows}} & X'_i \\ \downarrow & & \downarrow & & \downarrow \\ K_\iota & \longrightarrow & \prod X_r & \overset{pr_i}{\underset{X(\iota)pr_j}{\rightrightarrows}} & X_i \end{array}$$

for the morphism $\iota : i \to j$ in $\underset{n+1[p]}{\Delta^+}$ (notations as in §7), where the product is over all $\underset{n+1[p]}{\Delta^+}$. Moreover, $\cap K_\iota$ is rather $\times_S K_\iota$. By looking at the bi-simplical scheme $[X'_./X_.]$ with components $[X'_./X_.]_p := X'_. \times_{X_.} \ldots \times_{X_.} X'_.$ ($(p+1)$ times) and its base change with $X'_. \to X_.$, one sees that it suffices to replace $X'_. \to X_.$ by its base change with $[X'_./X_.]_p$ for all $p \geq 0$. This is again of the form (5.18), and has a section $s$. So $fs = id$, and because $sf$ is the identity on $sk_n(Z_.)$, it is homotopic to the identity ([SGA 4, Vbis], (3.0.2.4)). Therefore $f$ in this situation is a homotopy equivalence, and hence induces a quasi-isomorphism by (5.18). This finishes the proof of Lemma 5.17.

**Proof of Theorem 5.12** We construct the morphism of homology theories on $\mathcal{V}_{K,d}$,

$$\varphi : \overline{H}_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to H^W_*(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \tag{5.20}$$

using Proposition 5.16. We may replace $K$ by its perfect hull $K'$, because both cohomology theories do not change; the source of $\varphi$ by the topological invariance of étale cohomology, and the target by its definition. Obviously, one has a direct sum decomposition $\overline{C}(X_1 \coprod X_2, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cong \overline{C}(X_1, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \oplus \overline{C}(X_2, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ for varieties $X_1$, $X_2$, and a morphism $\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \overline{C}(K, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathbb{Q}_\ell/\mathbb{Z}_\ell$ induced by the structural morphism $X \to \mathrm{Spec}(K)$ for any variety. Applying this to the connected components of each smooth projective variety, we get a functorial map

$$\overline{C}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \to \bigoplus_{\pi_0(X)} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

for all smooth projective varieties over $K$. Thus by 5.9 and 5.16 and our assumptions we obtain the morphism of homology theories (5.20).

Property 4.22 (i) holds for $\overline{H}$ by remark 4.23, and property 4.22 (ii) holds for $H^W$ by theorem 5.10.

It remains to show property 4.22 (iii) for the morphism (5.20). Let $Z$ be an integral variety of dimension $d$ over $K$ and let $V \subset Z$ be any non-empty smooth subvariety. Let $U \subset V$ and $U \subset X$, $Y \subset X$ be as in property $\mathrm{RS}_{gc}(K, d)$ (which holds by assumption). By possibly removing a suitable further smooth hyperplane section we may assume that $X \setminus Y_1$ is affine. For a finite field $K$ be argue as in the proof of theorem 5.10.

If $W$ is irreducible of dimension $d$, then $H_d(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ can be identified with

$$\ker(H^d(K(W) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \to \bigoplus_{x \in W^1} H^{d-1}(K(x) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1))_{G_K}),$$

and if $W$ is irreducible and smooth of dimension $d$ the Bloch-Ogus theory gives a canonical morphism

$$\gamma_W : H^d(\overline{W}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \longrightarrow H_d(W, \mathbb{Q}_\ell/\mathbb{Z}_\ell).$$

**Lemma 5.21** There is a commutative diagram

$$\begin{array}{ccccccc}
0 \to & H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} & \stackrel{e}{\to} & H^0(\overline{Y^{[d]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))_{G_K} & \stackrel{d_2}{\to} & H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \\
& \downarrow \gamma_U & & \downarrow \gamma_{Y^{[d]}} & & \downarrow \gamma' \\
0 \to & H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \stackrel{e}{\to} & H_0(Y^{[d]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \stackrel{d_2}{\to} & H_0(Y^{[d-1]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\
& \downarrow \varphi_U & & \downarrow \varphi_{Y^{[d]}} & & \downarrow \varphi_{Y^{[d-1]}} \\
0 \to & H^W_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \stackrel{e}{\to} & H^W_0(Y^{[d]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \stackrel{d_2}{\to} & H^W_0(Y^{[d-1]}, \mathbb{Q}_\ell/\mathbb{Z}_\ell).
\end{array}$$

Here the maps $e$ and $d_2$ in the first row are those occuring in Theorem 3.1. The maps in the second and third row are the homological analogues: $e$ is the composition of the morphisms

$$H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \stackrel{\delta}{\to} H_{d-1}(Y_{i_d} \setminus (\bigcup_{i \neq i_d} Y_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \stackrel{\delta}{\to} \ldots$$

$$\dots \xrightarrow{\delta} H^1(Y_{i_2,\dots,i_d} \smallsetminus (\bigcup_{i \neq i_2,\dots,i_d} Y_i), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\delta} H^0(Y_{i_1,\dots,i_d}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

where each $\delta$ is the connecting morphism for the obvious localization sequence for the homology theory, and $d_2 = \sum_{\mu=1}^{d}(-1)^\mu \delta_\mu$, where $\delta_\mu$ is induced by the push-forward morphisms for the inclusions $Y_{i_1,\dots,i_d} \hookrightarrow Y_{i_1,\dots,\hat{i_\mu},\dots,i_d}$. Finally, $\delta' = \gamma_{Y^{[d-1]}} \circ tr'$ where $tr' : H^2(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))_{G_K} \to \sim H^0(\overline{Y^{[d-1]}}, \mathbb{Q}_\ell/$ is the map induced by the trace map.

**Proof** For any smooth irreducible variety $W$ of dimension $d$ over $K$, and any smooth irreducible divisor $i : W' \hookrightarrow W$, we have a commutative diagram

$$\begin{array}{ccc} H^d(\overline{W \smallsetminus W'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \xrightarrow{\delta} & H^{d-1}(\overline{W'}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)) \\ \downarrow & & \downarrow \\ H^d(K(W) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d)) & \xrightarrow{\delta} & H^{d-1}(K(W') \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d-1)) \end{array},$$

where the $\delta$ in the top line is the connecting morphism for the Gysin sequence for $W' \subseteq W \supseteq W \smallsetminus W'$, and the $\delta$ in the bottom line is the residue map for the point in $W^1$ corresponding to $W'$. The latter induces the connecting morphism

$$H_d(W \smallsetminus W', \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\delta} H_{d-1}(W', \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

for the localization sequence for $(W', W, W \smallsetminus W')$. This shows the commutativity of the top left square in 5.21, by definition of the maps $e$.

On the other hand, for a smooth projective curve $C$ over $\overline{K}$, and a closed point $P : \mathrm{Spec}(\overline{K} \to C$, the composition

$$H^0(\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0)) \xrightarrow{P_*} H^2(C, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \xrightarrow{tr} H^0(\overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(0))$$

is the identity. This shows the commutativity of the top right square in 5.21. The two bottom squares commute because $\varphi$ is a morphism of homology theories.

We proceed with the proof of property 4.22 (iii) for $\varphi$. The compositions of the vertical maps in the middle column and the right column of the diagram in 5.21 are isomorphisms, and the top row is exact by Theorem 3.1 (and our assumption on $U$). But the bottom line is exact as well: This follows in a similar (but simpler) way as in the proof of Theorem 3.1, by noting that $H_a^W(T, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$ for $a \neq 0$ if $T$ is smooth and projective of dimension $> 0$, by definition. It can also be deduced from the fact that $H_a^W(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is computed as the $a$-th homology of the complex

$$\bigoplus_{\pi_0(Y^{[d]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \bigoplus_{\pi_0(Y^{[d-1]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell \to \dots \quad \dots \to \bigoplus_{\pi_0(Y^{[0]})} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

as noted before.

This shows that the composition

$$H^d(\overline{U}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \xrightarrow{\gamma_U} H_d(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\varphi_U} H_d^W(U, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

figuring in the left column of 5.21 is an isomorphism. Because the subvarieties $U$ as constructed above form a cofinal family in the set of open subvarieties of $Z$, by passing to the limit we get an isomorphism

$$H^d(K(Z) \otimes_K \overline{K}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d))_{G_K} \xrightarrow{\gamma} H_d(K(Z), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\varphi} H_d^W(K(Z), \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

in which the first map $\gamma$ is an isomorphism by definition. Therefore $\varphi$ is an isomorphism as wanted.

## §6 The Kato conjecture over finite fields

Theorem 4.18 also gives the following conditional result on another conjecture by Kato [Ka]:

**Theorem 6.1** Let $k$ be a finite field, let $X$ be an irreducible smooth projective variety of dimension $d$ over $k$, and assume that the properties $\mathrm{RS}_{gc}(k,d)$ and $\mathrm{RS}_{pb}(k,d)$ hold. Let $\ell$ be a prime. Then

$$H_a^K(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) := H_a(C^{1,0}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)) = \begin{cases} \mathbb{Q}_\ell/\mathbb{Z}_\ell & , \quad a=0 \\ 0 & , \quad a \neq 0\,. \end{cases}$$

If in addition property $\mathrm{BK}(k,d,\ell)$ holds, then for any $n$ one has

$$H_a^K(X,\mathbb{Z}/\ell^n\mathbb{Z}) := H_a(C^{1,0}(X,\mathbb{Z}/\ell^n\mathbb{Z})) = \begin{cases} \mathbb{Z}/\ell^n\mathbb{Z} & , \quad a=0 \\ 0 & , \quad a \neq 0\,. \end{cases}$$

In fact, we have the following lemma.

**Lemma 6.2** Let $X$ be any variety over $k$.
(a) One has a canonical isomorphism of complexes

$$C^{1,0}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \cong C^{0,0}(X\times_k \overline{k},\mathbb{Z}/\ell^n\mathbb{Z})_{G_k}\,.$$

(b) If $\mathrm{BK}(k,d,\ell)$ holds, then the canonical sequence

$$0 \to C^{1,0}(X,\mathbb{Z}/\ell^n\mathbb{Z}) \to C^{1,0}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\ell^n} C^{1,0}(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell) \to 0\,.$$

is exact.

**Proof** Let $F$ be a function field of transcendence degree $m$ over $k$. Then one has canonical isomorphisms

$$H^{m+1}(F,\mathbb{Z}/\ell^n\mathbb{Z}(m)) \cong H^1(k,H^m(F\overline{k},\mathbb{Z}/\ell^n\mathbb{Z}(m))) \cong H^m(F\overline{k},\mathbb{Z}/\ell^n\mathbb{Z}(m))_{G_k}\,,$$

where $F\overline{k}$ is the function field over $\overline{k}$ deduced from $F$ (i.e., $F\overline{k} = F\otimes_{\{k\}}\overline{k}$, where $\{K\}$ is the algebraic closure of $k$ in $F$). In fact, the first isomorphism follows from the Hochschild-Serre spectral sequence, because $F\overline{k}$ has $\ell$-cohomological dimension $m$, and the second isomorphism comes from the canonical identification $H^1(k,M) = M_{G_k}$ for any $G_k$-module $M$, if $k$ is a finite field. By applying this to all fields $k(x)$ for $x\in X$, we obtain (a).

(b) follows similarly as in the proof of lemma 4.11.

By the Lemma, Theorem 6.1 follows immediately from Theorem 4.18, if $\ell$ is invertible in $k$. Now consider the prime $p = \mathrm{char}(k)$. Since $k$ is a finite field, we then can perform exactly the same steps of Theorem 3.1 as in the case of $\ell \neq \mathrm{char}(k)$ (compare also the arguments in the proof of [JS1], Theorem 3.5). In fact, one has Gysin sequences for closed immersions of smooth varieties [JSS], and one has a weak Lefschetz theorem by a result of Suwa [Su] lemma 2.1. Finally one can use weights by using Deligne's proof of the $\ell$-adic Weil conjecture and the transport to crystalline cohomology due to Katz and Messing, see the arguments in [GrS].

## §7 Appendix: Hyperenvelopes

In this section we collect some results on hyperenvelopes which are scattered or not explicit enough in the literature.

Let $\mathcal{C}$ be a category in which finite inverse (not necessarily filtered) limits exist, i.e., in which finite products and difference kernels exist. Let $\Delta^o\mathcal{C} = Hom(\Delta^o, \mathcal{C})$ be the category of simplicial objects in $\mathcal{C}$, i.e., contravariant functors from the category $\Delta$ of the sets $[m] = \{0, \ldots, m\}$ with monotone (non-decreasing) maps. As usual we write the objects $X$ in $\Delta^o\mathcal{C}$ as $X.$, where $X_n$ stands for $X([n])$; similarly for the truncated case. Let $\Delta_n \subset \Delta$ be the full subcategory formed by the objects $[m]$ with $m \leq n$, and let $\Delta_n^o\mathcal{C} = Hom(\Delta_n^o, \mathcal{C})$ be the category of $n$-truncated simplicial objects in $\mathcal{C}$, see [SGA V 2], exposé V bis. In these categories again finite inverse limits exist, and are formed 'componentwise', i.e., the functors $X. \mapsto X_m$ commute with the limits. The functor of restriction

$$sk_n : \Delta^o\mathcal{C} \longrightarrow \Delta_n^o\mathcal{C}\,,$$

called $n$-sceleton, commutes with finite inverse limits and has a right adjoint

$$cosk_n : \Delta_n^o\mathcal{C} \longrightarrow \Delta^o\mathcal{C}\,,$$

called $n$-cosceleton, which commutes with finite inverse limits again. One has

$$cosk_n(X)_m = \lim_{\substack{\longleftarrow \\ \underset{n[m]}{\Delta}}} X_q\,, \tag{7.1}$$

where $\underset{n}{\Delta}$ is the full subcategory of $\Delta$ formed by the objects $[q]$ with $q \leq n$ and $\underset{n[m]}{\Delta}$ is the category of arrows $[q] \to [m]$ in $\underset{n}{\Delta}$. From this it folllows that the adjunction morphisms $Z. \to sk_n(cosk_n(Z.))$ are isomorphisms for any $n$-truncated simplicial object $Z.$, i.e., one can identify $cosk_n(Z)_m = Z_m$ for $m \leq n$ (because then $\underset{n[m]}{\Delta}$ has the final object $id_{[m]}$).

If we apply this to the category Sch/$S$ of schemes over a fixed base scheme $S$, we obtain functors which we call $sk_n^S$ and $cosk_n^S$, respectively. (If we do not mention $S$ we can always take $S = \mathrm{Spec}(\mathbb{Z})$.) We may write $sk_n$ because this construction does not depend on $S$. More generally, fix a simplicial $S$-scheme $T.$ and consider the categories $\Delta^0\mathrm{Sch}/T.$ and $\Delta_n^0\mathrm{Sch}/sk_nT.$ of simplicial schemes over $T.$ and $n$-truncated simplicial schemes over $sk_nT.$, respectively. Then $sk_n$ induces a functor

$$sk_n : \Delta^0\mathrm{Sch}/T. \longrightarrow \Delta_n^0\mathrm{Sch}/sk_nT.\,,$$

and this functor has a right adjoint

$$cosk_n^{T.} : \Delta_n^0\mathrm{Sch}/sk_nT. \longrightarrow \Delta^0\mathrm{Sch}/T.$$

which we may call $n$-cosceleton over $T.$. In fact, it is easily checked that we have

$$cosk_n^{T.}(Z.) = cosk_n^S(Z.) \times_{cosk_n^S(sk_nT.)} T. \tag{7.2}$$

for an $n$-truncated simplicial scheme $Z.$ where the morphism $T. \to cosk_n^S(sk_nT.)$ comes from adjunction. In other words, we have $cosk_n^{T.}(Z.)_m = cosk_n^S(z.)_m \times_{cosk^S(T.)_m} T_m$ for all $m$. This does not depend on $S$ but only on $T.$, because we can always replace $S$ by Spec($\mathbb{Z}$): If $T.$ is a simplicial $S$-scheme, then any simplicial scheme over $T.$ is canonically a simplicial $S$-scheme, and the right hand side of (7.2) is the same for Spec($\mathbb{Z}$) and $S$. If $\underline{T}.$ is the constant simplicial scheme associated to an $S$-scheme $T$ (which is the constant functor $\Delta \to \mathrm{Sch}/S$ with value $T$), then a simplicial scheme $X$ over $\underline{T}.$ is the same as a simplicial $T$-scheme, and one has $cosk_n^{\underline{T}.}(X) = cosk_n^T(X)$.

The following definitions can be found in [Gi] p. 89 and [GS] 1.4.1.

**Lemma 7.3** (a) A morphism $f : X \to Y$ of schemes is called an envelope, if it is proper, and for any $y \in Y$ there is a point $x \in X$ such that $f(x) = y$ and the morphism $k(x) \to k(y)$ is an isomorphism.

(b) For a scheme $S$, a simplicial $S$-scheme $X.$ is called a hyperenvelope of $S$, if for all $n \geq 0$ the canonical morphism

$$can : X_n \to cosk_{n-1}^S(sk_{n-1}(X.))_n$$

coming from adjunction is an envelope (where $cosk_{-1}^S(X.) := \underline{T}.$).

(c) More generally, a morphism of simplicial schemes $f : X. \to Y.$ is called a hyperenvelope, if for all $n \geq 0$ the canonical adjunction morphism

$$can : X_n \to cosk_{n-1}^{Y.}(sk_{n-1}(X.))_n$$

is an envelope (where $cosk_{-1}^{Y.}(X.) := Y.$).

Here (b) is a special case of (c), because (b) means that $X. \to \underline{S}.$ is a hyperenvelope. For the following we note that, by (7.2), we have

(7.4)
$$cosk_m^{Y.}(sk_m(X.) = Cosk_m(X.) \times_{Cosk_m(Y.)} Y.\,,$$

where we define $Cosk_m(X.) = cosk_m(sk_m(X.))$.

**Lemma 7.5** (a) If $X. \to Y.$ is a hyperenvelope and $Y'. \to Y.$ is any morphism, then $X. \times_{Y.} Y'. \to Y'.$ is a hyperenvelope.

(b) If $f : X. \to Y.$ and $g : Y. \to Z.$ are hyperenvelopes, then $gf : X. \to Z.$ is a hyperenvelope.

**Proof** The base change of an envelope is an envelope, and the composition of envelopes is an envelope. Therefore both claims follow from (7.4) and the fact that $sk_n$ and $cosk_n$ and hence $Cosk_n$ commute with (fiber) products and that a base change of an envelope is an envelope. Indeed, for (a) one uses the formula

$$Cosk_m(X. \times_{Y.} Y'.) \times_{Cosk_m(Y'.)} Y'. = Cosk_m(X.) \times_{Cosk_m(Y.)} Y'.$$

and for (b) the factorization of the morphism $can : X. \to Cosk_m(X.) \times_{Cosk_m(Z.)} Z.$ into

$$X. \xrightarrow{can} Cosk_m(X.) \times_{Cosk_m(Y.)} Y. \xrightarrow{id\times can} Cosk_m(X.) \times_{Cosk_m(Y.)} Cosk_m(Y.) \times_{Cosk_m(Z.)} Z.$$

and the obvious contraction isomorphism.

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