Rigidity theorems for $\mathcal{K}$- and $\mathcal{H}$-cohomology and other functors

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Abstract

Suslin [Su] proved that algebraic K-theory with finite coefficients is the same for an algebraically closed field K and for an algebraically closed extension L. In this paper we generalize this to other functors and other field extensions.

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0 Introduction

Consider the following contravariant functors on schemes with values in abelian groups.

(1) $V(Y) = H^i(Y_{ét}, \mathbb{Z}/n\mathbb{Z}(j))$ (étale cohomology), where $i, j \in \mathbb{Z}$, $n \in \mathbb{N}$ is invertible on Y, and $\mathbb{Z}/n\mathbb{Z}(j) = \mu_n^{\otimes j}$, for the sheaf μ_n of n-th roots of unity on V.

(2) $V(Y) = K_m(V)$ (algebraic K-theory) for $m \in \mathbb{N}_0$.

(3) $V(Y) = CH^m(Y, n)$ (Bloch's higher Chow groups) for $m, n \in \mathbb{N}_0$.

(4) $V(Y) = H^\nu(Y, \mathcal{K}_m)$ ($\mathcal{K}$-cohomology) for $m, n \in \mathbb{N}_0$, the n-th Zariski cohomology of the Zariski sheaf associated to the presheaf $U \mapsto K_m(U)$.

(5) $V(Y) = H^\nu(Y, \mathcal{H}^i(\mathbb{Z}/n\mathbb{Z}(j)))$ ($\mathcal{H}$-cohomology), the Zariski cohomology of the Zariski sheaf associated to the presheaf associated to the presheaf $U \mapsto H^i(U_{ét}, \mathbb{Z}/n\mathbb{Z}(j))$, for n invertible on Y and $i \in \mathbb{N}$ and $j \in \mathbb{Z}$.

On the other hand, consider field extensions $L^0 \subset L$ and $n \in \mathbb{N}$ such that one of the following conditions hold

(i) Both fields are algebraically closed, or

(ii) both fields are real closed, or

(iii) L is a complete discrete valuation field, L^0 is algebraically closed in L and dense in L for the valuation topology, and n is invertible in L.

An application we have in mind for *(iii)* is the situation where K is a global field (i.e., a number field or a function field in one variable over a finite field), and where $L = K_v$ is the completion of K with respect to a discrete valuation v of K, and L^0 is the Henselization $K_{(v)}$ of K at v. An example for *(ii)* is the situation where L is the completion of K at a real place v (hence isomorphic to the field $\mathbb{R}$ of real numbers), and L^0 is the associated real closure of K in L, the algebraic elements in the extension L/K. An example for *(i)* is the situation where L is the completion of K at a complex place (hence isomorphic to $\mathbb{C}$), and L^0 is the algebraic closure of K in L.

Theorem 0.1 *If V is one of the functors (1) - (5) and the field extension $L^0 \subset L$ is one of the field extensions (i) - (iii), and X is a smooth projective variety over L^0, then, with n as in the respective cases (1) - (5), the restriction maps*

$$V(X)[n] \xrightarrow{\cong} V(X_L)[n] \qquad \text{and} \qquad V(X)/n \xrightarrow{\cong} V(X_L)/n$$

are isomorphisms, where $X_L = X \times_{L^0} L$, $A[n] = ker(A \xrightarrow{n} A)$, and $A/n = coker(A \xrightarrow{n} A)$.

Remark 0.2 *(a) This is well-known for (1), where case (i) is well-known and implies (ii) and (iii) by Galois descent.*

(b) For (2), case (i), and $Y = Spec(L^0)$ this is the result of Suslin quoted in the beginning.

(c) For (4) and case (i) this was proved by F. Lecomte [Le].

(d) The cases (ii) and (iii) of (b) and (c) do not follow - because these theories do not have Galois descent.

However, with our methods explained below, we can treat more cases.

Example 0.3 *Let K be a global field, let v be a non-archimedean place.*

(a) For case (3) and $m = n$ we consider the Chow groups $H^m(X, \mathcal{K}_m) = CH^m(X)$, i.e., the Chow groups of a smooth projective variety X over a field. In case (iii) we get isomorphisms, for a global field K, and an integer n invertible in K, and a non-archimedean place v of K:

$$CH^m(X_{K_{(v)}})[n] \xrightarrow{\cong} CH^m(X_{K_v})[n] \qquad \text{and} \qquad CH^m(X_{K_{(v)}})/n \xrightarrow{\cong} CH^m(X_{K_v})/n\,.$$

(b) Let X be a smooth projective curve over K and consider the "residue map" for n invertible in K

$$H^i(K(X), \mathbb{Z}/n\mathbb{Z}(j)) \xrightarrow{\alpha} \bigoplus_{x \in |X|} H^{i-1}(k(x), \mathbb{Z}/n\mathbb{Z}(j))\,,$$

where $\mid X \mid$ *is the set of closed points of* X*. Then one has* $\ker(\alpha) = H^0(X, \mathcal{H}^i_n(j))$*, and by the above this group for* $X_{K_{(v)}}$ *is isomorphic to the one for* X_{K_v}*. The same holds for* $\mathrm{coker}(\alpha) = H^1(X, \mathcal{H}^i_n(j))$*.*

(c) For a field F *one has an isomorphism* $K^M_m(F) \cong CH^m(F, m)$ *between the Milnor* K*-group and the written Bloch higher Chow group, by work of Nesterenko and Suslin [NeSu], see also Totaro [To]. Hence one gets rigidity for Milnor* K*-theory mod* n *and the* n*-torsion of Milnor* K*-theory.*

The second main result of this paper is the following.

Theorem 0.4 *Let* k *be a field and let* R *be the Henselization of the local ring of a smooth* k*-variety at a* k*-rational point. Then, for any smooth* k*-scheme* X *and any integer* n *invertible in* k *the restriction maps*

$$V(X \times_k R)[n] \longrightarrow V(X)[n] \qquad \text{and} \qquad V(X \times_k R)/n \xrightarrow{\cong} V(X)/n$$

are isomorphisms, if V *is one of the functors (1) - (5).*

Example 0.5 *(a) This result is well-known for example (1) (étale cohomology with coefficients invertible in* k*), where it is known by the base change theorem in étale cohomology.*

(b) For example (2) (algebraic K*-theory) the result is known by results of Gabber, Gillet, and Thomason*

The above results are implied by some more general theorems. We introduce a notion of rigid functors (see Definition 1.1) and sufficiently rigid functors (see Definition **??**, and prove that the above rigidity result hold for such functors (which is not obvious).

1 Rigid Functors

Let $\mathcal{S}$ be a category of schemes.

Definition 1.1 *A contravariant functor* V *on* $\mathcal{S}$ *with values in the category* Ab *of abelian groups is called rigid, if it satisfies the following properties, provided the occurring schemes and morphisms are in* $\mathcal{S}$*.*

(a) For any flat finite morphism $\pi : X \to Y$ *there is a transfer morphism* $\pi_* : V(X) \to V(Y)$*, such that for another flat finite morphism* $\rho : Y \to Z$ *one has* $(\rho\pi)_* = \rho_*\pi_*$*.*

(b) For every cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y \end{array} \quad ,$$

with π finite and flat, one has $f^*\pi_* = \pi'_* f'^* : V(X) \to V(Y')$.

(c) If $X = X_1 \amalg X_2$, then the immersions $\pi_i : X_i \hookrightarrow X$ $(i = 1, 2)$ induce an isomorphism

$$(\pi_1^*, \pi_2^*) : V(X) \xrightarrow{\sim} V(X_1) \oplus V(X_2)$$

with inverse $(\pi_1)_* + (\pi_2)_*$.

(d) If $X_m = X \times_{\mathbb{Z}} Spec\ (\mathbb{Z}[T]/(T^m))$ is the m-fold thickening of X, then for the morphism $\pi : X_m \to X$ one has $\pi^*\pi_* =$ multiplication by m.

(e) If $\mathbb{A}^1_X = X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}$ is the affine line over X, then the projection $p : \mathbb{A}^1_X \to X$ induces an isomorphism $p^* : V(X) \xrightarrow{\sim} V(\mathbb{A}^1_X)$.

(f) Let $i \mapsto X_i$ be a filtered projective system of schemes such that the transition morphisms $X_i \to X_j$ are affine, and let $X = \varprojlim X_i$. Then the canonical map

$$\varinjlim V(X_i) \longrightarrow V(X)$$

is an isomorphism.

Theorem 1.2 *Let K be $\mathbb{R}$, $\mathbb{C}$ or a complete discrete valuation field. Let K^0 be a dense subfield of K which is algebraically closed in K, and assume that K is separable over K^0 if char $(K) > 0$. If V is a rigid functor on the category of all noetherian K^0-schemes, such that the value groups $V(Y)$ are torsion groups of order prime to char (K), then the restriction map*

$$V(K^0) \longrightarrow V(K)$$

is an isomorphism (where we write $V(L)$ for $V(Spec\ L)$ if L is a field).

Proof For the surjectivity it suffices to prove

Claim 1: If F is a function field over K^0, contained in K, then $Im\ (V(F) \to V(K) \subseteq Im\ (V(K^0) \to V(K))$.

In fact, by the limit property 1.1 (f), $V(K)$ is generated by the images of the maps $V(F) \to V(K)$ for all such fields F.

We prove Claim 1 by induction on $f = deg.tr.(F/K^0)$, the degree of transcendence of F over K^0. If $d = 0$, then necessarily $F = K^0$, since K^0 is algebraically closed in K, and the claim is trivially true. If $d > 0$, then by Noether normalization and separability of F over K^0, there exists a function field F_1 with $deg.tr.(F_1/K^0) = d - 1$ and a smooth, geometrically irreducible curve C_1 over F_1 such that $F = F_1(C_1)$, the function field of C_1. Let $\tilde{F}_1$ be the algebraic closure of F_1 in K and let $\tilde{C}_1 = C_1 \times_{F_1} \tilde{F}_1$, and $\tilde{F} = \tilde{F}_1(\tilde{C}_1)$. Then it suffices to show

Claim 2: $Im(V(\tilde{F}) \to V(K) \subseteq Im(V(\tilde{F}_1) \to V(K))$.

In fact, by 1.1 (f) every $\alpha \in Im(V(\tilde{F}_1) \to V(K))$ lies in $Im(V(F_2) \to V(K))$ for a function field F_2 over K with $deg.tr.(F_2/K^0) = d-1$, so by induction α lies in $Im(V(K^0) \to V(K))$.

In other words, it suffices to prove Claim 1 for a field $L \subseteq K$ in place of K^0, which is dense and algebraically closed in K but for which K is not necessarily separable over L, and for a function field F with $deg.tr.(F/L) = 1$ which is separable over L.

Let $\alpha \in Im(V(F) \to V(K))$. By 1.1 (f) there is a smooth, geometrically irreducible curve C over L with function field $L(C) = F$ such that $\alpha \in Im(V(C) \to V(K))$. Let $Div(C)$ be the group of divisors on C, i.e., the free abelian group on the closed points $x \in C$. Consider the bilinear pairing

$$Div(C) \times V(C) \longrightarrow V(L)$$

defined by sending (x, β) to $(\pi_x)_* \varphi_x^*(\beta)$, where $\varphi_x : Spec\ \kappa(x) \to C$ and $\pi_x : Spec\ \kappa(x) \to Spec\ L$ are the canonical morphisms. Denote by $\overline{C}$ the regular proper model of C and set $C_\infty = \overline{C} \setminus C$. Let f be a meromorphic function on $\overline{C}$ which is defined and equal to one on C_∞ (i.e., for $x \in C_\infty$, f lies in $\mathcal{O}_{\overline{C},x}$ and its image in $\kappa(x)$ is 1). Then the principal divisor (f) lies in the kernel of the above pairing. In fact, f defines a covering $\pi : C' \to \mathbb{A}^1_L = \mathbb{P}^1_L - 1$, where C' is obtained from C by deleting the points where f is defined and equal to one. Now one has a commutative diagram of pairings

$$\begin{array}{ccccc} Div(C) & \times & V(C) & \to & V(L) \\ \cup| & & \downarrow & & \| \\ Div(C') & \times & V(C') & \to & V(L) \\ \pi* \uparrow & & \downarrow \pi_* & & \| \\ Div(\mathbb{A}^1_L) & \times & V(\mathbb{A}^1_L) & \to & V(L)\ . \end{array}$$

Indeed, recall that for a closed point $x \in \mathbb{A}^1_L$ one has

$$\pi^*(x) = \sum_{\pi(y)=x} e(y/x) \cdot x \quad ,$$

where $e(y/x) = \text{length}(\mathcal{O}_{C,y} \otimes \kappa(x))$ is the ramification index of y over x (the tensor product is over $\mathcal{O}_{\mathbb{A}^1_L,x}$). Consider the following cartesian diagram

$$\begin{array}{ccc} C'_x & \xrightarrow{\varphi'_x} & C' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Spec\,\kappa(x) & \xrightarrow{\varphi_x} & \mathbb{A}^1{}_F \end{array} \quad .$$

Then $\varphi_x^* \pi_* = (\pi')_* (\varphi'_x)^*$ by 1.1 (b), and it suffices to show that

$$(\pi')_* = \sum_{\pi(y)=x} e(y/x)(\pi' \alpha_y)_* \alpha_y^* \quad ,$$

where $\alpha_y : Spec\ \kappa(y) \to C'_x$ is the canonical morphism. But since C' is smooth over L, we have

$$C'_x \cong \coprod_{\pi(y)=x} Spec(\kappa(y)[T]/(T^{e(y/x)}))\ ,$$

so the wanted equality follows easily from 1.1 (c) and (d), cf. also the proof of Lemma 1.8 below.

Now by definition $(f) = \pi^*(0-\infty)$; hence the commutative diagram of pairings shows that $\psi((f), \beta) \in V(L)$ coincides with $\psi'(0-\infty, \pi_*(\beta))$, which is zero in view of the homotopy invariance 1.1 (e).

We have proved that the pairing factors through $Pic(\overline{C}, C_\infty) \otimes V(C)$, where

$$Pic(\overline{C}, C_\infty) = Div(C)/\{f \in L(C)^\times \mid f = 1 \text{ on } C_\infty\}$$

is the divisor class group of modulus C_∞, where the closed subscheme $C_\infty \subset \overline{C}$ is identified with the corresponding Cartier divisor and hence with the Weil divisor $\sum\limits_{y \in C_\infty} y$.

Let $Pic^0(\overline{C}, C_\infty)$ be the subgroup defined by the divisors of degree zero, where $deg\ x = [\kappa(x) : L]$. Then we have a canonical isomorphism

$$Pic^0(\overline{C}, C_\infty) = J_{C_\infty}(\overline{C})(L) \ ,$$

where the right hand side is the group of L-rational points of the Rosenlicht generalized Jacobian $J_{C_\infty}(\overline{C})$ (cf. [Ro], [Se] Chap. V and [CC]), which is a commutative, smooth, geometrically connected group variety over L. Now

(1.2.1) $$J_{C_\infty}(\overline{C}) \times_L K = J_{C_{\infty,K}}(\overline{C}_K) = J_{\tilde{C}_\infty}(\tilde{C}_K) \ ,$$

where $C_{\infty,K} = C_\infty \times_L K = C_\infty \times_{\overline{C}} \overline{C}_K \subseteq \overline{C}_K = \overline{C} \times_L K$ as a closed subscheme, $\tilde{C}_K$ is the complete regular model of the smooth curve $C_K = C \times_L K$, and $\tilde{C}_\infty = C_\infty \times_{\tilde{C}} \tilde{C}_K$ as a closed subscheme (Note that $\overline{C}_K$ is not necessarily regular and that $C_{\infty,K}$ is not necessarily reduced, if $char\ K > 0$ and K/L is not separable). In the language of schemes the equalities (1.2.1) can most easily be seen by the fact that $J_D(X)$, for a geometrically integral curve X over a field k and a Cartier divisor $D \subseteq X$ such that $X - D$ is smooth, represents the Picard functor $Pic_{X_D/k}$, where X_D is the curve obtained by contracting D to a point, i.e., X_D is the scheme theoretic amalgamated sum $X \amalg_D Spec\ k$ (cf. [Se] p. 85 and [CC]). In fact, one has $(\overline{C}_K)_{C_{\infty,K}} = (\tilde{C}_K)_{\tilde{C}_\infty}$, since the diagram

$$\begin{array}{ccc} \tilde{C}_\infty & \longrightarrow & \tilde{C}_K \\ \downarrow & & \downarrow \\ C_{\infty,K} & \hookrightarrow & \overline{C}_K \end{array}$$

is cartesian and cocartesian.

If the pairing

$$\psi_K : Div(C_K) \times V(C_K) \longrightarrow V(K)$$

is defined for C_K over K in the same way as for C over L above, then by the same argument as there, this pairing factors through $Pic(\tilde{C}_K, (\tilde{C}_\infty)_{red}) \otimes V(C_K)$ and therefore also through $Pic(\tilde{C}_K, \tilde{C}_\infty) \otimes V(C_K)$, where

$$Pic(\tilde{C}_K, \tilde{C}_\infty) = Div(\tilde{C}_K)/\{f \in K(C_K)^\times \mid f \equiv 1 mod\ \tilde{C}_\infty\}$$

is the divisor class group of modulus $\tilde{C}_\infty$ ($f \equiv 1 mod\ \tilde{C}_\infty$ meaning that f lies in $\Gamma(U, \mathcal{O}_U)$ for an open neighbourhood of $(\tilde{C}_\infty)_{red}$ and has image 1 in $\Gamma(\tilde{C}_\infty, \mathcal{O}_{\tilde{C}_\infty})$). Let $p : C_K \to$

C be the projection induced by $p : Spec\ K \to Spec\ L$. Then by 1.1 (b) we have a commutative diagram of pairings

$$\begin{array}{lcccccc} \psi_K : & Div(C_K) & \times & V(C_K) & \longrightarrow & V(K) \\ & {\scriptstyle p*}\uparrow & & \uparrow{\scriptstyle p*} & & \uparrow{\scriptstyle p*} \\ \psi : & Div(C) & \times & V(C) & \longrightarrow & V(L)\,, \end{array}$$

where the left hand map is the pull-back of Cartier divisors (which sends $x \in\mid C \mid$ to the unique $x' \in\mid C_K \mid$ with $p(x') = x$; note that $\kappa(x) \otimes_L K$ is a field, since $\kappa(x)$ is separable over L and L is separably closed in K).

For the following we may assume that V is annihilated by a natural number n invertible in K, since it suffices to prove Theorem 1.2 for all subfunctors $V[n]$ for such n. Then we get an induced diagram

$$\begin{array}{ccccc} Pic(\tilde{C}_K, \tilde{C}_\infty)/n & \otimes & V(C_K) & \longrightarrow & V(K) \\ {\scriptstyle p*}\uparrow & & \uparrow{\scriptstyle p*} & & \uparrow{\scriptstyle p*} \\ Pic(\overline{C}, C_\infty)/n & \otimes & V(C) & \longrightarrow & V(L)\,, \end{array}$$

To prove Claim 1 here it then suffices to show

Claim 3: $p^* :\ Pic^0(\overline{C}, C_\infty)/n \longrightarrow Pic^0(\tilde{C}_K, \tilde{C}_\infty)/n$ is surjective.

In fact, consider an element $\beta \in V(C)$ mapping to our element $\alpha \in Im(V(C) \to V(K))$. Then $\alpha = \psi_K(y_0, p^*(\beta))$, where y_0 is the K-rational point of C_K corresponding to the generic point $Spec\ K \to Spec\ F \to C$. By Lemma 1.3 below, there exists an L-rational point x_0 of C. Then $y_0 - p^*(x_0)$ lies in $Div^0(C_K)$, and by Claim 3 there exists an element $z \in Div(C)$ with $\psi_K(y_0 - p^*(x_0), p^*(\beta)) = \psi_K(p^*(z), p^*(\beta))$. Thus $\alpha = \psi_K(y_0, p^*(\beta)) = p^*\psi(x_0 + z, \beta)$ lies in the image of $p^* : V(L) \to V(K)$ as wanted.

To prove Claim 3, note that the considered map can be identified with the natural map

$$J_{C_\infty}(C)(L)/n \longrightarrow J_{C_\infty}(C)(K)/n\ ,$$

since $J_{C_\infty}(C)(K) = (J_{C_\infty}(C) \times_L K)(K) = J_{\tilde{C}_\infty}(\tilde{C}_K)(K) = Pic(\tilde{C}_K, \tilde{C}_\infty)$ (and since $J_D(X)$ represents $Pic_{X_D/k}$ as mentioned above). Now $n\ J_{C_\infty}(C)(K)$ is open in $J_{C_\infty}(C)(K)$ for the strong topology on this set (i.e., the topology coming from the topology of K), since the morphism $n :\ J_{C_\infty}(C) \to J_{C_\infty}(C)$ is étale, n being invertible in L. Therefore the claim follows from

Lemma 1.3 *Let K be $\mathbb{R}$, $\mathbb{C}$, or a complete discrete valuation field, let L be a dense subfield, and let X be a scheme of finite type over L. If K is separable over L or if X is smooth over L, then $X(L)$ is dense in $X(K)$ for the strong topology.*

This is proved in [K-S] Lemma 4, where the first case is explicitly stated and reduced to the second case in the proof, and where the reader may also find a definition of the strong topology (called the usual topology there).

It remains to prove the injectivity of the map $V(K^0) \to V(K)$. Since $K = \varinjlim A_i$ for smooth K^0-algebras A_i, by 1.1 (f) it suffices to show that

$$q_i^* : V(Spec\ K^0) \longrightarrow V(X_i)$$

is injective for every $q_i : X_i = Spec\ A_i \to K^0$. But every X_i has a K-rational point, hence a K^0-rational point $s_i : Spec\ K^0 \to X_i$ by lemma 1.3. Since $s_i^* q_i^* = (q_i s_i)^* = id$, q_i^* must be injective.

Corollary 1.4 *It K, K^0 and V are as in theorem* 1.2, *then for every scheme X of finite type over K^0 the restriction map*

$$V(X) \longrightarrow V(X_K)$$

is an isomorphism.

Indeed, the functor V_X with $V_X(Y) = V(X \times_{K^0} Y)$ is again a rigid functor on the category of all noetherian K^0-schemes.

Remark 1.4 *Let R be an excellent discrete valuation ring, let $\hat{R}$ be its completion, and denote by k and K the fraction fields of R and $\hat{R}$, respectively. Then by definition ([EGA IV](2),7.8.2), K is separable over k. Hence, if K^0 is the algebraic (= separable) closure of k in K, then the assumptions of theorem 1.1 are fulfilled for K and K^0. Note that K^0 is the fraction field of the Henselization $\tilde{R}$ of R ([EGA IV](4),18.9.3). In particular, if K is a global field and v is a place of K, then theorem 1.1 holds for the pair $(K_{(v)}, K_v)$, where K_v is the completion of K at v and $K_{(v)}$ is the algebraic closure of K in K_v. In fact, if v is non-archimedean, then the corresponding valuation ring is excellent ([EGA IV](2),7.8.3(ii),(iii). If K_s is a separable closure of K and w is a place of K_s extending v, then $K_{(v)}$ can also be replaced by the isomorphic decomposition field $K_s^{G_w}$, where $G_w = \{\sigma \in Gal(K_s/K) \mid \sigma w = w\}$ is the decomposition group of w, which is the classical Henselian field associated to K and w.*

In view of the applications we have in mind, we note that the full strength of the axioms in 1.1 was not needed in the proof of theorem 1.2. Consider the following weakening.

Definition 1.5 *Let $\mathcal{S} = Sch^{noeth}/k$ be the category of noetherian k-schemes, for a field k. A contravariant functor $V : \mathcal{S} \mapsto Ab$ is called sufficiently rigid, if it satisfies the following properties.*

(a) Call a morphism $\pi : X \to Y$ in $\mathcal{S}$ admissible, if it is a finite and flat morphism of smooth L-schemes, for a field extension L of k. Then for any admissible morphism $\pi : X \to Y$ there is a transfer map $\pi_* : V(X) \to V(Y)$, such that $(\varrho\pi)_* = \varrho_*\pi_*$ for another admissible morphism $\varrho : Y \to Z$.

(b) For every cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle\pi'}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y' & \xrightarrow{f} & Y \end{array}$$

lying in $\mathcal{S}$, with π and π' admissible, one has $f^*\pi_* = \pi'_* f'^* : V(X) \to V(Y')$.

(c) *Let $\pi : C \to D$ be a finite flat morphism of smooth curves over an extension L of k. For $y \in D$ and $x \in C$ with $\pi(x) = y$ consider the commutative (in general not cartesian!) diagram*

$$\begin{array}{ccc} Spec\ (\kappa(x)) & \xrightarrow{\varphi_x} & C \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi} \\ Spec\ (\kappa(y)) & \xrightarrow{\varphi_y} & D \end{array}$$

where φ_x, φ_y and π_x are the canonical morphisms, and for every smooth L-scheme X denote by

$$\begin{array}{ccc} X \times_L \kappa(x) & \xrightarrow{\varphi_x} & X \times_L C \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi} \\ X \times_L \kappa(y) & \xrightarrow{\varphi_y} & X \times_L D \end{array} \tag{1.6.1}$$

also the diagram obtained by base change with X over L. Then

$$\varphi_y^* \pi_* = \sum_{\pi(x)=y} e(x/y)(\pi_x)_*\ (\varphi_x)^* : V(X \times_L C) \to V(X \times_L \kappa(y)),$$

where $e(x/y)$ is the ramification index of x over y, and the same equality holds after base change with an open subscheme U of $X \times_L D$.

(d) *If X is a smooth L-scheme, for an extension L of k, then the projection $p : \mathbb{A}^1_X \to X$ induces an isomorphism $p^* : V(X) \xrightarrow{\sim} V(\mathbb{A}^1_X)$.*

(e) *Let $i \mapsto X_i$ be a filtered projective system of schemes in $\mathcal{S}$ such that the transition morphisms $X_i \to X_j$ are affine, and assume that $X = \varprojlim X_i$ is in $\mathcal{S}$. Then the canonical map*

$$\varinjlim V(X_i) \to V(X)$$

is an isomorphism.

Then we have

Theorem 1.6 *Let $K^0 \subset K$ be fields, and let $V : Sch^{noeth}/K^0 \to Ab$ be a sufficiently rigid functor. Then for every smooth K^0-scheme X the restriction map*

$$V(X) \to V(X_K)$$

is an isomorphism, provided we are in one of the following two situations:

(i) The fields K^0 and K are algebraically closed, and V has values in torsion abelian groups.

(ii) The fields K^0 and K are as in theorem 1.2, and V has values in torsion abelian groups whose order is prime to char (K).

Proof The proof of theorem 1.2 applies to the functor V_X with $V_X(Y) = V(X \times_{K^0} Y)$. In fact, by directly using 1.6 (c), applied to $X \times_{K^0} C \overset{id_X \times \pi}{\longrightarrow} X \times_{K^0} \mathbb{A}^1_L$, we need only consider transfer maps for admissible morphisms, and we only need the homotopy axiom 1.1 (e) for $\mathbb{A}^1_{X\times_{K^0}L} = X \times_{K^0} \mathbb{A}^1_L \to X \times_{K^0} L$, with L and $\pi : C \to \mathbb{A}^1_L$ as in the proof of theorem 1.2. Note that $X \times_{K^0} Y = (X \times_{K^0} L) \times_L Y$ is a smooth L-scheme for a smooth L-scheme Y, L an extension of K^0.

If $K^0 \subset K$ are algebraically closed, then Suslin's proof of [Su] Prop. 2.3 and Cor. 2.3.3 applies to V_X. In the notations of the proof of our theorem 1.2, we have $\overline{C}_K = \tilde{C}_K$ and $C_{\infty,K} = \tilde{C}_\infty$, and claim 3 is replaced by the fact that $Pic^0(\tilde{C}_K, \tilde{C}_\infty)/n$ is zero for every $\mathbb{N}$, since $\tilde{C}_\infty$ is reduced, so that $J_{\tilde{C}_\infty}(\tilde{C}_K)$ is a semi-abelian variety.

Of course, here we only needed 1.6 (c) for diagrams (1.6.1), and not for their localizations. This additional property will be needed in section 2. Therefore we record the following, not completely obvious fact.

Lemma 1.8 Let k be a field. A rigid functor $V : Sch^{noeth}/k \to Ab$ is also a sufficiently rigid functor.

Proof We only have to show that 1.6 (c) holds for V. With the notations of 1.6 (c) we have a commutative diagram

$$\begin{array}{ccccc}
\coprod\limits_{\pi(x)=y} X \times_L \kappa(x) & \xrightarrow{\amalg\alpha_x} & X \times_L C_y & \xrightarrow{\varphi'_y} & X \times_L C \\
& \searrow^{\amalg\pi_x} & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
& & X \times_L \kappa(y) & \xrightarrow{\varphi_y} & X \times_L D
\end{array}$$

such that the square is cartesian and $\varphi'_y \alpha_x = \varphi_x$. By assumption, the triangle can be rewritten and factored as

$$\begin{array}{ccc}
\coprod\limits_{\pi(x)=y} X \times_L \kappa(x) & \xrightarrow{(\alpha_x)} & \coprod\limits_{\pi(x)=y} X \times_L \left(\kappa(x)[T]/(T^{e(x/y)})\right) \\
& \searrow^{id} & \downarrow{\scriptstyle (\gamma_x)} \\
& & \coprod\limits_{\pi(x)=y} X \times_L \kappa(x) \\
& & \downarrow{\scriptstyle \amalg\pi_x} \\
& & X \times_L \kappa(y).
\end{array}$$

By forming the base change of the diagrams with an open subscheme U of $X \times_L D$, we obtain a commutative diagram

$$\begin{array}{ccccc}
\coprod\limits_{\pi(x)=y} Y_x & \xrightarrow{(\alpha_x)} & \coprod\limits_{\pi(x)=y} Y_x[T]/(T^{e(x/y)+1}) & \xrightarrow{\varphi'_y} & U' = U \times_D C \\
& \searrow^{id} & \downarrow{\scriptstyle (\gamma_x)} & & \\
& & \coprod\limits_{\pi(x)=y} Y_x & & \downarrow{\scriptstyle \pi} \\
& & \downarrow{\scriptstyle \amalg\pi_x} & & \\
& & Y & \xrightarrow{\varphi_y} & U
\end{array} ,$$

with cartesian square, and we want to show that

$$\varphi_y^* \pi_* = \sum_{\pi(\times)=y} e(x/y)(\pi_x)_* \varphi_x^* : V(U') \to V(Y),$$

where $\varphi_x = \varphi'_y \alpha_x$. Since $\varphi'^*_y \pi_* = (\amalg \pi_x)_*(\gamma_x)_*(\varphi'_y)^*$ by 1.1 (b), it suffices to show that

$$(\gamma_x)_* = e(x/y)\alpha_x^*,$$

by 1.1 (c). Since $\gamma_x \alpha_x = id_{Y_x}$, we have $\alpha_x^* \gamma_x^* = id$, and γ_x^* is injective. Hence it suffices to show the equality

$$\gamma_x^*(\gamma_x)_* = e(x/y)\gamma_x^* \alpha_x^* = e(x/y)id,$$

which holds by 1.1 (d).

2 Examples of rigid functors

The first two examples of rigid functors are given by étale cohomology and algebraic K-theory.

Proposition 2.1 *Let $\mathcal{S}$ be a category of quasi-compact schemes, and let $\mathcal{F}$ be an étale torsion sheaf on $\mathcal{S}$ whose torsion is invertible in $\mathcal{S}$. Assume that $\mathcal{F} \mid_X = f^*\mathcal{F} \mid_Y$ for every morphism $f : X \to Y$ in $\mathcal{S}$. (e.g., for any natural number n we may take $\mathcal{S} = Sch^{qc}/\mathbb{Z}[1/n]$, the category of quasi-compact schemes on which n is invertible, and $\mathcal{F} = \mathbb{Z}/n(j) = \mu_n^{\otimes j}$, the j-th Tate twist of the constant sheaf $\mathbb{Z}/n$, cf. [Mi], p. 163). Then for any integer $i \geq 0$ the functor*

$$V(Y) = H^i(Y, \mathcal{F})$$

given by the i-th étale cohomology with coefficients in $\mathcal{F}$ is a rigid functor on $\mathcal{S}$.

Proof This is well-known: The contravariance of V is induced by the adjunction maps $ad_f : \mathcal{F} \to f_* f^* \mathcal{F}$ for morphisms $f : X \to Y$ (cf. [Mi] III 1.6(c)), viz., f^* is the composition

$$f^* : H^i(Y, \mathcal{F}) \xrightarrow{ad_f} H^i(Y, f_* f^* \mathcal{F}) \xrightarrow{can} H^i(X, f^*\mathcal{F}) = H^i(X, \mathcal{F}) \ .$$

For the limit property 1.1(f) cf. [Mi] III 1.16, and for the homotopy property 1.1 (e), which is related to smooth base change, cf. [Mi] VI 4.15. The transfer π_* for a finite flat morphism $\pi : X \to Y$ is defined as follows. By [SGA 4] XVII 6.2.3 there is a canonical trace morphism

$$Tr_\pi \ : \ \pi_* \pi^* \mathcal{F} \ \longrightarrow \ \mathcal{F}$$

for every abelian sheaf $\mathcal{F}$ on Y (note that $\pi_* = \pi_!$ for a finite morphism). Then π_* is defined as the composition

$$\pi_* : H^i(X, \pi^*\mathcal{F}) \underset{\sim}{\xleftarrow{can}} H^i(Y, \pi_* \pi^* \mathcal{F}) \xrightarrow{Tr_\pi} H^i(Y, \mathcal{F})$$

in which the first map is an isomorphism since π is finite (so that $\mathcal{G} \mapsto \pi_* \mathcal{G}$ is exact, cf. [Mi] II 3.6). The functoriality $(\rho\pi)_* = \rho_* \pi_*$ in 1.1 (a) then follows from the transitivity of the trace morphism ([SGA 4] XVII 6.2.3 ($Var3$)). For property 1.1 (d) we first note that the composition

$$\mathcal{F} \xrightarrow{ad_\pi} \pi_*\pi^*\mathcal{F} \xrightarrow{Tr_\pi} \mathcal{F}$$

is multiplication by d for *every* finite flat morphism $\pi : X \to Y$ of (constant) rank d (loc.cit. (Var 4)). Hence $\pi_*\pi^*$ is multiplication by d for such π. Now let $\pi : X_d = X[T]/(T^{d+1}) \to X$ be as in 1.1 (d), and let $s : X \to X_d$ be the obvious “zero” section ($T \mapsto 0$) of π. Then s^* is an isomorphism by the topological invariance of étale cohomology ([SGA 4] VIII 1.2), hence π^* is an isomorphism as well. Thus the equality $\pi^*\pi_*\pi^* = d\pi^*$ implies that $\pi^*\pi_* = d$ as wanted. If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a cartesian diagram, with π (and hence π') finite and flat, then the base change property $f^*\pi_* = \pi'_*f'^*$ of 1.1 (b) follows from the commutativity of the diagram

$$\begin{array}{ccccc} \pi_*f'_*f'^*\pi^*\mathcal{G} & \xleftarrow{\pi_* ad_{f'}} & \pi_*\pi^*\mathcal{G} & \xrightarrow{Tr_\pi} & \mathcal{G} \\ \| & (1) & \downarrow{\scriptstyle ad_f} & (2) & \downarrow{\scriptstyle ad_f} \\ f_*\pi'_*f'^*\pi^*\mathcal{G} & \xleftarrow{f_*\Phi} & f_*f^*\pi_*\pi^*\mathcal{G} & \xrightarrow{f_*f^*Tr_\pi} & f_*f^*\mathcal{G} \\ \| & & (3) & & \| \\ f_*\pi'_*\pi'^*f^*\mathcal{G} & & \xrightarrow{\quad f_*Tr_{\pi'} \quad} & & f_*f^*\mathcal{G} \end{array}$$

for any torsion sheaf $\mathcal{G}$ on Y, where $\Phi : f^*\pi_* \to \pi'_*f'^*$ is the base change morphism. Here (3) is commutative by loc. cit. (Var.2), (2) commutes by the functoriality in sheaves of ad_f, and (1) commutes by the definition of the base change morphism (cf., e.g, [Mi] p. 223). Note that we get an induced commutative diagram

$$\begin{array}{ccccccc} H^i(Y, \pi_*\pi^*\mathcal{G}) & \longrightarrow & H^i(Y, \pi_*f'_*f'^*\pi^*\mathcal{G}) & = & H^i(Y, f^*\pi'_*\pi'^*f^*\mathcal{G}) & \longrightarrow & H^i(Y, f_*f^*\mathcal{G}) \\ {\scriptstyle can}\downarrow\wr & & {\scriptstyle can}\downarrow\wr & & {\scriptstyle can}\downarrow\wr & & {\scriptstyle can}\downarrow\wr \\ H^i(X, \pi^*\mathcal{G}) & \xrightarrow{ad_{f'}} & H^i(X, f'_*f'^*\pi^*\mathcal{G}) & & H^i(Y', \pi'_*\pi'_*\pi'^*f^*\mathcal{G}) & \xrightarrow{Tr_{\pi'}} & H^i(Y', f^*\mathcal{G}) \\ & \searrow^{f'^*} & {\scriptstyle can}\downarrow & & {\scriptstyle can}\downarrow\wr & \nearrow_{\pi'_*} & \\ & & H^i(X', f'^*\pi^*\mathcal{G}) & = & H^i(X', \pi'^*f^*\mathcal{G}) & & \end{array}$$

Finally, property 1.1 (c) is a straightforward consequence of [SGA 4] 6.2.3.1: it implies that for $X = X_1 \amalg X_2$ with open and closed immersions $\alpha_i : X_i \hookrightarrow X$ ($i = 1, 2$) one has $(\alpha_i)^*(\alpha_j)_* = \delta_{ij} id_{X_i}$.

Proposition 2.2 *For every integer $m \geq 0$, the functor*

$$V(X) = K_m(X)$$

given by the m-th algebraic K-group is a rigid functor on the category $\mathcal{S}$ of all noetherian schemes, except that the homotopy axiom 1.1 (e) possibly only holds for a regular base X.

Proof This follows from the results of Quillen in [Qui]: By [Qui] §7,2, $X \mapsto K_m(X)$ is a contravariant functor, and the limitproperty 1.1(f) is proved in loc. cit. §7, 2.2. The transfer map for a finite flat morphism $\pi : X \to Y$ is defined as follows. Recall that $K_m(X) = K_m(P(X))$, the m-th K-group of the exact category $P(X)$ of locally free coherent $\mathcal{O}_X$-modules. Then $\pi_* : K_m(X) \to K_m(Y)$ is induced by the exact functor $\pi_* : P(X) \to P(Y)$ sending an $\mathcal{O}_X$-module $\mathcal{P}$ in $P(X)$ to the $\mathcal{O}_Y$-module $\pi_*\mathcal{P}$ in $P(Y)$ (cf. loc. cit. §7). The functoriality $(\rho\pi)_* = \rho_*\pi_*$ is immediate. For 1.1 (b) recall that the pull-back $f^* : K_m(Y) \to K_m(Y')$ for a morphism $f : Y' \to Y$ is induced by the exact functor $f^* : P(Y) \to P(Y')$ sending $\mathcal{Q}$ in $P(Y)$ to $f^*\mathcal{Q} = \mathcal{O}_{Y'} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{Q}$ (coherent pull-back) in $P(Y)$. If now

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y \end{array}$$

is a cartesian diagram, with π finite and flat, then the base change morphism $f^*\pi_* \to \pi'_* f'^*$ is an isomorphism of exact functors from $P(X)$ to $P(Y')$; therefore $f^*\pi_* = \pi'_* f'^*$ on the level of K-groups ([Qui] §1 Prop. 2). The additivity property 1.1 (c) follows from [Qui] §1, (8) and the fact that for $X = X_1 \amalg X_2$ the immersions $\alpha_i : X_i \hookrightarrow X$ induce an equivalence of exact categories $(\alpha_1^*, \alpha_2^*) : P(X) \to P(X_1) \times P(X_2)$ with $(\alpha_i)^*(\alpha_j)_* \xrightarrow{\sim} \delta_{ij} id : P(X_j) \to P(X_i)$. Property 1.1 (d) follows from the more general fact that for any flat finite morphism $\pi : X \to Y$ for which $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y^d$ as an $\mathcal{O}_Y$-module one has a functorial isomorphism

$$\pi_*\pi^* P \xrightarrow{\sim} P \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X \xrightarrow{\sim} P^d$$

for P in $P(Y)$. This implies that $\pi_*\pi^*$ is the multiplication by d on $K_m(Y)$ by [Qui] §3 Corollary 1. Finally, the homotopy property 1.1(e) for $V(X) = K_m(X)$ is proved in [Qui] §7, Proposition 4.1, for a regular base X.

We now turn to K- and $\mathcal{H}$-cohomologies. Let $\mathcal{S}$ be a category of schemes such that for a morphism $f : X \to Y$ in $\mathcal{S}$ and an open immersion $j : U \hookrightarrow Y$ all morphisms of the cartesian diagram

(2.3.1)
$$\begin{array}{ccc} U' & \overset{j'}{\hookrightarrow} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ U & \overset{j}{\hookrightarrow} & Y \end{array}$$

lie in $\mathcal{S}$. Thus $\mathcal{S}$ becomes a site, if we endow it with the Zariski topology. Let V be a contravariant functor on $\mathcal{S}$ with values in the category Ab of abelian groups, i.e., a presheaf (of abelian groups) on $\mathcal{S}$, and let $\mathcal{V}$ be the associated Zariski sheaf on $\mathcal{S}$. If V_X and $\mathcal{V}_X$ denote the restrictions of V and $\mathcal{V}$ to the small Zariski site X_{Zar} for a scheme X in $\mathcal{S}$ (which consists of all open immersions $(U \hookrightarrow X)$, then $\mathcal{V}_X = a\ V_X$, where $a = a_X$ is the functor mapping a presheaf on X_{Zar} to its associated sheaf (on X_{Zar}).

For a scheme X in $\mathcal{S}$, let $H^i(X, \mathcal{V})$ be the i-th Zariski cohomology of $\mathcal{V}$ on X. This is equivalently, the i-th derived functor of the functor $\mathcal{F} \mapsto H^0(X, \mathcal{F}) = \mathcal{F}(X)$, from sheaves on $\mathcal{S}$ to Ab, evaluated at $\mathcal{V}$, and also equal to $H^i(X, \mathcal{V}_X)$, the i-th cohomology of $\mathcal{V}_X$ on X_{Zar} (cf. [Mi] III 1.5 (b), 1.10, and 3.1 (c)). It is clear from the first description that

$X \rightsquigarrow H^i(X,\mathcal{V})$ is a contravariant functor from $\mathcal{S}$ to Ab. In the second description, this functoriality can be described as follows. Since $\mathcal{V}$ is a sheaf on $\mathcal{S}$, one gets a canonical morphism

$$\alpha_f : \mathcal{V}_Y \to f_*\mathcal{V}_*$$

for every morphism $f : X \to Y$ in $\mathcal{S}$, defined via the maps

$$\mathcal{V}_Y(U) = \mathcal{V}(U) \xrightarrow{(f')^*} \mathcal{V}(U') = \mathcal{V}_X(U') = f_*\mathcal{V}_*(U)$$

for each diagram (2.3.1). Then the wanted pull-back is the composition

$$H^i(Y,\mathcal{V}_Y) \xrightarrow{\alpha_f} H^i(Y, f_*\mathcal{V}_*) \xrightarrow{can} H^i(X,\mathcal{V}_X). \tag{2.3.2}$$

Theorem 2.3 *Let $\mathcal{S} = Sch^{noeth}/k$ be the category of noetherian k-schemes, for a field k, and let V be a sufficiently rigid functor on $\mathcal{S}$. Assume that the following properties hold for rings A which are localizations of (the coordinate rings of) smooth affine L-schemes Y, where L is an extension field of k.*

(i) If A is semi-local, then the natural map $V(Spec\ A) \to H^0(Spec\ A,\mathcal{V})$ is an isomorphism, and $H^i(Spec\ A,\mathcal{V}) = 0$ for $i > 0$.

(ii) If A is local, then the restriction map $H^0(Spec\ A,\mathcal{V}) \to H(SpecA[T],\mathcal{V})$ is an isomorphism, and $H^i(Spec\ A,[T],\mathcal{V}) = 0$ for $i > 0$.

Then for every $\nu \geq 0$ the functor

$$Y \quad \rightsquigarrow \quad H^\nu(Y,\mathcal{V})$$

is a sufficiently rigid functor on $\mathcal{S}$.

Proof For a scheme Y denote by $P(X_{Zar})$ and $S(X_{Zar})$ the category of abelian presheaves and sheaves on the small Zariski site X_{Zar}, respectively, let $i : S(X_{Zar}) \to P(X_{Zar})$ be the inclusion, and let $a = a_x : P(X_{Zar}) \to S(X_{Zar})$ be its left adjoint, mapping a presheaf P to its associated sheaf. For a morphism $f : X \to Y$ let $f_P : P(X_{Zar}) \to P(Y_{Zar})$ be the direct image functor (defined by $(f_P P)(U) = P(f^{-1}(U))$), and let $f^P : P(Y_{Zar}) \to P(X_{Zar})$ be its left adjoint.

We first show the limit property 1.6 (e) for the functors $H^\nu(-,\mathcal{V})$. Let $(X_i)_{i\in I}$ be a filtered projective system of schemes in $\mathcal{S}$ with affine transition maps. Assume that $X = \varprojlim X_i$. is in $\mathcal{S}$, and let $u_i : X \to X_i$ be the canonical morphism. Then the morphism

$$\varinjlim_i \ u_i^* \ \mathcal{V}_{X_i} \longrightarrow \mathcal{V}_X$$

is an isomorphism. In fact, the stalk at a point $x \in X$ with images x_i in X_i is the map

$$\varinjlim_i V(\mathcal{O}_{X_i,x_i}) \to V(\mathcal{O}_{X,x}) \ ,$$

which is an isomorphism by 1.6 (e) for V, since $\mathcal{O}_{X,x} = \varinjlim \mathcal{O}_{X_i,x_i}$.

It now follows that the map

$$\varinjlim H^\nu(X_i, \mathcal{V}_{X_i}) \to H^\nu(X, \mathcal{V}_X)$$

is an isomorphism since X_{Zar} is the limit of the sites $(X_i)_{Zar}$ (cf. [SGA 4] VII 5.7 for the case of étale sites; the case of Zariski sites is much simpler and follows from [EGA IV] 3, 8.6.3 and 8.10.5 (vi), cf. the proof of [SGA 4] VII 5.6).

Now let $\pi : X \to Y$ be an admissible finite and flat morphism in $\mathcal{S}$. The transfer morphisms for π and all base changes with open immersions $U \hookrightarrow Y$ define a morphism

$$\pi_P V_X \to V_Y \quad .$$

We get an induced diagram

$$\begin{array}{ccccc} & & a\ \pi_P V_X & \longrightarrow & a\ V_Y = \mathcal{V}_Y \\ & & \downarrow \phi & & \\ \pi_* \mathcal{V}_X & = & \pi_* a\ V_X\ , & & \end{array}$$

where the morphism ϕ is the canonical map, which for instance arises from the adjunction map $id \to i\ a$ and the equality $\pi_* = a\ \pi_P i$. We claim that ϕ is an isomorphism, Let $y \in Y$. Then the stalk of ϕ at y is the canonical map

$$\begin{array}{ccc} \varinjlim\ V(U \times_Y X) & \longrightarrow & \varinjlim\ H^0(U \times_Y X, \mathcal{V}) \\ \wr\downarrow & & \downarrow\wr \\ V(Spec\ \mathcal{O}_{Y,y} \times_Y X) & \xrightarrow{\phi_y} & H^0(Spec\ \mathcal{O}_{Y,y} \times_Y X, \mathcal{V}) \end{array} \quad ,$$

where U runs through all open neighborhoods of y in Y. The vertical maps are isomorphisms since 1.6 (e) holds for V and $H^0(-, \mathcal{V})$. Since π is finite, $Spec\ \mathcal{O}_{Y,y} \times_Y X = Spec\ A$ for a semi-local ring A, which is a localization of X. Since π is admissible, X is smooth over an extension L of k. Hence ϕ_y is an isomorphism by our assumption (i) on V. As a consequence, we have a canonical morphism

$$Tr_\pi : \pi_*\ \mathcal{V}_X \to \mathcal{V}_Y \quad .$$

Next we claim that $R^i\pi_*\ \mathcal{V}_X = 0$ for $i > 0$. In fact, the stalk at $y \in Y$ of $R^i\pi_*\ \mathcal{V}_X$ is

$$\varinjlim\ H^i(U \times_Y X, \mathcal{V}),$$

where the limit is over all open neighborhoods of y in Y. This coincides with $H^i(Spec\ \mathcal{O}_{Y,y} \times_Y X, \mathcal{V})$ by the limit property 1.6 (e) for $H^i(-, \mathcal{V})$, and this is zero for $i > 0$ by the assumption (i) on V and the same arguments as before.

As a consequence, the canonical map $H^\nu(Y, \pi_*\mathcal{V}_X) \overset{can}{\to} H^\nu(X, \mathcal{V}_X)$ is an isomorphism for all $\nu \geq 0$, and we define the transfer maps for π as the compositions

$$\pi_* : H^\nu(X, \mathcal{V}_X) \overset{\sim}{\longleftarrow} H^\nu(Y, \pi_*\mathcal{V}_X) \xrightarrow{Tr_\pi} H^\nu(Y, \mathcal{V}_Y) \quad ,$$

for all $\nu \geq 0$. That $(\varrho\pi)_* = \varrho_*\pi_*$, for an admissable morphism $\varrho : Y \to Z$, is a straightforward consequence of the fact that $Tr_{\varrho\pi}$ coincides with the composition

$$\varrho_*\pi_*\mathcal{V}_X \xrightarrow{\varrho_* Tr_\pi} \varrho_*\mathcal{V}_Y \xrightarrow{Tr_\varrho} \mathcal{V}_Z \quad ,$$

which easily follows from the definitions and the observation that $a\varrho_P\pi_P V_X \to \varrho_* a\pi_P V_X$ is an isomorphism.

Next we show 1.6 (b). Let

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{f} & Y \end{array}$$

be a cartesian diagram in $\mathcal{S}$, with π and π' admissible. Then the diagram

$$\text{(2.3.3)} \qquad \begin{array}{ccccc} \pi_*\,\mathcal{V}_X & \xrightarrow{\pi_*\alpha_{f'}} & \pi_* f'_*\mathcal{V}_{X'} & = & f_*\pi'_{\forall}\mathcal{V}_{X'} \\ {\scriptstyle Tr_\pi}\downarrow & & & & \downarrow{\scriptstyle f_* Tr_{\pi'}} \\ \mathcal{V}_Y & & \xrightarrow{\qquad\alpha_f\qquad} & & f_*\mathcal{V}_{Y'} \end{array}$$

is commutative. In fact, by passing to the stalks at $y \in Y$ we get the diagram

$$\begin{array}{ccc} V(Spec\ \mathcal{O}_{Y,y} \times_Y X) & \xrightarrow{f'^*} & V(Spec\ \mathcal{O}_{Y,y} \times_Y X') \\ {\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi'_*} \\ V(Spec\ \mathcal{O}_{Y,y}) & \xrightarrow{f^*} & V(Spec\ \mathcal{O}_{Y,y} \times_Y Y') \end{array}$$

which is commutative by 1.6 (b) for V. The morphism $\pi : Spec\ \mathcal{O}_{Y,y} \times_Y X \to Spec\ \mathcal{O}_{Y,y}$ is only the localization of an admissible morphism, but the map π_* can be defined via passing to the limit over the maps $\pi_* : V(U \times_Y X) \to V(U)$ for U running through the open neighborhoods of y in Y. The analogous statement holds for π'_* and the equality $f^*\pi_* = \pi'_* f'^*$ then follows from the corresponding equalities for the U.

The equality $f^*\pi_* = \pi'_* f'^*$ for the functors $H^\nu(-, \mathcal{V})$ now follows from the commutative diagram

$$\begin{array}{ccccccc} H^\nu(X, \mathcal{V}_X) & \xrightarrow{\alpha_{f'}} & H^\nu(X, f'_*\mathcal{V}_{X'}) & & \xrightarrow{\qquad can \qquad} & & H^\nu(X', \mathcal{V}_{X'}) \\ {\scriptstyle \wr}\uparrow{\scriptstyle can} & & \uparrow{\scriptstyle can} & & & & {\scriptstyle \wr}\uparrow{\scriptstyle can} \\ H^\nu(Y, \pi_*\mathcal{V}_X) & \xrightarrow{\pi_*\alpha_{f'}} & H^\nu(Y, \pi_* f'_*\mathcal{V}_{X'}) & = & H^\nu(Y, f_*\pi'_*\mathcal{V}_{X'}) & \xrightarrow{can} & H^\nu(Y', \pi'_*\mathcal{V}_{X'}) \\ \downarrow{\scriptstyle Tr_\pi} & & \text{(2.3.3)} & & \downarrow{\scriptstyle f_* Tr_\pi} & & \downarrow{\scriptstyle Tr_{\pi'}} \\ H^\nu(Y, \mathcal{V}_Y) & & \xrightarrow{\qquad \alpha_f \qquad} & & H^\nu(Y, f_*\mathcal{V}_{Y'}) & \xrightarrow{can} & H^\nu(Y', \mathcal{V}_{Y'}). \end{array}$$

The proof of 1.6 (c) for the $H^\nu(-, \mathcal{V})$ is similar. By similar arguments as above, it suffices to show that the diagram

$$\begin{array}{ccccc} \pi_*\mathcal{V}_{X\times C} & \xrightarrow{\bigoplus \pi_*\alpha_{\varphi_x}} & \bigoplus\limits_{\pi(x)=y} \pi_*(\varphi_x)_*\ \mathcal{V}_{X\times\kappa(x)} & = & \bigoplus\limits_{\pi(x)=y} (\varphi_y)_*)(\pi_x)_*\ \mathcal{V}_{X\times\kappa(x)} \\ {\scriptstyle Tr_\pi}\downarrow & & & & \downarrow{\scriptstyle \sum e(x/y)(\varphi_y)_* Tr_{\pi_*}} \\ \mathcal{V}_{X\times D} & & \xrightarrow{\qquad \alpha_{\varphi_y} \qquad} & & (\varphi_y)_*\ \mathcal{V}_{X\times\kappa(y)} \end{array}$$

commutes, where the notations are as in 1.6 (c). But by taking the stalks at $t \in X \times \kappa(y)$ we obtain the diagram

$$\begin{array}{ccc} V(Spec\ (R) \times_D C) & \xrightarrow{\oplus \varphi_x^*} & \bigoplus_{\pi(y)=x} V\ (Spec\ (R) \times_D \kappa(x)) \\ \downarrow{\pi_*} & & \downarrow{\Sigma\ e(x/y)(\pi_x)_*} \\ V(Spec\ R) & \xrightarrow{\varphi_y^*} & V\ (Spec\ (R) \times_D \kappa(x)) \end{array}$$

with $R = \mathcal{O}_{X \times D,t}$, which is commutative by 1.6 (c) for V (for π_* and the $(\pi_x)_*$ the same remarks as before apply).

Finally, we prove the homotopy property 1.6 (d) for the functors $H^\nu(-, \mathcal{V})$. Let Y be smooth over an extension L of k, and let $p : \mathbb{A}^1_Y \to Y$ be the affine line over Y. Then

$$\alpha_p : \mathcal{V}_Y \to p_*\mathcal{V}_{\mathbb{A}^1_Y}$$

is an isomorphism, and $R^i p_* \mathcal{V}_{\mathbb{A}^1_Y} = 0$ for $i > 0$, by assumption (ii) on V. In fact, for $y \in Y$, the stalk of α_f at y is the pull-back map

$$H^0(Spec\ \mathcal{O}_{Y,y}, \mathcal{V}) \longrightarrow H^0(\mathbb{A}^1_{Spec\ \mathcal{O}_{Y,y}}, \mathcal{V}) \quad ,$$

and the stalk $R^i p_* \mathcal{V})_y$ is isomorphic to $H^i(\mathbb{A}^1_{Spec\ \mathcal{O}_{Y,y}}, \mathcal{V})$ for all $i \geqslant 0$. As a consequence, we obtain the bijectivity of the maps in the composition

$$p^* : H^\nu(Y, \mathcal{V}_Y) \underset{\sim}{\overset{\alpha_p}{\to}} H^\nu(Y, p_*\mathcal{V}_{\mathbb{A}^1_Y}) \underset{\sim}{\overset{can}{\to}} H^\nu(\mathbb{A}^1_Y, \mathcal{V}_{\mathbb{A}^1_Y}),$$

for every $\nu \geq 0$. q.e.d.

Remark 2.4 *(a) It seems unlikely that one can define natural transfer maps $\pi_* : H^\nu(X, \mathcal{V}) \to H^\nu(Y, \mathcal{V})$ for arbitrary finite and flat maps $\pi : X \to Y$, even if one has transfer maps $\pi_* : V(X) \to V(Y)$ for all such π.*
(b) Fix an extension L of k, and let $p : \mathbb{A}^1_Y \to Y$ be the affine line over a smooth L-scheme Y. Given the limit property 1.6 (e) for V (and hence for the $H^\nu(-, \mathcal{V})$) the following statements are equivalent.

(1) $p^* : H^\nu(U, \mathcal{V}) \to H^\nu(\mathbb{A}^1_U, \mathcal{V})$ is an isomorphism for all $\nu \geqslant 0$ and all open subschemes U of Y.

(2) The morphism $\mathcal{V}_Y \to RR_{p_*}\mathcal{V}_{\mathbb{A}^1_Y}$ is a quasi-isomorphism.

(3) Property 2.3 (ii) holds for all local rings $\mathcal{O}_{Y,y}$ of Y.

Theorem 2.5 *Let k be any field. The functor*

$$Y \mapsto H^\nu(Y, \mathcal{K}_m) \quad ,$$

where K_m is the Zariski sheaf associated to the presheaf $U \mapsto K_m(U)$ given by the m-th algebraic K-groups, is a sufficiently rigid functor on the category of all noetherian k-schemes.

Proof We have to show the properties 2.3 (i) and (ii). The first property is a direct consequence of results of Quillen. In fact, let Y be a noetherian scheme. In [Qui] §7,5.4 Quillen constructed a spectral sequence

$$E_1^{p,q}(Y) = \bigoplus_{x \in Y^{(p)}} K_{-p-q}(\kappa(x)) \Rightarrow K'_{-p-q}(Y) \quad ,$$

which is contravariant for flat morphisms. Here $K'_m(Y)$ is the m-th K-group of the category $M(Y)$ of coherent $\mathcal{O}_Y$-modules, but for a regular noetherian scheme Y, this group is canonically isomorphic to $K_m(Y)$ ([Qui] §7, (1.1)), functorially for flat pull-backs (both functorialities are induced by the exact functor mapping an $\mathcal{O}_Y$-module $\mathcal{M}$ to its coherent pull-back $f^*\mathcal{M} = \mathcal{O}_{Y'} \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{M}$ for a flat morphism $f : Y' \to Y$). If Y is smooth over a field L, or any localization of such a scheme, then Quillen constructed a canonical isomorphism

$$E_2^{p,q}(Y) = H^p(Y, \mathcal{K}_{-q}) \quad (p, q \in \mathbb{Z}), \tag{2.5.1}$$

compatible with flat pull-backs, by showing that for any ring A obtained by localizing (an open affine subscheme of) Y in finitely many points, the edge morphism

$$K_{-m}(A) \longrightarrow E_2^{0,m}(Spec\ A)$$

is an isomorphism for all $m \in \mathbb{Z}$ and

$$E_2^{p,q}(Spec\ A) = 0 \quad \text{for } p \neq 0$$

(loc. it. 5.6, 5.8 and 5.10). This proves 2.3 (i) for such A (by definition, the composition $K_{-m}(Spec\ A) \to E_2^{o,m}(Spec\ A) = H^0(Spec A, \mathcal{K}_{-m})$ is the canonical map).

In view of (2.5.1) and remark 2.4 (b), property 2.3 (ii) follows from results of Gillet. For it follows from [Gi] Thm. 8.3 that for the affine line $p : \mathbb{A}^1_Y \to Y$ over a smooth L-scheme Y, the pull-back

$$p^* : E_2^{p,q}(Y) \longrightarrow E_2^{p,q}(\mathbb{A}^1_Y)$$

is an isomorphism for all $p, q \in \mathbb{Z}$ (without restriction, Y is equidimensional of dimension m; then the above map is the map $p^* : CH_{m-p,\, m+g}(Y) \to CH_{m+1-p,\, m+1+g}(\mathbb{A}^1_Y)$ of loc. cit.). q.e.d.

The following observation will be useful in the next section.

Lemma 2.6 *Let V be a sufficiently rigid functor on the category Sch^{noeth}/k, where k is a perfect field. Then properties 2.3 (i) and (ii) hold if and only if they hold for rings A which are localizations of smooth k-schemes.*

Proof (cf. [Qui] §7, Proof of thm. 5.11) Let L be an extension of $k, Y = Spec\ R$ a smooth affine L-scheme, and A a semi-local ring obtained by localizing R in a finite set of primes S. Then there exists a subfield L' of L finitely generated over k, a smooth L'-Algebra R' and a finite subset S' of $Spec\ R'$ such that $R = L \otimes_{L'} R'$ and such that the primes in S are the base extensions of the primes in S'. This follows by applying [EGA IV] 8.8.2, 8.10.5, 8.7.3 and 17.7.8 to the family $L_i \quad (i \in I)$ of subfields of L which are finitely generated over k. Let $L_j \quad (j \in J)$ be the subfamily of those fields which contain

L', and for each $j \in J$ let S_j be the set of primes in $R_j = L_j \otimes_{L'} R'$ obtained by tensoring the primes in S' with L_j, and let A_j be the localization of R_j in S_j. Then $A = \varinjlim A_j$, so by the limit property 1.6 (e) for V and the $H^\nu(-, \mathcal{V})$, we see that it suffices to show 2.3 (i) and (ii) for the localization A of a smooth L-scheme Y, where L is finitely generated over k. But since k is perfect, every such L is the function field of a smooth k-scheme; hence A is the localization of a smooth k-scheme, and the claim follows.

3 Poincaré duality theories and $\mathcal{H}$-cohomology

To show the properties 2.3 (ii) and (iii) for $\mathcal{H}$-cohomology, where $\mathcal{H}$ is the Zariski sheaf associated to étale cohomology, it is most convenient to use the notion of a twisted Poincaré duality theory (with supports) as introduced by Bloch and Ogus [BO] (1.3). It encodes the usual properties of a cohomology theory with supports for algebraic schemes over a field k, with an associated (Borel-Moore type) homology theory. We recall the axioms, since we need to consider them more closely.

Definition 3.1 *Let k be a field, and let $\mathcal{S}$ be a category of algebraic k-schemes such that $Y \in Ob(\mathcal{S})$ if $X \in Ob(\mathcal{S})$ and $Y \subset X$ is locally closed.*

(1) Let $\mathcal{S}^*$ be the category whose objects are the closed immersions $Y \subset X$ in $\mathcal{S}$ and whose morphisms are cartesian squares

$$\begin{array}{ccc} Y' & \subset & X' \\ \downarrow & & \downarrow \\ Y & \subset & X \end{array} .$$

A twisted cohomology theory (with supports) is a sequence (indexed by $j \in \mathbb{Z}$) of contravariant functors

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \text{(graded abelian groups)} \\ (Y \subset X) & \mapsto & \oplus_i H^i_Y(X, j) \end{array}$$

satisfying the following properties

(a) (long exact cohomology sequence) For $Z \subset Y \subset X$, there is a long exact sequence

$$\ldots \to H^i_Z(X, j) \to H^i_Y(X, j) \to H^i_{Y-Z}(X \smallsetminus Z, j) \to H^{i+1}_Z(X, j) \to \ldots,$$

functorial with respect to morphisms

$$\begin{array}{ccccc} Z' & \subset & Y' & \subset & X' \\ \downarrow & & \downarrow & & \downarrow \\ Z & \subset & Y & \subset & X \end{array}$$

in the obvious way.

(b) (excision) If $Z \subset X \in$ $Ob\ \mathcal{V}^*$ and if $U \subset X$ is open in X and contains Z, then the map $H^i_Z(X, j) \to H^i_Z(U, j)$ is an isomorphism.

(2) Let $\mathcal{S}_*$ be the category with $Ob(\mathcal{S}_*) = Ob\mathcal{S}$ but whose arrows are only the proper morphisms in $\mathcal{S}$. A twisted homology theory is a sequence (indexed by $b \in \mathbb{Z}$ of covariant functors

$$\begin{array}{ccc} \varphi_* & \to & \text{(graded abelian groups)} \\ X & \rightsquigarrow & \bigoplus_a H_a(X,b) \end{array}$$

such that the following properties hold.

(c) If $\alpha : X' \to X$ is étale, there is a map

$$\alpha^* : H_a(X,b) \to H_a(X',b) \quad ,$$

such that $(\alpha\ \alpha')^* = \alpha'^*\alpha^*$ for $\alpha' : X'' \to X'$ étale.

(d) If the diagram below on the left is cartesian, with proper f and g, and étale α and β, then the diagram on the right commutes.

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & Y \end{array} \qquad\qquad \begin{array}{ccc} H_a(X,b) & \xrightarrow{\beta^*} & H_a(X',b) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ H_a(Y,b) & \xrightarrow{\alpha^*} & H_a(Y',b) \end{array} \quad .$$

(e) If $i : Z \hookrightarrow X$ is a closed immersion in $\mathcal{V}$, with open complement $\alpha : V \hookrightarrow X$, then there are long exact sequences

$$\ldots \to H_a(Z,b) \stackrel{i_*}{\to} H_a(X,b) \stackrel{\alpha^*}{\to} H_a(U,b) \to H_{a-1}(Z,b) \to \ldots \quad .$$

which satisfy the following compatibilities (NB only the commutativity of the squares (1) and (2) below is a new statement):

(f) If $f : X' \to X$ is a proper morphism, restricting to $f' : Z' \to Z$ for a closed subscheme $Z' \subseteq X$ then the diagram

$$\begin{array}{ccccccccc} \ldots \to & H_a(Z',b) & \to & H_a(X',b) & \to & H_a(U',b) & \to & H_{a-1}(Z',b) & \to \ldots \\ & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*\alpha^*} & (1) & \downarrow{\scriptstyle f'_*} & \\ \ldots \to & H_a(Z,b) & \to & H_a(X,b) & \to & H_a(U,b) & \to & H_{a-1}(Z,b) & \to \ldots \end{array}$$

commutes, where $\alpha : f^{-1}(U) = X' - f^{-1}(Z) \to X' - Z' = U'$ is the open immersion.

(g) If $\alpha : X' \to X$ is étale, then the diagram

$$\begin{array}{ccccccccc} \ldots \to & H_a(Z',b) & \to & H_a(X',b) & \to & H_a(U',b) & \to & H_{a-1}(Z',b) & \to \ldots \\ & \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \alpha^*} & & \uparrow{\scriptstyle \alpha^*} & (2) & \uparrow{\scriptstyle \alpha^*} & \\ \ldots \to & H_a(Z,b) & \to & H_a(X,b) & \to & H_a(U,b) & \to & H_{a-1}(Z,b) & \to \ldots \end{array} \quad ,$$

commutes, where $Z' = f^{-1}(Z)$ and $U' = f^{-1}(U)$.

(3) A Poincaré duality theory is given by a cohomology and homology theory as above, together with the following structures.

(h) (cap product) For $Y \subset X \in$ Ob φ^* there is a pairing

$$H_a(X,b) \otimes H^i_Y(X,j) \to H_{a-i}(Y,b-j) \ ,$$

compatible with étale pull-backs, in the obvious way.

(i) (projection formula) For a cartesian diagram below on the left, with proper f, the diagram on the right commutes.

$$\begin{array}{ccc} Y' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ Y & \hookrightarrow & X \end{array} \qquad \begin{array}{ccccc} H_a(X',b) & \otimes & H^i_{Y'}(X',j) & \longrightarrow & H_{a-i}(Y',b-j) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_a(X,b) & \otimes & H^i_Y(X,j) & \longrightarrow & H_{a-i}(Y,b-j) \end{array} \ .$$

(j) (fundamental class) If $X \in$ Ob φ is irreducible and of dimension d, then there is a canonical element $\eta_X \in H_{2d}(X,d)$. If $\alpha : X' \to X$ is étale, then $\alpha^*\eta_X = \eta'_X$.

(k) (Poincaré duality) If $X \in$ Ob φ is smooth of pure dimension d, and $Y \subset X$ is a closed subscheme, then cap-product with η_X induces an isomorphism

$$\eta_X \cap \ : \ H^i_Y(X,j) \xrightarrow{\sim} H_{2d-i}(Y,d-j).$$

(l) (Principal triviality) Let $i : W \hookrightarrow X$ be a smooth principal divisior in the smooth scheme X. Then $i_*\eta_W = 0$.

Write $H^i(X,j)$ for $H^i_X(X,j)$. By results of Bloch and Ogus we then have:

Proposition 3.2 *Let k be a perfect field, and let*

$$(Z \subset X) \mapsto H^*_Z(X,*) \quad , \quad X \mapsto H_*(X,*)$$

be a twisted Poincaré duality theory on the category Sch^{alg}/k of all algebraic k-schemes. Let $\mathcal{H}^i(j)$ be the Zariski sheaf associated to the presheaf $U \mapsto H^i(U,j)$. Then for all $i,j \in \mathbb{Z}$ the functor $V : (Sch^{alg}/k)^0 \to Ab$ defined by

$$V(Y) = H^i(Y,j)$$

satisfies property 2.3 (i) for semi-local regular rings A which are localizations of smooth k-schemes.

Since V is a priori only defined on k-schemes of finite type, this statement has to be interpreted in the following way: the functors V and $H^\nu(-,\mathcal{V})$ are defined on *Spec A* by taking the limit of the value groups at all opens $U \subset X$ containing *Spec A* (or, equivalently, the maximal ideals of A).

Proof of 3.2 For any algebraic k-scheme X denote by $X_{(p)}$ the set of points $x \in X$ of dimension p, and for $x \in X$ put

$$H_a(x,b) = \underset{U \subset Z \text{ open}}{\overset{\lim}{\longrightarrow}} H_a(U,b) \quad ,$$

where $Z = \overline{\{x\}}$ is the Zariski closure of x. Then Bloch and Ogus construct a homological spectral sequence

$$E^1_{p,q} = E^1_{p,q}(X,b) = \underset{x \in X_{(p)}}{\oplus} H_{p+q}(x,b) \;\Rightarrow\; H_{p,q}(X,b) \tag{3.2.1}$$

as follows ([BO] (3.7)). Let $Z_p = Z_p(X)$ be the set of all closed subsets $Z \subseteq X$ of dimension $\leq p$, ordered by inclusion, and put

$$H_a(Z_p(X),b) = \underset{Z \in Z_p}{\overset{\lim}{\longrightarrow}} H_a(Z,b) \quad .$$

Then by 3.1 (e) one gets exact sequences

$$\ldots \to H_a(Z_{p-1},b) \stackrel{i}{\to} H_a(Z_p,b) \stackrel{j}{\to} \underset{x \in X_{(p)}}{\oplus} H_a(x,b) \stackrel{k}{\to} H_{a-1}(Z_{p-1},b) \to \ldots \,. \tag{3.2.2}$$

The method of exact couples now gives the desired spectral sequence. Note that by definition the differential $d^1_{p,q}$ is the composition

$$d^1_{p,q} : E^1_{p,q} \stackrel{k}{\to} H_{p+q-1}(Z_{p-1},b) \stackrel{j}{\to} E^1_{p-1,q} \quad .$$

Moreover, if $dim\ X = d$, one has a complex

$$0 \to H_a(X,b) \stackrel{\varepsilon}{\to} E^1_{d,a-d}(X,b) \stackrel{d^1}{\to} E^1_{d-1,a-d}(X,b) \stackrel{d^1}{\to} \ldots \quad , \tag{3.2.3}$$

in which ε is the edge morphism. By 3.1 (g) the sequences (3.2.3) for all opens $U \subseteq X$ form a complex of Zariski presheaves. Let $\mathcal{H}_a(b)$ and $\mathcal{E}^1_{p,q}(b)$ be the Zariski sheaves associated to $U \mapsto H_a(U,b)$ and $U \mapsto E^1_{p,q}(U,b) = \underset{x \in U_{(p)}}{\oplus} H_{p+q}(x,b)$, respectively, so that we get a complex of sheaves

$$0 \to \mathcal{H}_a(b) \to \mathcal{E}^1_{d,a-d}(b) \to \mathcal{E}^1_{d-a,a-d}(b) \to \ldots \quad . \tag{3.2.4}$$

It is clear from the definition that $U \mapsto E^1_{p,q}(U)$ is already a sheaf, and is flabby.

Now let X be smooth. Then Bloch and Ogus show that for any finite set $S \subset X$ which is contained in an affine open, the sequence (3.2.3) becomes exact after passing to the limit over all opens $V \subset X$ containing S. In fact, this is equivalent to the statement that all maps $i : H_a(Z_{p-1}(X),b) \to H_a(Z_p(X),b)$ vanish after passing to the limit over such U, and this is shown in [B O], section 4 and 5 (in the claim on p.191 loc. cit. only the case of a one-element set S is stated, but the proof works more generally, since the trick of Quillen quoted loc. cit. is valid for a finite S as above). In particular, (3.2.4) is an exact sequence of Zariski sheaves, hence a resolution of $\mathcal{H}_a(b)$ by the complex with flabby components

$$R^{\cdot}_{a-d}(b) = \mathcal{E}^1_{d-\cdot,a-d}(b) \quad . \tag{3.2.5}$$

As a consequence, we get a canonical isomorphism

$$H^\nu(U, \mathcal{H}_a(b)) = E^2_{d-\nu, a-d}(U, b) \tag{3.2.6}$$

for every open U in X and $\nu \geq 0$, if X is smooth. Moreover, if A is a semi-local ring obtained by localizing X, then $H^\nu(Spec\ A, \mathcal{H}_a(b)) = 0$ for all $\nu > 0$, and the map

$$H_a(Spec\ A, b) \to E^2_{a,d-a}(Spec\ A, b) = H^0(Spec\ A, \mathcal{H}_a(b))$$

is an isomorphism. Since the presheaves $U \mapsto H_a(U, b)$ and $U \mapsto H^{2d-a}(U, d-b)$ are isomorphic by 3.1 (h), (j) and (k), the proposition follows.

For the treatment of the homotopy property 2.3 (ii) for $\mathcal{H}$-cohomology we need the following extended version of a Poincaré duality theory.

Definition 3.3 *Let k be a field. A twisted Poincaré duality theory*

$$(Z \subset X) \mapsto H^*_Z(X, *) \quad , \quad X \mapsto H_*(X, *)$$

on Sch^{alg}/k is called an extended Poincaré duality theory, if for every flat morphism $f : X' \to X$ which is equidimensional of dimension m (i.e., whose fibres are either empty or equidimensional of dimension m , cf. [EGA IV], 13.3) there are functorially associated maps

$$f^* : H_a(X, b) \to H_{a+2m}(X', b+m) \ ,$$

agreeing with the pull-back maps in 3.1 (c) for étale f and $m = 0$, such that the following properties hold.

(m) If X and X' are irreducible, then $f^*\eta_X = \eta_{X'}$.

(n) If $Z \subset X$ is a closed subscheme, and $Z' = Z \times_X X' \subset X'$, then the following diagram commutes.

$$\begin{array}{ccccc}
H_{a+2m}(X', b+m) & \otimes & H^{i'_Z}(X', j) & \longrightarrow & H_{a-i+2m}(Z', b-j+m) \\
\uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\
H_a(X', b) & \otimes & H^i_{Z'}(X', j) & \longrightarrow & H_{a-i}(Z', b-j)
\end{array} \quad .$$

(o) If Z is closed in X and U is the open complement, then one has a commutative diagram

$$\begin{array}{ccccccccc}
.. \longrightarrow & H_{a+2m}(Z', b+m) & \longrightarrow & H_{a+2m}(X', b+m) & \longrightarrow & H_{a+2m}(U', b+m) & \longrightarrow & H_{a-1+2m}(Z', b+m) & \\
& \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} & \\
.. \longrightarrow & H_a(Z, b) & \longrightarrow & H_a(X, b) & \longrightarrow & H_a(U, b) & \longrightarrow & H_{a-1}(Z, b) & \longrightarrow ..
\end{array}$$

where $Z' = Z \times_X X'$ and $U' = f^{-1}(U)$.

We shall give some examples below. First we note

Proposition 3.4 *Let k be a perfect field, and let $(Z \subset X) \mapsto H_Z^{\cdot}(X,\cdot), X \mapsto H_{\cdot}(X,\cdot)$ be an extended Poincaré duality theory on Sch^{alg}/k. Assume that the following "homotopy invariance" holds:*

(p) For every smooth k-scheme X the maps

$$p^* : H^i(X,j) \longrightarrow H^i(\mathbb{A}^1_X, j)$$

induced by the projection $p : \mathbb{A}^1_X \to X$ are isomorphisms for all $i,j \in \mathbb{Z}$.

Then for all $i,j \in \mathbb{Z}$ the contravariant functor V on Sch^{alg}/k given by

$$V(Y) \quad = \quad H^i(Y,j)$$

satisfies property 2.3 (ii) for local rings A which are local rings $\mathcal{O}_{Y,y}$ of smooth k-schemes Y.

Proof (The values on *Spec A* are defined as in Proposition 3.2). Let $f : X' \to X$ be a flat morphism in Sch^{alg}/k which is equidimensional of dimension m.

We first note that by 3.3 (o) the flat pull-backs induce maps between the sequences (3.2.2) for X and X', with appropriate shift of degrees (for $Z \subset X$ of dimension p the preimage $f^{-1}(Z) = Z \times_X X'$ is of dimension $p+m$). This induces a map of spectral sequences (with the indicated shift of degrees)

(3.4.1)
$$\begin{array}{ccc} E^1_{p+m,q+m}(X', b+m) & \Longrightarrow & H_{p+q+zm}(X', b+m) \\ \uparrow f^* & & \uparrow f^* \\ E^1_{p,q}(X,b) & \Longrightarrow & H_{p+q}(X,b) \end{array} \quad .$$

In particular, there are natural pull-back maps

(3.4.2)
$$f^* : E^2_{p,q}(X,b) \to E^2_{p+m,q+m}(X', b+m) \quad .$$

On the other hand, if X and X' are irreducible of dimensions d and d', respectively, (so that $m = d' - d$), then by 3.3 (m) and (n) the diagram

(3.4.3)
$$\begin{array}{ccc} H^i(X',j) & \xrightarrow{\eta_{X'}^{\cap}} & H_{2d'-i}(X', d'-j) \\ f^* \uparrow & & \uparrow f^* \\ H^i(X,j) & \xrightarrow{\eta_X^{\cap}} & H_{2d-i}(X, d-j) \end{array}$$

is commutative. Now let X and X' be smooth. Then the horizontal maps in the above diagram are isomorphisms by Poincaré duality 3.1 (k). The same is true for open subschemes; hence the pull-back map $f^* : H^\nu(X, \mathcal{H}^i(j)) \to H^\nu(X', \mathcal{H}^i(j))$ can be identified with a pull-back map

(3.4.4)
$$f^* : H^\nu(X, \mathcal{H}_{2d-i}(d-j)) \to H^\nu(X', \mathcal{H}_{2d'-i}(d'-j))$$

which is defined in a way analogous to (2.3.2), using the compatibility of flat pull-backs with open immersions. Here $\mathcal{H}_a(b)$ is a Zariski sheaf on X or X' associated to the presheaf $U \mapsto H_a(U, b)$.

Furthermore, it follows from (3.4.1) that, via the isomorphisms (3.2.6) for X and X', the pull-back map (3.4.4) can be identified with the pull-back map (3.4.2) for $(p, q, b, m) = (d - \nu, d - i, d - j, d' - d)$. Thus, in view of remark 2.4 (b), the proposition follows from part (ii) of the lemma below, applied to smooth k-schemes X.

Lemma 3.5 *In the situation of proposition 3.4, let X be any algebraic k-scheme (not necessarily smooth), and let $p : \mathbb{A}^1_X \to X$ be the affine line over X. Then the following holds.*

(i) The flat pull-back maps $p^* : H_a(X, b) \to H_{a+2}(\mathbb{A}^1_X, b + 1)$ are isomorphisms for all $a, b \in \mathbb{Z}$.

(ii) The pull-back maps $p^* : E^2_{p,q}(X,\, b) \to E^2_{p+1,\,q+1}(\mathbb{A}^1_X,\, b + 1)$ are isomorphisms for all $p, q, b \in \mathbb{Z}$.

Proof We proceed by induction on *dim* (X), and we may and will consider only reduced schemes (since $H_a(X,\, b) = H_a(X_{red},\, b)$ by 3.1 (e)). For $\dim(X) = 0$ we then may assume that $X = Spec\ K$ for a finite extension K of k, necessarily separable since k is perfect. Then (3.4.1) gives a commutative diagram with exact top row

$$(3.5.1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & E^2_{0,a}(\mathbb{A}^1_K, b+1) & \longrightarrow & H_{a+2}(\mathbb{A}^1_K, b+1) & \longrightarrow & E^2_{1,a-1}(\mathbb{A}^1_K, b+1) & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle p^*} & & \uparrow{\scriptstyle p^*} & & \\ & & & & H_a(Spec\ (K), b) & \xrightarrow{\sim} & E^2_{0,a}(Spec\ (K), b) & & \end{array}$$

in which the middle vertical map is an isomorphism by the assumption 3.4 (p) and Poincaré duality (cf. (3.4.3)). On the other hand, the right hand vertical map is injective: it can be identified with

$$p^* : H^0(Spec\ (K), \mathcal{H}^{-a}(-b)) \to H^0(\mathbb{A}^1_K, \mathcal{H}^{-a}(-b)) \quad ,$$

and any K-rational point $s : Spec\ K \to \mathbb{A}^1_K$ gives a left inverse s^* of p^*. Putting this together, we deduce that both vertical maps are isomorphisms and that $E^2_{0,a}(\mathbb{A}^1_K, b + 1) = 0$. This shows (i) and (ii) in this case, since $E^2_{p,q}(Spec\ (K), b) = 0$ for $p \neq 0$ and $E^2_{p,q}(\mathbb{A}^1_K, b) = 0$ for $p \neq 0$ or 1.

If X has positive dimension (and is reduced), then there is a dense open subscheme $U \subseteq X$ which is smooth (because k is perfect). Since (i) holds for U by Poincaré duality and assumption, and for $Z = X - U$ by induction hypothesis, it holds for X by 3.3 (o) and the five-lemma.

For (ii) we observe the following: if $Z \overset{i}{\hookrightarrow} X$ is a closed subscheme and $U = X - Z \overset{j}{\hookrightarrow} X$ is the open complement, then one has an exact sequence of complexes

$$(3.5.2)\qquad 0 \to E^1_{\cdot,q}(Z, b) \xrightarrow{i_*} E^1_{\cdot,q}(X, b) \xrightarrow{j^*} E^1_{\cdot,q}(U, b) \to 0.$$

Here j^* comes from (3.4.1) for $f = j$, and i_* comes from the contravariance of the spectral sequence (3.2.1) for the proper morphism i, which is an immediate consequence of 3.1 (f). The exactness of the sequence follows from the easily verified fact that in degree p the sequence is given by

$$0 \to \bigoplus_{x\in Z_{(p)}} H_{p+q}(\kappa(x), b) \xrightarrow{i_*} \bigoplus_{x\in X_{(p)}} H_{p+q}(\kappa(x), b) \xrightarrow{j^*} \bigoplus_{x\in U_{(p)}} H_{p+q}(\kappa(x), b) \longrightarrow 0 \ ,$$

where i_* and j^* are the obvious inclusion and projection, respectively. There is a corresponding exact sequence for the triple $\mathbb{A}^1_Z \hookrightarrow \mathbb{A}^1_X \hookleftarrow \mathbb{A}^1_U$, and both exact sequences are connected by the pull-back maps for the projections $p_X : \mathbb{A}^1_X \to X$, p_U and p_Z, in a commutative map. Passing to the cohomology, we obtain a commutative diagram with long exact rows

(3.5.3)

$$\begin{array}{ccccccc}
..E^2_{p+1,q+1}(\mathbb{A}^1_Z, b{+}1) & \longrightarrow & E^2_{p+1,q+1}(\mathbb{A}^1_X, b{+}1) & \longrightarrow & E^2_{p+1,q+1}(\mathbb{A}^1_U, b{+}1) & \longrightarrow & E^2_{p,q+1}(\mathbb{A}^1_Z, b{+}1).. \\
\uparrow {\scriptstyle p_Z^*} & & \uparrow {\scriptstyle p_X^*} & & \uparrow {\scriptstyle p_U^*} & & \uparrow {\scriptstyle p_Z^*} \\
..E^2_{p,q}(Z, b) & \longrightarrow & E^2_{p,q}(X, b) & \longrightarrow & E^2_{p,q}(U, b) & \longrightarrow & E^2_{p-1,q}(Z, b) \to ..
\end{array}$$

By induction hypothesis we may assume that all p_Z^* are isomorphisms for $\dim Z < \dim X$. Hence it suffices to show that the p_U^* become isomorphisms after passing to the limit over all dense opens $U \subset X$. Moreover, we may restrict to the case that we consider the limit over all open neighbourhoods of a fixed generic point η. If $K = \kappa(\eta)$ is the function field of the corresponding connected component of X, then we obtain formally the same diagram as (3.5.1), by putting $H_a(K, b) = \overset{\lim}{\longrightarrow} H_a(U, b), H_a(\mathbb{A}^1_K, b) = \overset{\lim}{\longrightarrow} H_a(\mathbb{A}^1_U, b)$ etc. . By reasoning in a completely similar way as in the case of a finite extension K of k, we deduce that the pull-back map

$$p^* : E^2_{p,q}(K, b) \to E^2_{p+1,q+1}(\mathbb{A}^1_K, b + 1)$$

is an isomorphism as wanted (the map is injective since $p^* : E^2_{p,q}(U, b) \to E^2_{p+1,q+1}(\mathbb{A}^1_U, b{+}1)$ is injective for each smooth open $U \subseteq X$).

Remark 3.6 The proof of 3.5 follows very much Gillet's proof of the corresponding statement for the Chow groups $CH_{r,s}$ (X) ([Gi] Thm. 8.3), except that the proof for $X = Spec\ (K)$ via (3.5.1) is more direct than the recursion to the projective bundle theorem in [Gi] Lemma 8.4.

We want to apply the above to the following example. For other examples of extended Poincaré duality theories with homotopy invariance, we refer to the appendix.

Proposition 3.7 Let k be a field, and fix $n \in \mathbb{N}$ invertible in k The following functors form an extended Poincaré duality theory on Sch^{alg}/k, and the properties 3.5 (i) and (ii) hold for them (In particular, by Poincaré duality, the homotopy invariance 3.4 (p) holds for them):

$$\begin{array}{lcll}
H^i_Z(X, j) & = & H^i_Z(X, \mathbb{Z}/n(j)) & \text{(étale cohomology with supports)}, \\
H_a(X, b) & = & H_a(X, \mathbb{Z}/n(b)) := H^{-a}(X, a^!_X \mathbb{Z}/n(-b)) & \text{(étale homology)}
\end{array}$$

Here $\mathbb{Z}/n(j) = \mu_a^{\otimes j}$ as in 2.1, $a_X : X \to Spec\ (k)$ is the structural morphism, and $a_X^! : D_c^b(Spec(k), \mathbb{Z}/n) \to D_c^b(X, \mathbb{Z}/n)$ (= derived category of bounded complexes of étale $\mathbb{Z}/n$-sheaves on X with constructible cohomology) is the right adjoint of $R(a_X)_!$ constructed in Grothendieck-Verdier duality [SGA 4] XVII, 3.

Proof That étale cohomology and homology from a twisted Poincaré duality theory, follows from the results in [SGA 4], cf. the sketch in [BO] 2.1. We now describe flat pullbacks in the homology, for a flat morphism $f : X' \to X$ which is equidimensional of dimension m (cf. the discussion for an algebraically closed field k in Laumon's article [DV] VIII, 5). By [SGA 4] XVIII 2.9, there is a canonical trace morphism

$$(3.7.1) \qquad Tr_f : R^{2m} f_! f^* \mathcal{F}(m) \to \mathcal{F} \quad ,$$

for any étale $\mathbb{Z}/n$-sheaf $\mathcal{F}$ on X, coinciding with the trace morphism used in 2.1 for finite f (in which case $m = 0$ and $f_! = f_*$). Since $R^i f_! \mathcal{F} = 0$ for $i > 2m$, this trace morphism can be regarded as a morphism

$$Tr_f : \quad Rf_! f^* \mathcal{F}(m)[2m] \to \mathcal{F}$$

in $D_c^b(X, \mathbb{Z}/n)$. This can be extended to arbitrary complexes $\mathcal{L}$ in $D_c^b(X, \mathbb{Z}/n)$ as follows. If let $\mathcal{F} = \mathbb{Z}/n$ and tensor with $\mathcal{L}$, then by the "projection formula" isomorphism

$$(3.7.2) \qquad \gamma_f : \quad Rf_! \mathcal{K} \otimes^L_{\mathbb{Z}/n} f^* \mathcal{L}) \stackrel{\sim}{\to} Rf_!(\mathcal{K} \otimes^L_{\mathbb{Z}/n} f^* \mathcal{L})$$

([SGA 4] XVII 5.2.9), we obtain a trace morphism

$$(3.7.3) \qquad Tr_f : Rf_! f^* \mathcal{L}(m)[zm] \to \mathcal{L}$$

(cf. [SGA 4] XVIII 2.13.2). By adjunction between $Rf_!$ and $f!$, we now get a morphism

$$t_f : f^* \mathcal{L}(m)[2m] \to f^! \mathcal{L}.$$

(cf. loc. cit. 3.2.3). Applied to $\mathcal{L} = a_X^! \mathbb{Z}/n(-b)$, for which $f^! \mathcal{L} = a_{X'}^! \mathbb{Z}/n(-b)$, this induces the wanted pull-back maps

$$\begin{array}{cccccc} f^* : & H^{-a}(X, a_X^! \mathbb{Z}/n(-b)) & \longrightarrow & H^{-a}(X', f^* a_X^! \mathbb{Z}/n(-b)) & \xrightarrow{t_f} & H^{-a-2m}(X', a_{X'}^!, \mathbb{Z}/n(-b-m)) \\ & \| & & & & \| \\ & H_a(X, \mathbb{Z}/n(b)) & & & & H_{a+2m}(X', \mathbb{Z}/n(b+m)), \end{array}$$

where the first arrow in the restriction morphism which exists for any complex of sheaves $\mathcal{K}$ (i.e., the composition $H^\cdot(\chi, \mathcal{K}) \stackrel{ad_f}{\to} H^\cdot(X, Rf_* f^* \mathcal{K}) = H^\cdot(X', f^* \mathcal{K})$, cf. also 2.1).

If f is étale, then $f^!$ is identified with f^* via t_f, and the pull-back by definition is the one used for étale morphisms in homology (cf. [BO] 2.1). The functoriality of flat pull-backs is a direct consequence of the "transitivity" of the trace maps (3.7.1) (cf. [SGA 4] XVIII 2.9, (Var 3)). This in turn implies 3.3 (m), because η_X is the image of $1 \in \mathbb{Z}/n = H_0(Spec(k), \mathbb{Z}/n)$ under

$$a_X^* : H_0(Spec(k), \mathbb{Z}/n) \to H_{2d}(X, \mathbb{Z}/n(d) \quad ,$$

if X is irreducible of dimension d (cf. [SGA $4\frac{1}{2}$], [cycle], 2.3).

For 3.3 (n) we recall that the cap product is induced by a pairing $a_X^!\mathbb{Z}/n(r) \otimes a_X^*\mathbb{Z}/n(s) \to a_X^!\mathbb{Z}/n(r+s)$ (cf. [BO] 2.1) which is a special case of the following, more general one. For any morphism $g : X \to Y$ in Sch^{alg}/k and any $\mathbb{Z}/n$-sheaves $\mathcal{F}, \mathcal{G}$ on Y (in fact, any objects $\mathcal{F}, \mathcal{G}$ in $D_c^b(Y, \mathbb{Z}/n)$), one has a pairing

$$\varphi_g : g^!\mathcal{F} \otimes^L g^*\mathcal{G} \to g^!(\mathcal{F} \otimes^L \mathcal{G}) \quad ,$$

which by adjunction corresponds to the horizontal morphism making the diagram

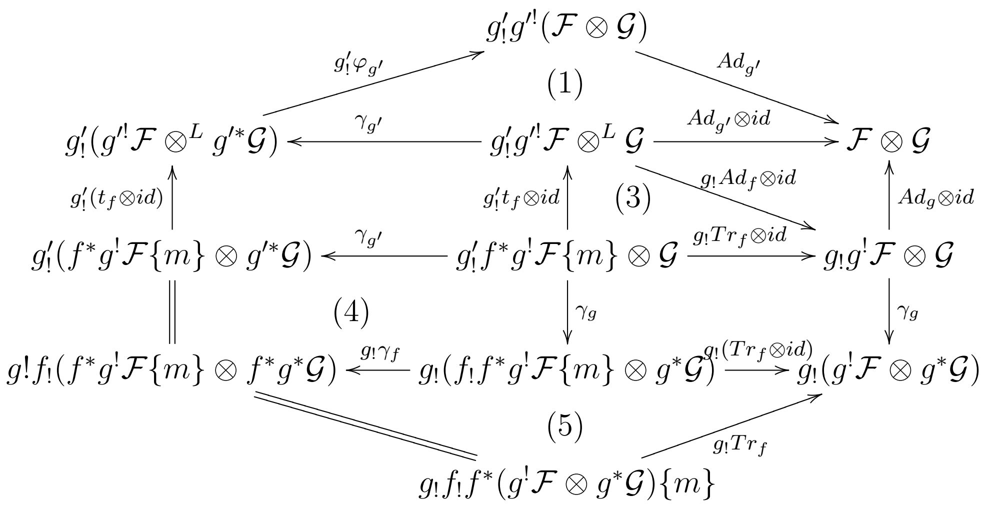
$$\begin{array}{ccc} R\, g_!(g^!\mathcal{F} \otimes^L g^*\mathcal{G}) & \longrightarrow & \mathcal{F} \otimes^L \mathcal{G} \\ {\scriptstyle \gamma_g}\big\downarrow & \nearrow{\scriptstyle Ad_g \otimes id} & \\ R\, g_! g^!\mathcal{F} \otimes^L \mathcal{G} & & \end{array}$$

commutative. Here the vertical isomorphism is the projection formula isomorphism (3.7.2), and $Ad_g : Rg_!g^!\mathcal{F} \to \mathcal{F}$ is the adjunction map.

Now let $f : X' \to X$ be a flat morphism which is equidimensional of dimension m, and put $g' = gf : X' \to Y$. Then we claim that the diagram

$$(3.7.4) \qquad \begin{array}{ccccc} f^!g^!\mathcal{F} \otimes^L f^*g^*\mathcal{G} & = & g'^!\mathcal{F} \otimes^L g'^*\mathcal{G} & \xrightarrow{\varphi_{g'}} & g'^!(\mathcal{F} \otimes^L \mathcal{G}) \\ {\scriptstyle t_f \otimes id}\big\uparrow & & & & \big\uparrow{\scriptstyle t_f} \\ f^*g^!\mathcal{F}\{m\} \otimes^L f^*g^*\mathcal{G} & = & f^*(g^!\mathcal{F} \otimes^L g^*\mathcal{G})\{m\} & \xrightarrow{f^*\varphi_g} & f^*g^!(\mathcal{F} \otimes^L \mathcal{G})\{m\} \end{array}$$

commutes, where we put $\mathcal{H}\{m\} = \mathcal{H}(m)[2m]$ for a complex $\mathcal{H}$. By adjunction, this amounts to the commutating of the following two diagrams (where we have written $f_!$ for $Rf_!$, etc.)



$$\begin{array}{ccccc}
g'_!g'^!(\mathcal{F}\otimes\mathcal{G}) & \xrightarrow{Ad_{g'}} & \mathcal{F}\otimes\mathcal{G} & \xleftarrow{Ad_g\otimes id} & \\
g'_!t_f \uparrow \quad (3) & \searrow^{g_!Ad_f} & (2) \quad \uparrow Ad_g & & \\
g'_!f^*g^!(\mathcal{F}\otimes\mathcal{G})\{m\} & \xrightarrow{g_!Tr_f} & g_!g^!(\mathcal{F}\otimes\mathcal{G}) & (1) & g_!g^!\mathcal{F}\otimes\mathcal{G} \\
g'^!f^*\varphi_g \uparrow & & \uparrow g_!\varphi_g & \swarrow_{\gamma_g} & \\
g_!f_!f^*(g^!\mathcal{F}\otimes g^*\mathcal{G})\{m\} & \xrightarrow{g_!Tr_f} & g_!(g^!\mathcal{F}\otimes g^*\mathcal{G}) & &
\end{array}$$

Here the diagrams (1) and (3) commute by the definitions of φ and t, respectively, and the identification $g'^! = f^!g^!$ is just the one for which the diagrams (2) commute (these morphisms being defined as the adjoints of $g'_!$, $f_!$ and $g_!$). The commutativity of (4) is easily checked (cf. also [SGA 4] XVII 5.2.4), and (5) commutes by our definition of (3.7.3), together with an obvious "associativity" for φ_f. The remaining squares commute by functoriality.

Now let $Y = Spec\ (k)$. Then by definition the map product for X is the composition

$$\begin{array}{ccccc}
H^{-a}(X, a^!_X\mathbb{Z}/n(-b) & \otimes & H^i_Z(X,\ a^*_X\mathbb{Z}/n(j)) & \longrightarrow & H^{i-a}_Z(X,\ a^!_X\mathbb{Z}/n(j-b)) \\
\| & & \| & & \| \\
H_a(X,\ \mathbb{Z}/n(b)) & & H^i_z(X,\ \mathbb{Z}/n(j)) & & H_{a-i}(Z,\ \mathbb{Z}/n(b-j))
\end{array}$$

induced by the usual cup product (with supports) and φ_{a_X}, together with the identification $H^{\cdot}_Z(X,\ a^!_X\mathcal{F}) = H^{\cdot}(Z,\ Ri^!a^!_X\mathcal{F}) = H^{\cdot}(Z,\ a^!_z\mathcal{F})$ for $i : Z \hookrightarrow X$. Thus 3.3 (n) follows from the commutativity of (3.7.4).

Next we consider 3.3 (o). Let $i : Z \hookrightarrow X$ be a closed immersion with open complement $j : U = X - Z \hookrightarrow X$. For a flat morphism $f : X' \to X$, which is equidimensional of dimension m, consider the cartesian squares

$$\begin{array}{ccccc}
Z' & \xhookrightarrow{i'} & X' & \xhookleftarrow{j'} & U' \\
\downarrow f_Z & & \downarrow f & & \downarrow f_U \\
Z & \xhookrightarrow{i'} & X & \xhookleftarrow{j} & U
\end{array} \quad .$$

The relative sequence 3.1 (e) for the triple $(Z,\ X,\ U)$ is obtained by taking the cohomology on X of the canonical exact triangle

$$i_*Ri^!a^!_X\mathbb{Z}/n(-b) \xrightarrow{Ad_i} a^!_X\mathbb{Z}/n(-b) \xrightarrow{ad_i} Rj_*j^*a^!_X\mathbb{Z}/n(-b) \to ,$$

and identifying $Ri^!a^!_X = a^!_Z$ and $j^*a^!_X = a^!_U$ (Here we used that $Ri_* = i_*$, and have written $Ri^!$ since $i^!$ may be misinterpreted as the functor " sections with support in Z" whose derivative $Ri^!$ is). Hence 3.3 (o). follows from the fact that one has natural identifications of exact triangles

$$\begin{array}{ccccccc}
i'_*Ri'^!a^!_{X'}\mathcal{F} & \longrightarrow & a^!_{X'}\mathcal{F} & \longrightarrow & Rj'_*j'^*a^!_{X'}\mathcal{F} & \longrightarrow & \\
\| & & \| & & \| & & \\
i'_*f^!_ZRi^!a^!_X\mathcal{F} & \longrightarrow & f^!a^!_X\mathcal{F} & \longrightarrow & Rj'_*f^!_Uj^*a^!_X\mathcal{F} & \longrightarrow & \\
\beta' \downarrow \wr & & \| & & \beta'' \downarrow \wr & & \\
f^!i_*Ri'a^!_X\mathcal{F} & \longrightarrow & f^!a^!_X\mathcal{F} & \longrightarrow & f^!Rj_*j^*a^!_X\mathcal{F} & \longrightarrow &
\end{array} \quad ,$$

for any complex of sheaves $\mathcal{F}$ on $Spec\ (k)$, where β' and β'' are obtained from the "adjoint base change isomorphism" [SGA 4] XVIII 3.1.12.3. Note that by definition the diagram

$$\begin{array}{ccc} i'_* R i^! f^! \mathcal{G} & \xrightarrow{Ad'_i} & f^! \mathcal{G} \\ \| & & \uparrow{\scriptstyle f^! Ad_i} \\ i'_* f_z^! R i^! \mathcal{G} & \xrightarrow{\beta'} & f^! i_* R i^! \mathcal{G} \end{array}$$

commutes, similarly for β'' (where $j^! = j^*$ and $j'^! = j'^*$).

Finally we show that the homotopy invariance 3.5 (i) holds for étale homology (this implies property 3.5 (ii) as well, as is clear from the proof of 3.5). Since $p : \mathbb{A}^1_X \to X$ is acyclic (cf. [Mi] VI 4.20), the restriction map

$$H^{-a}(X,\, a_X^! \mathbb{Z}/n(-b)) \to H^{-a}(\mathbb{A}^1_X,\, p^* a_X^! \mathbb{Z}/n(-b))$$

is an isomorphism for all a, b, $\in \mathbb{Z}$. We conclude by recalling that

$$t_p : p^* \mathcal{L}(1)[2] \to p^! \mathcal{L}$$

is an isomorphism for any $\mathcal{L}$ in $D^b_c(X,\, \mathbb{Z}/n)$ (for this it suffices that p is smooth of relative dimension 1, cf. [SGA 4] XVIII proof of 3.2.5).

We can now collect the fruits of our efforts.

Theorem 3.8 Let k be a field, and let n be a natural number invertible in k. For i, $j \in \mathbb{Z}$ let $\mathcal{H}^i(j)$ be the Zariski sheaf on the category $\mathrm{Sch}^{\mathrm{noeth}}/k$ of all noetherian k-schemes associated to the presheaf

$$U \leadsto H^i(U,\, \mathbb{Z}/n(j))$$

given by étale cohomology. Then for all ν, i, $j \in \mathbb{Z}$ the functor

$$X \leadsto H^\nu(X,\, \mathcal{H}^i(j))$$

is a sufficiently rigid functor on Sch^{noeth}/k.

Proof By 2.1 and 1.8, étale cohomology is a sufficiently rigid functor on $\mathrm{Sch}^{\mathrm{noeth}}\ /k$. In view of theorem 2.3, we then have to show 2.3 (i) and (ii) for the $\mathcal{H}^i(j)$. Since it suffices to consider the bigger category $\mathrm{Sch}^{\mathrm{noeth}}\ /k_0$, where k_0 is the prime field, we may assume that k is perfect, and by 2.6 we may restrict our attention to algebraic k-schemes. By proposition 3.7, $X \mapsto H^{\cdot}(X)$, $\mathbb{Z}/n(\cdot)$ is part of an extended Poincaré duality theory with homotopy invariance. Therefore the claim follows from proposition 3.2 and 3.4.

By theorem 1.7, we obtain case (5) of Theorem 0.1 in the introduction.

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