Canonical embedded and non-embedded resolution of singularities for excellent two-dimensional schemes

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Contents

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0 Introduction

The principal aim of this paper is to show the following three theorems on the resolution of singularities of an arbitrary reduced excellent scheme X of dimension at most two.

Theorem 0.1 *(Canonical controlled resolution) There exists a canonical finite sequence of morphisms*

$$\pi : X' = X_n \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 = X$$

such that X' is regular and, for each i, $X_{i+1} \to X_i$ is the blow-up of X_i in a permissible center $D_i \subset X_i$ which is contained in $(X_i)_{sing}$, the singular locus of X_i. This sequence is functorial in the sense that it is compatible with automorphisms of X and (Zariski or étale) localizations.

We note that this implies that π is an isomorphism over $X_{reg} = X - X_{sing}$, and we recall that a subscheme $D \subset X$ is called permissible, if D is regular and X is normally flat along D (see 2.1). The compatibility with automorphisms means that every automorphism of X extends to the sequence in a unique way. The compatibility with the localizations means that the pull-back via a localization $U \to X$ is the canonical resolution sequence for U after suppressing the morphisms which become isomorphisms over U. It is well-known that Theorem 0.1 implies:

Theorem 0.2 *(Canonical embedded resolution) Let $i : X \hookrightarrow Z$ be a closed immersion, where Z is a regular excellent scheme. Then there is a canonical commutative diagram*

$$
\begin{array}{ccc}
X' & \overset{i'}{\longrightarrow} & Z' \\
\pi \downarrow & & \downarrow \pi_Z \\
X & \overset{i}{\longrightarrow} & Z
\end{array}
$$

where X' and Z' are regular, i' is a closed immersion, and π and π_Z are proper and surjective morphisms inducing isomorphisms $\pi^{-1}(X - X_{sing}) \overset{\sim}{\longrightarrow} X - X_{sing}$ and $\pi_Z^{-1}(Z - X_{sing}) \overset{\sim}{\longrightarrow} Z - X_{sing}$. Moreover, the morphisms π and π_Z are compatible with automorphisms of (X, Z) and (Zariski or étale) localizations in Z.

In fact, starting from Theorem 0.1 one gets a canonical sequence $Z' = Z_n \to \ldots Z_1 \to Z_0 = Z$ and closed immersions $X_i \hookrightarrow Z_i$ for all i, such that $Z_{i+1} \to Z_i$ is the blow-up in $D_i \subset (X_i)_{sing} \subset Z_i$ and $X_{i+1} \subseteq Z_{i+1}$ is identified with the strict transform of X_i in the blow-up $Z_{i+1} \to Z_i$. Then $Z_{i+1} \to Z_i$ is proper (in fact, projective) and surjective, and Z_{i+1} is regular since Z_i and D_i are.

For several applications the following refinement is useful:

Theorem 0.3 *(Canonical embedded resolution with boundary) Let $i : X \hookrightarrow Z$ be a closed immersion into a regular scheme Z, and let $B \subset Z$ be a simple normal crossings divisor such that no irreducible component of X is contained in B. Then there is a canonical commutative diagram*

$$
\begin{array}{ccc}
X' & \overset{i'}{\longrightarrow} & Z' \supset B' \\
\pi_X \downarrow & & \pi_Z \downarrow \\
X & \overset{i}{\longrightarrow} & Z \supset B
\end{array}
$$

where i' is a closed immersion, π_X and π_Z are projective, surjective, and isomorphisms outside $X_{sing} \cup (X \cap B)$, and $B' = \pi_Z^{-1}(B) \cup E$, where E is the exceptional locus of π_Z (which is a closed subscheme such that π_Z is an isomorphism over $Z - \pi_Z(E)$). Moreover, X' and Z' are regular, B' is a simple normal crossings divisor on Z', and X' intersects B' transversally on Z'. Furthermore, π_X and π_Z are compatible with automorphisms of (Z, X, B) and with (Zariski or étale) localizations in Z.

More precisely, we prove the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
\pi_B: & B' = B_m & & B_{m-1} & & \ldots & & B_1 & & B_0 = B \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_Z: & Z' = Z_m & \longrightarrow & Z_{m-1} & \longrightarrow & \ldots & \longrightarrow & Z_1 & \longrightarrow & Z_0 = Z \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & \downarrow \\
\pi: & X' = X_m & \longrightarrow & X_{m-1} & \longrightarrow & \ldots & \longrightarrow & X_1 & \longrightarrow & X_0 = X
\end{array}
$$

where the vertical morphisms are closed immersions and, for each i, $X_{i+1} = B\ell_{D_i}(X_i) \to X_i$ is the blow-up of X_i in a permissible center $D_i \subset (X_i)_{sing}$, $Z_{i+1} = B\ell_{D_i}(Z_i) \to Z_i$ is the blow-up of Z_i in D_i (so that Z_{i+1} is regular and X_{i+1} is identified with the strict transform of X_i in Z_{i+1}), and B_{i+1} is the complete transform of B_i, i.e., the union of the strict transform of B_i in Z_{i+1} and the exceptional divisor of the blow-up $Z_{i+1} \to Z_i$. Furthermore, D_i is B_i-permissible, i.e., $D_i \subset X_i$ is permissible, and normal crossing with B_i (see Definition 3.1), which implies that B_{i+1} is a simple normal crossings divisor on Z_{i+1} if this holds for B_i on Z_i.

In fact, the second main theorem of this paper, Theorem 5.9, states a somewhat more general version, in which B can contain irreducible components of X. Then one can assume that D_i is not contained in the strongly B_i-regular locus $X_{\mathcal{B}\text{sreg}}$ (see Definition 5.1), and one gets that X' is normal crossing with B (Definition 3.1). This implies that π is an isomorphism above $X_{\mathcal{B}\text{sreg}} \subseteq X_{reg}$, and, in particular, again over $X_{reg} - B$. In addition, this Theorem also treats non-reduced schemes X, in which case $(X')_{red}$ is regular and normal crossing with B and X' is normally flat along $(X')_{red}$.

Moreover, we obtain a variant, in which we only consider strict transforms for the normal crossings divisor, i.e., where B_{i+1} is the strict transform of B_i. Then we only get the normal crossing of X' (or X'_{red} in the non-reduced case) with the strict transform $\tilde{B}$ of B in Z'.

Theorem 0.1, i.e., the case where we do not assume any embedding for X, will also be proved in a more general version: Our first main theorem, Theorem 5.6, allows a non-reduced scheme X as well as a so-called boundary on X, a notion which is newly introduced in this paper (see section 4). Again this theorem comes in two versions, one with complete transforms and one with strict transforms.

Our approach implies that Theorem 5.6 implies Theorem 5.9. In particular, the canonical resolution sequence of Theorem 5.9 for $B = \emptyset$ and strict transforms (or of Theorem 0.3 for this variant) coincides with the intrinsic sequence for X from Theorem 0.1. Thus, the readers only interested in Theorems 0.1 and 0.2 can skip sections 3 and 4 and ignore any mentioning of boundaries/normal crossings divisors (by assuming them to be empty).

We note the following corollary.

Corollary 0.4 *Let Z be a regular excellent scheme (of any dimension), and let $X \subset Z$ be a reduced closed subscheme of dimension at most two. Then there exists a projective surjective morphism $\pi : Z' \longrightarrow Z$ which is an isomorphism over $Z - X$, such that $\pi^{-1}(X)$, with the reduced subscheme structure, is a simple normal crossings divisor on Z'.*

In fact, applying Theorem 0.3 with $B = \emptyset$, we get a projective surjective morphism $\pi_1 : Z_1 \longrightarrow Z$ with regular Z_1, a regular closed subscheme $X_1 \subset Z_1$ and a simple normal crossings divisor B_1 on Z_1 such that π_1 is an isomorphism over $Z - X$ (in fact, over $Z - (X_{sing})$), and $\pi_1^{-1}(X) = X_1 \cup B_1$. Moreover, X_1 and B_1 intersect transversally. In particular, X_1 is normal crossing with B_1 in the sense of Definition 3.1. Hence we obtain the wanted situation by composing π_1 with $\pi_2 : Z' \longrightarrow Z_1$, the blow-up of Z_1 in the B_1-permissible (regular) subscheme X_1, and letting $X' = \pi_2^{-1}(X_1 \cup B_1)$ which is a simple normal crossings divisor, see Lemma 3.2.

Moreover we mention that Theorem 0.3 is applied in a paper of the second and third author [JS], to prove a conjecture of Kato and finiteness of certain motivic cohomology groups for varieties over finite fields. This was a main motivation for these authors to work on this subject.

To our knowledge, none of the three theorems is known, at least not in the stated generality. Even for $\dim(X) = 1$ we do not know a reference for these results, although they may be well-known. For X integral of dimension 1, Theorem 0.1 can be found in [Be] section 4, and a proof of Theorem 0.3 is written in [Ja].

In 1939 Zariski [Za1] proved Theorem 0.1 (without discussing canonicity or functoriality) for irreducible surfaces over algebraically closed fields of characteristic zero. Five years later, in [Za3], he proved Corollary 0.4 (again without canonicity or functoriality) for surfaces over fields of characteristic zero which are embedded in a non-singular threefold. In 1966, in his book [Ab4], Abhyankar extended this last result to all algebraically closed fields, making heavy use of his papers [Ab3] and [Ab5]. Around the same time, Hironaka [H6] sketched a shorter proof of the same result, over all algebraically closed ground fields, based on a different method. Recently a shorter account of Abhyankar's results was given by Cutkosky [Cu2]. For all excellent schemes of characteristic zero, i.e., whose residue fields all have characteristic zero, and of arbitrary dimension, Theorems 0.1 and 0.3 were proved by Hironaka in his famous 1964 paper [H1] (Main Theorem 1*, p. 138, and Corollary 3, p. 146), so Theorem 0.2 holds as well, except that the approach is not constructive, so it does not give canonicity or functoriality. These issues were addressed and solved in the later literature, especially in the papers by Villamayor, see in particular [Vi], and by Bierstone-Millman, see [BM1], by related, but different approaches. In these references, a scheme with a fixed embedding into a regular scheme is considered, and in [Vi], the process depends on the embedding. The last issue is remedied by a different approach in [EH]. In positive characteristics, canonicity was addressed by Abhyankar in [Ab6].

There are further results on a weaker type of resolution for surfaces, replacing the blowups in regular centers by different techniques. In [Za2] Zariski showed how to resolve a surface

over a not necessarily algebraically closed field of characteristic zero by so-called local uniformization which is based on valuation-theoretic methods. Abhyankar [Ab1] extended this to all algebraically closed fields of positive characteristics, and later [Ab2] extended several of the results to more general schemes whose closed points have perfect residue fields. In 1978 Lipman [Li] gave a very simple procedure to obtain resolution of singularities for arbitrary excellent two-dimensional schemes X in the following way: There is a finite sequence $X_n \to X_n \to \ldots X_1 \to X$ of proper surjective morphisms such that X_n is regular. This sequence is obtained by alternating normalization and blowing up in finitely many isolated singular points. But the processes of uniformization or normalization are not controlled in the sense of Theorem 0.1, i.e., not obtained by permissible blow-ups, and it is not known how to extend them to an ambient regular scheme Z like in Theorem 0.2. Neither is it clear how to get Theorem 0.3 by such a procedure. In particular, these weaker results were not sufficient for the mentioned applications in [JS]. This is even more the case for the weak resolution of singularities proved by de Jong [dJ].

It remains to mention that there are some results on weak resolution of singularities for threefolds over a field k by Zariski [Za3] (char(k) = 0), Abhyankar [Ab4] (k algebraically closed of characteristic $\neq 2, 3, 5$ – see also [Cu2]), and by Cossart and Piltant [CP1], [CP2] (k arbitrary), but this is not the topic of the present paper.

Our approach is roughly based on the strategy of Levi-Zariski used in [Za1], but more precisely follows the approach (still for surfaces) given by Hironaka in the paper [H6] cited above. The general strategy is very common by now: One develops certain invariants which measure the singularities and aims at constructing a sequence of blow-ups for which the invariants are non-increasing, and finally decreasing, so that in the end one concludes one has reached the regular situation. The choices for the centers of the blow-ups are made by considering the strata where the invariants are the same. In fact, one blows up 'the worst locus', i.e., the strata where the invariants are maximal, after possibly desingularising these strata. The main point is to show that the invariants do finally decrease. In characteristic zero this is done by a technique introduced by Hironaka in [H1], which is now called the method of maximal contact (see [AHV] and Giraud's papers [Gi2] and [Gi3] for some theoretic background), and an induction on dimension.

But it is known that the theory of maximal contact does not work in positive characteristic. There are some theoretic counterexamples in [Gi3], and some explicit counterexamples for threefolds in characteristic two by Narasimhan [Na1], see also [Co2] for an interpretation in our sense. It is not clear if the counterexamples in [Na2], for threefolds in any positive characteristic, can be used in the same way. But in section 15 of this paper, we show that maximal contact does not even exists for surfaces, in any characteristic, even if maximal contact is considered in the weakest sense. Therefore the strategy of proof has to be different, and we follow the one outlined in [H6], based on certain polyhedra (see below). That paper only considers the case of a hypersurface, but in another paper [H3] Hironaka develops the theory of these polyhedra for ideals with several generators, in terms of certain 'standard bases' for them (which also appear in [H1]). The introduction of [H3] expresses the hope that this theory of polyhedra will be useful for the resolution of singularities, at least for surfaces. Our paper can be seen as a fulfilment of this program.

In his fundamental paper [H1], Hironaka uses two important invariants for measuring the singularity at a point x of an arbitrary scheme X. The primary is the ν-invariant $\nu_x^*(X) \in$

$\mathbb{N}^\mathbb{N}$, and the secondary one is the dimension $e_x(X) \in \mathbb{N}$ (with $0 \leq e_x(X) \leq \dim(X)$) of the so-called directrix $\mathrm{Dir}_x(X)$ of X at x. Both only depend on the cone $C_x(X)$ of X at x. Hironaka proves that for a permissible blow-up $X' \to X$ and a point $x' \in X'$ with image $x \in X$ the ν-invariant is non-increasing: $\nu_{x'}^*(X') \leq \nu_x^*(X)$. If equality holds here (one says x' is near to x), then the (suitably normalized) e-invariant is non-decreasing. So the main problem is to show that there cannot be an infinite sequence of blow-ups with 'very near' points $x' \mapsto x$ (which means that they have the same ν- and e-invariants).

To control this, Hironaka in [H3] and [H6] introduces a tertiary, more complex invariant, the polyhedron associated to the singularity, which lies in $\mathbb{R}_{\geq 0}^e$. It depends not just on $C_x(X)$, but on the local ring $\mathcal{O}_{X,x}$ of X at x itself, and also on various choices: a regular local ring R having $\mathcal{O}_{X,x}$ as a quotient, a system of regular parameters $y_1, \ldots, y_r, u_1, \ldots, u_e$ for R such that $u_1, \ldots, u_e$ are 'parameters' for the directrix $\mathrm{Dir}_x(X)$, and equations $f_1, \ldots, f_m$ for $\mathcal{O}_{X,x}$ as a quotient of R (more precisely, a (u)-standard base of $J = \ker(R \to \mathcal{O}_{X,x})$). In the situation of Theorem 0.2, R is naturally given as $\mathcal{O}_{Z,x}$, but in any case, such an R always exists after completion, and the question of ruling out an infinite sequence of very near points only depends on the completion of $\mathcal{O}_{X,x}$ as well. In the case considered in section 13, it is not a single strictly decreasing invariant which comes out of these polyhedra, but rather the behavior of their shape which tells in the end that an infinite sequence of very near points cannot exits. This is sufficient for our purpose, but it might be interesting to find a strictly decreasing invariant also in this case. In the particular situation considered in [H6] (a hypersurface over an algebraically closed field), this was done by Hironaka; see also [Ha] for a variant.

As a counterpart to this local question, one has to consider a global strategy and the global behavior of the invariants, to understand the choice of permissible centers and the global improvement of regularity. Since the ν-invariants are nice for local computations, but their geometric behavior is not so nice, we use the Hilbert-Samuel invariant $H_{\mathcal{O}_{X,x}} \in \mathbb{N}^\mathbb{N}$ as an alternative primary invariant here. They were extensively studied by Bennett [Be], who proved similar non-increasing results for permissible blow-ups, which was then somewhat improved by Singh [Si1]. Bennett also defined global Hilbert-Samuel functions $H_X : X \to \mathbb{N}^\mathbb{N}$, which, however, only work well and give nice strata in the case of so-called weakly biequidimensional excellent schemes. We introduce a variant (Definition 1.28) which works for arbitrary (finite dimensional) excellent schemes. This solves a question raised by Bennett. The associated Hilbert-Samuel strata

$$X(\nu) = \{x \in X \mid H_X(x) = \nu\} \qquad \text{for} \quad \nu \in \mathbb{N}^\mathbb{N},$$

are then locally closed, with closures contained in $X(\geq \nu) = \{x \in X \mid H_X(x) \geq \nu\}$. In particular, $X(\nu)$ is closed for maximal ν (here $\mu \geq \nu$ if $\mu(n) \geq \nu(n)$ for all n).

Although our main results are for two-dimensional schemes, the major part of this paper is written for schemes of arbitrary dimension, in the hope that this might be useful for further investigations. Only in part of section 5 and in sections 10 through 14 we have to exploit some specific features of the low-dimensional situation. According to our understanding, there are mainly two obstructions against the extension to higher-dimensional schemes: The fact that in Theorem 2.14 (which gives crucial information on the locus of near points) one has to assume $\mathrm{char}(k(x)) = 0$ or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$, and the lack of good invariants of the polyhedra for $e > 2$, or of other suitable tertiary invariants in this case.

5

We have tried to write the paper in such a way that it is well readable for those who are not experts in resolution of singularities (like two of us) but want to understand some results and techniques and apply them in arithmetic or algebraic geometry. This is also a reason why we did not use the notion of idealistic exponents [H7]. This would have given the extra burden to recall this theory, define characteristic polyhedra of idealistic exponents, and rephrase the statements in [H5]. Equipped with this theory, the treatment of the functions defining the scheme and the functions defining the boundary would have looked more symmetric; on the other hand, the global algorithm clearly distinguishes these two.

We now briefly discuss the contents of the sections. In section 1 we discuss the primary and secondary invariants (local and global) of singularities mentioned above. In section 2 we discuss permissible blow-ups and the behavior of the introduced invariants for these, based on the fundamental results of Hironaka and Bennett (and Singh).

In section 3 we study similar questions in the setting of Theorem 0.3, i.e., in a 'log-situation' $X \subset Z$ where one has a 'boundary': a normal crossings divisor $\mathbb{B}$ on Z. We define a class of log-Hilbert-Samuel functions H_X^O, depending on the choice of a 'history function' $O : X \to \{$ subdivisors of $\mathbb{B}\}$ characterizing the 'old components' of $\mathbb{B}$ at $x \in X$. Then $H_X^O(x) = (H_X(x), n)$, where n is the number of old components at x. This gives associated log-Hilbert-Samuel strata

$$X(\widetilde{\nu}) = \{x \in X \mid H_X^O(x) = \widetilde{\nu}\} \qquad \text{for} \quad \widetilde{\nu} \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \ .$$

For a $\mathbb{B}$-permissible blow-up $X' \to X$, we relate the two Hilbert-Samuel functions and strata, and study some transversality properties.

In section 4 we extend this theory to the situation where we have just an excellent scheme X and no embedding into a regular scheme Z. It turns out that one can also define the notion of a boundary $\mathcal{B}$ on X: it is just a tuple $(B_1, \ldots, B_r)$ (or rather a multiset, by forgetting the ordering) of locally principal closed subschemes B_i of X. In the embedded situation $X \subset Z$, with a normal crossings divisor $\mathbb{B}$ on Z, the associated boundary $\mathcal{B}_X$ on X is just given by the traces of the components of $\mathbb{B}$ on X and we show that they carry all the information which is needed. Moreover, this approach makes evident that the constructions and strategies defined later are intrinsic and do not depend on the embedding. All results in section 3 can be carried over to section 4, and there is a perfect matching (see Lemma 4.21). We could have started with section 4 and derived the embedded situation in section 3 as a special case, but we felt it more illuminating to start with the familiar classical setting; moreover, some of the results in section 4 (and later in the paper) are reduced to the embedded situation, by passing to the local ring and completing (see Remark 4.22, Lemma 4.21 and the applications thereafter).

In section 5 we state the Main Theorems 5.6 and 5.9, corresponding to somewhat more general versions of Theorem 0.1 and 0.3, respectively, and we explain the strategy to prove them. Based on an important theorem by Hironaka (see [H2] Theorem (1,B) and the following remark), it suffices to find a succession of permissible blowups for which the Hilbert-Samuel invariants decrease. Although this principle seems to be well-known, and might be obvious for surfaces, we could not find a suitable reference and have provided a precise statement and (short) proof of this fact in any dimension (see Corollaries 5.18

6

and 5.19 for the case without boundary, and Corollaries 5.26 and 5.27 for the case with boundary). The problem arising is that the set of Hilbert functions is ordered by the total (or product) order ($H \leq H'$ iff $H(n) \leq H(n)$ for all n), and that with this order there are infinite decreasing sequences in the set $\mathbb{N}^{\mathbb{N}}$ of all functions $\nu : \mathbb{N} \to \mathbb{N}$. This is overcome by the fact that the subset of Hilbert functions of quotients of a fixed polynomial ring $k[x_1, \ldots, x_n]$ is a noetherian ordered set, see Theorem 15.3. After these preparations we define a canonical resolution sequence (see Remark 5.29 for the definition of so-called $\widetilde{\nu}$-eliminations, and Corollaries 5.18 and 5.19 for the definition of the whole resolution sequence out of this).

We point out that in Remark 5.29, we define these canonical resolution sequences, i.e., an explicit strategy for resolution of singularities for any dimension. It would be interesting to see if this strategy always works.

The proof of the finiteness of these resolution sequences for dimension two is reduced to two key theorems, Theorem 5.35 and 5.40, which exclude the possibility of certain infinite chains of blow-ups with near (or O-near) points. The key theorems concern only isolated singularities and hence only the local ring of X at a closed point x, and they hold for X of arbitrary dimension, but with the condition that the 'geometric' dimension of the directrix is ≤ 2 (which holds for $\dim(X) = 2$). As mentioned above, for this local situation we may assume that we are in an embedded situation.

As a basic tool for various considerations, we study a situation as mentioned above, where a local ring $\mathcal{O}$ (of arbitrary dimension) is a quotient R/J of a regular local ring R. In section 6 we discuss suitable systems $(y, u) = (y_1, \ldots, y_n, u_1, \ldots, u_e)$ of regular parameters for R and suitable families $f = (f_1, \ldots, f_m)$ of generators for J. A good choice for (y, u) is obtained if u is admissible for J (Definition 6.1) which means that $u_1, \ldots, u_e$ are affine parameters of the directrix of $\mathcal{O}$ (so that e is the e-invariant recalled above). We study valuations associated to (y, u) and initial forms (with respect to these valuations) of elements in J and their behavior under change of the system of parameters. As for f, in the special case that J is generated by one element (case of hypersurface singularities), any choice of $f = (f_1)$ is good. In general, some choices of $f = (f_1, \ldots, f_m)$ are better than the other. A favorable choice is a standard basis of J (Definition 1.17) as introduced in [H1]. In [H3] Hironaka introduced the more general notion of a (u)-standard base of J which is more flexible to work with and plays an important role in our paper.

In section 7 we recall, in a slightly different way, Hironaka's definition [H3] of the polyhedron $\Delta(f, y, u)$ associated to a system of parameters (y, u) and a (u)-standard basis f, and the polyhedron $\Delta(J, u)$ which is the intersection of all $\Delta(f, y, u)$ for all choices of y and f as above (with fixed u). We recall Hironaka's crucial result from [H3] that $\Delta(f, y, u) = \Delta(J, u)$ if u is admissible and (f, y, u) is what Hironaka calls well-prepared, namely normalized (Definition 7.12) and not solvable (Definition 7.13) at all vertices. Also, there is a certain process of making a given (f, y, u) normalized (by changing f) and not solvable (by changing y) at finitely many vertices, and at all vertices, if R is complete. One significance of this result is that it provides a natural way of transforming a (u)-standard base into a standard base under the assumption that u is admissible.

As explained above, it is important to study permissible blow-ups $X' \to X$ and near points $x' \in X'$ and $x \in X$. In this situation, to a system (f, y, u) at x we associate certain new systems (f', y', u') at x'. A key result proved in section 8 is that if f is a

standard base, then f' is a (u')-standard base. The next key result is that the chosen u' is admissible. Hence, by Hironaka's crucial result mentioned above, we can transform (f', y', u') into a system (g', z', u'), where g' is a standard base.

The Key Theorems 5.35 and 5.40 concern certain sequences of permissible blowups, which arise naturally from the canonical resolution sequence. We call them fundamental sequences of B-permissible blowups (Definition 5.34) and fundamental units of permissible blowups (Definition 5.38) and use them as a principal tool. These are sequences of B-permissible blowups

$$X_m \to \ldots \to X_1 \to X_0 = X \,,$$

where the first blowup is in a closed point $x \in X$ (the initial point), and where the later blowups are in certain maximal B-permissible centers C_i, which map isomorphically onto each other, lie above x, and consist of points near to x. For a fundamental sequence there is still a B-permissible center $C_m \subset X_m$ with the same properties; for a fundamental unit there is none, but only a chosen closed point $x_m \in X_m$ (the terminal point) which is near to x. In section 9 we study some first properties of these fundamental sequences. In particular we show a certain bound for the δ-invariant of the associated polyhedra. This suffices to show the first Key Theorem 5.35 (dealing with the case $e_x(X) = 1$), but is also used in section 14.

For the second Key Theorem 5.40 (dealing with the case $e_x(X) = 2$), one needs some more information on the (2-dimensional) polyhedra, in particular, some additional invariants. These are introduced in section 10. Then Theorem 5.40 is proved in the next three sections. It states that there is no infinite sequence

$$\ldots \to \mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$$

of fundamental units of blow-ups such that the closed initial points and terminal points match and are isolated in their Hilbert-Samuel strata. After some preparations in section 11, section 12 treats the case where the residue field extension $k(x')/k(x)$ is trivial (or separable). This is very much inspired by [H6], which however only treats the special situation of a hypersurface in a regular threefold over an algebraically closed field and does not contain proofs for all claims. Then section 13 treats the case where there occur inseparable residue field extensions $k(x')/k(x)$. This case was basically treated in [Co1] but we give a more detailed account and fill gaps in the original proof, with the aid of the results of section 8, and Giraud's notion of the ridge [Gi1], [Gi3] (faîte in French) a notion which generalizes the directrix.

In section 14, we show that maximal contact does not exist for surfaces in positive characteristic p. For each p a counterexample is given which then works for any field of that characteristic.

In section 15 we give a more algebro-geometric proof of the fact that it suffices to show how to eliminate the maximal Hilbert-Samuel stratum.

Finally, in section 16 we give a re-interpretation of the functoriality we obtain for our resolution, for arbitrary flat morphisms with regular fibers, and we apply this to show resolution for exccellent algebraic stacks with atlas of dimension at most two.

It will be clear from the above how much we owe to all the earlier work on resolution of singularities, in particular to the work of Hironaka which gave the general strategy but also the important tools used in this paper.

Convention: All schemes are assumed to be finite dimensional. Regular schemes are always assumed to be locally noetherian.

1 Basic invariants for singularities

In this section we introduce some basic invariants for singularities.

1.1 Invariants of graded rings and homogeneous ideals in polynomial rings

Let k be a field and $S = k[X_1, \ldots, X_n]$ be a polynomial ring with n variables. Let $S_\nu \subset S$ be the k-subspace of the homogeneous polynomials of degree n (including 0). Fix a homogeneous ideal $I \subseteq S$.

Definition 1.1 *For integers $i \geq 1$ we define $\nu^i(I) \in \mathbb{N} \cup \{\infty\}$ as the supremum of the $\nu \in \mathbb{N}$ satisfying the condition that there exist homogeneous $\varphi_1, \ldots, \varphi_{i-1} \in I$ such that*

$$S_\mu \cap I = S_\mu \cap < \varphi_1, \ldots, \varphi_{i-1} > \quad \text{for all } \mu < \nu.$$

By definition we have $\nu^1(I) \leq \nu^2(I) \leq \ldots$. We write

$$\nu^*(I) = (\nu^1(I), \nu^2(I), \ldots, \nu^m(I), \infty, \infty, \ldots)$$

and call it the ν-invariant of I. We have the following result (cf. [H1] Ch. III §1, Lemma 1).

Lemma 1.2 *Let $I = < \varphi_1, \ldots, \varphi_m >$ with homogeneous elements φ_i of degree ν_i such that:*

(i) $\varphi_i \notin < \varphi_1, \ldots, \varphi_{i-1} >$ for all $i = 1, \ldots, m$,

(ii) $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_m$.

Then we have

$$\nu^i(I) = \begin{cases} \nu_i & , \quad i \leq m, \\ \infty & , \quad i > m. \end{cases}$$

Definition 1.3 *Let $\varphi = (\varphi_1, \ldots, \varphi_m)$ be a system of homogeneous elements in S and $I \subset S$ be the ideal that it generates.*

(1) φ is weakly normalized if it satisfies the condition (i) of Lemma 1.2.

(2) φ is a standard base of I if it satisfies the conditions (i) and (ii) of Lemma 1.2.

We have the following easy consequence of 1.2.

Corollary 1.4 *Let $I \subset S$ be a homogeneous ideal and let $\psi = (\psi_1, \ldots, \psi_j)$ be a system of homogeneous elements in I which is weakly normalized.*

(1) The following conditions are equivalent:

(ii) $\deg \psi_i = \nu^i(I)$ for $i = 1, \ldots, j$.

(ii') For all $i = 1, \ldots, j$, ψ_i has minimal degree in I such that $\psi_i \notin < \psi_1, \ldots, \psi_{i-1} >$.

(2) If the conditions of (1) are satisfied, then ψ can be extended to a standard base of I.

By the lemma a standard base of I and $\nu^*(I)$ are obtained as follows:

Put $\nu_1 := \min\{\nu \mid \exists \, \varphi \in S_\nu \cap I \setminus \{0\}\}$ and pick $\varphi_1 \in S_{\nu_1} \cap I - \{0\}$.

Put $\nu_2 := \min\{\nu \mid \exists \, \varphi \in (S_\nu \cap I) \setminus (S_\nu \cap < \varphi_1 >)\}$ and pick $\varphi_2 \in (S_{\nu_2} \cap I) \setminus (S_{\nu_2} \cap < \varphi_1 >)$.

Proceed until we get $I = < \varphi_1, \ldots, \varphi_m >$. Then $(\varphi_1, \ldots, \varphi_m)$ is a standard base of I and $\nu^*(I) = (\nu_1, \ldots, \nu_m, \infty, \infty, \ldots)$.

Remark 1.5 *Let $\psi_1, \ldots, \psi_\ell \in I$ be homogeneous generators of I such that $\nu_1 \le \nu_2 \le \cdots \le \nu_\ell$, where $\nu_i = \deg(\psi_i)$. Then the above considerations show that*

$$(\nu_1, \nu_2, \ldots, \nu_\ell, \infty, \ldots) \le \nu^*(I) \,,$$

because a standard base of I is obtained by possibly omitting some of the ψ_i for $i \ge 2$.

In what follows, for a k-vector space (or a k-algebra) V and for a field extension K/k we write $V_K = V \otimes_k K$. From Lemma 1.2 the following is clear.

Lemma 1.6 *For the ideal $I_K \subseteq S_K$ we have*

$$\nu^*(I) = \nu^*(I_K) \,.$$

A second invariant is the directrix. By [H1] Ch. II §4, Lemma 10, we have:

Lemma 1.7 *Let K/k be a field extension. There is a smallest K-subvector space $\mathcal{T}(I, K) \subseteq (S_1)_K = \oplus_{i=1}^n KX_i$ such that*

$$(I_K \cap K[\mathcal{T}(I, K)]) \cdot S_K = I_K,$$

where $K[\mathcal{T}(I, K)] = \mathrm{Sym}_K(\mathcal{T}(I, K)) \subseteq \mathrm{Sym}_K((S_1)_K) = S_K$. In other words $\mathcal{T}(I, K) \subset (S_1)_K$ is the minimal K-subspace such that I_K is generated by elements in $K[\mathcal{T}(I, K)]$. For $K = k$ we simply write $\mathcal{T}(I) = \mathcal{T}(I, k)$.

Recall that $C(S/I) = \mathrm{Spec}(S/I)$ is called the cone of the graded ring S/I.

Definition 1.8 *For any field extension K/k, the closed subscheme*

$$\mathrm{Dir}_K(S/I) = \mathrm{Dir}(S_K/I_K) \subseteq C(S_K/I_K) = C(S/I) \times_k K$$

defined by the surjection $S_K/I_K \twoheadrightarrow S_K/\mathcal{T}(I, K)S_K$ is called the directrix of S/I in $C(S/I)$ over K. By definition

$$\mathrm{Dir}_K(S/I) \cong \mathrm{Spec}(\mathrm{Sym}_K((S_1)_K/\mathcal{T}(I, K))) \,.$$

We define

$$e(S/I)_K = \dim(\mathrm{Dir}_K(S/I)) = \dim(S) - \dim_K(\mathcal{T}(I, K)) = n - \dim_K(\mathcal{T}(I, K)) \,,$$

so that $\mathrm{Dir}_K(S/I) \cong \mathbb{A}_K^{e(S/I)_K}$, and simply write $e(S/I)$ for $e(S/I)_K$ with $K = k$.

Remark 1.9 (a) By definition $\mathrm{Dir}_K(S/I)$ is determined by the pair $S_K \supseteq I_K$, but indeed it has an intrinsic definition depending on A_K for $A = S/I$ only: Let $S_A = \mathrm{Sym}_k(A_1)$, which is a polynomial ring over k. Then the surjection $S_K \to A_K$ factors through the canonical surjection $\alpha_{A,K} : S_{A,K} \to A_K$, and the directrix as above is identified with the directrix defined via $S_{A,K}$ and $\ker(\alpha_{A,K})$. In this way $\mathrm{Dir}(A_K)$ is defined for any graded k-algebra A which is generated by elements in degree 1.

(b) Similarly, for any standard graded k-algebra A, i.e., any finitely generated graded k-algebra which is generated by A_1, we may define its intrinsic ν-invariant by $\nu^*(A) = \nu^*(\ker(\alpha_A))$, where $\alpha_A : \mathrm{Sym}_k(A_1) \twoheadrightarrow A$ is the canonical epimorphism. In the situation of Definition 1.1 we have

$$\nu^*(I) = (1, \ldots, 1, \nu^1(S/I), \nu^2(S/I), \ldots),$$

with $n - t$ entries of 1 before $\nu^1(S/I)$, where $t = \dim_k(A_1)$, and $\nu^1(S/I) > 1$.

(c) If X is a variable, then obviously $\nu^*(A[X]) = \nu^*(A)$ and $\nu^*(I[X]) = \nu^*(I)$ in the situation of 1.1. On the other hand, $\mathrm{Dir}_K(A[X]) \cong \mathrm{Dir}_K(A) \times_K \mathbb{A}_K^1$, i.e., $e_K(A[X]) = e_K(A) + 1$.

Lemma 1.10 Let the assumptions be as in 1.8.

(1) $e(S/I)_K \leq \dim(S/I)$.

(2) For field extensions $k \subset K \subset L$, we have $e(S/I)_K \leq e(S/I)_L$.

(3) The equality holds if one of the following conditions holds:

(i) L/K is separable (not necessarily algebraic).

(ii) $e(S/I)_K = \dim(S/I)$.

In particular it holds if K is a perfect field.

Proof The inequality in (1) is trivial, and (2) follows since $\mathcal{T}(I, K) \otimes L \subseteq \mathcal{T}(I, L)$, This in turn implies claim (3) for condition (ii). Claim (3) for condition (i) is proved in [H1] II, Lemma 12, p. 223, for arbitrary degree of transcendence. The case of a finite separable extension is easy: It is obviously sufficient to consider the case where L/K is finite Galois with Galois group G. Then, by Hilbert's theorem 90, for any L-vector space V on which G acts in a semi-linear way the canonical map $V^G \otimes_K L \to V$ is an isomorphism. This implies that $\mathcal{T}(I, L)^G \otimes_K L \xrightarrow{\sim} \mathcal{T}(I, L)$ and the claim follows. $\square$

Finally we recall the Hilbert function (not to be confused with the Hilbert polynomial) of a graded algebra. Let $\mathbb{N}$ be the set of the natural numbers (including 0) and let $\mathbb{N}^{\mathbb{N}}$ be the set of the functions $\nu : \mathbb{N} \to \mathbb{N}$. We endow $\mathbb{N}^{\mathbb{N}}$ with the product order defined by:

(1.1) $$\nu \geq \mu \Leftrightarrow \nu(n) \geq \mu(n) \quad \text{for any } n \in \mathbb{N}.$$

Definition 1.11 For a finitely generated graded k-algebra A its Hilbert function is the element of $\mathbb{N}^{\mathbb{N}}$ defined by

$$H^{(0)}(A)(n) = H(A)(n) = \dim_k(A_n) \quad (n \in \mathbb{N}).$$

For an integer $t \geq 1$ we define $H^{(t)}(A)$ inductively by:

$$H^{(t)}(A)(n) = \sum_{i=0}^{n} H^{(t-1)}(A)(i).$$

We note

$$H^{(t-1)}(A)(n) = H^{(t)}(A)(n) - H^{(t)}(A)(n-1) \leq H^{(t)}(n).$$

Remark 1.12 *(a) Obviously, for any field extension K/k we have*

$$H^{(0)}(A_K) = H^{(0)}(A).$$

(b) For a variable X we have

$$H^{(t)}(A[X]) = H^{(t+1)}(A).$$

(c) For any $\nu \in \mathbb{N}^{\mathbb{N}}$ and any $t \in \mathbb{N}$ define $\nu^{(t)}$ inductively by $\nu^{(t)}(n) = \sum_{i=0}^{n} \nu^{(t-1)}(i)$.

In a certain sense, the Hilbert function measures how far A is away from being a polynomial ring:

Definition 1.13 *Define the function $\Phi = \Phi^{(0)} \in \mathbb{N}^{\mathbb{N}}$ by $\Phi(0) = 1$ and $\Phi(n) = 0$ for $n > 0$. Define $\Phi^{(t)}$ for $t \in \mathbb{N}$ inductively as above, i.e., by $\Phi^{(t)}(n) = \sum_{i=0}^{n} \Phi^{(t-1)}(i)$.*

Then one has

$$\Phi^{(t)}(n) = H^{(0)}(k[X_1, \ldots, X_t])(n) = \binom{n+t-1}{n} \quad \text{for all } n \geq 0.$$

and:

Lemma 1.14 *Let A be a finitely generated graded algebra of dimension d over a field k, which is generated by elements in degree one (i.e., is a standard graded algebra).*

(a) *Then $H^{(0)}(A) \geq \Phi^{(d)}$, and equality holds if and only if $A \cong k[X_1, \ldots, X_d]$.*

(b) *For a suitable integer $m \geq 1$ one has $m\Phi^{(d)} \geq H^{(0)}$.*

(c) *If $H^{(0)}(A) = \Phi^{(N)}$ for $N \in \mathbb{N}$, then $N = d$.*

Proof (a): We may take a base change with an extension field K/k, and therefore may assume that k is infinite. In this case there is a Noether normalization $i : S = k[X_1, \ldots, X_d] \hookrightarrow A$ such that the elements $X_1, \ldots, X_d$ are mapped to A_1, the degree one part of A, see [Ku], Chap. II, Theorem 3.1 d). This means that i is a monomorphism of graded k-algebras. But then $H^{(0)}(A) \geq H^{(0)}(k[X_1, \ldots, X_d]) = \Phi^{(d)}$, and equality holds if and only if i is an isomorphism.

(b): Since A is a finitely generated S-module, it also has finitely many homogenous generators $a_1, \ldots, a_m$ of degrees $d_1, \ldots, d_m$. This gives a surjective map of graded S-modules

$$\oplus_{i=1}^{m} S[-d_i] \twoheadrightarrow A,$$

where $S[m]$ is S with grading $S[m]_n = S_{n+m}$, and hence

$$H(A)(n) \leq \sum_{i=1}^{m} \Phi^{(d)}(n - d_i) \leq m\Phi^{(d)}(n) \, .$$

(c) follows from (a) and (b), because $\Phi^{(d)}$ and $\Phi^{(N)}$ have different asymptotics for $N \neq d$.

$\square$

We shall need the following property.

Theorem 1.15 *Let k be a field and for $n \in \mathbb{N}$, let $HF_n \subset \mathbb{N}^{\mathbb{N}}$ be the set of all Hilbert functions $H(A)$ of all standard graded k-algebras A with $H(A)(1) \leq n$.*
(a) HF_n is independent of k.
(b) HF_n is a noetherian ordered set, i.e.,

> *(i) HF_n is well-founded, i.e., every strictly descending sequence $H_1 > H_2 > H_3 > \ldots$ in HF_n is finite.*

> *(ii) For every infinite subset $M \subset HF_n$ there are elements $H, H' \in M$ with $H < H'$.*

Proof If A is a standard graded k-algebra, then $H(A)(1) \leq n$ holds if and only if A is a quotient of $S_n = k[X_1, \ldots, X_n]$, i.e., $A = k[X_1, \ldots, X_n]/I$ for a homogeneous ideal I. On the other hand, it is known that $H(S_n/I) = H(S_n/I')$ for some monomial ideal $I' \subset S_n$, i.e., an ideal generated by monomials in the variables $X_1, \ldots, X_n$ (see [CLO] 6 §3, where this is attributed to Macaulay; to wit, one may take for I' the ideal of leading terms for I, with respect to the lexicographic order on monomials, loc. cit. Proposition 9). This proves (a). Moreover, for (b) we may assume that all considered Hilbert functions are of the form $H(S_n/I)$ for some monomial ideal $I \subset S_n$. Thus in (b) (ii) we are led to the consideration of an infinite set N of monomial ideals in S_n. But then the main theorem in [Macl] (Theorem 1.1) says that there are $I, I' \in N$ with $I \subsetneq I'$, so that $H(S_n/I) > H(S_n/I')$. For (b) (i) we may again assume that all H_i are of the form $H(S_n/I_i)$ for some monomial ideals $I_i \subset S_n$, and by [Macl] 1.1 that one finds an infinite chain $I_{i(1)} \subsetneq I_{i(2)} \subsetneq I_{i(3)} \subsetneq \ldots$ among these ideals, necessarily strictly increasing since the sequence of the $H_{i(\nu)}$ is, which is a contradiction because S_n is noetherian.

Remark 1.16 *(a) For the functions $H^{(1)}(A)$ instead of the functions $H(A)$ property (b) (i) was shown in [BM2] Theorem 5.2.1.*
(b) Our proof was modeled on [AP] Corollary 3.6, which is just formulated for monomial ideals. See also [AP] Corollary 1.8 for another argument that the set of monomial ideals in S_n, with respect to the reverse inclusion, is a noetherian ordered set. Finally, in [AH] it is shown, by more sophisticated methods, that even the set HF of all Hilbert functions is noetherian.

1.2 Invariants for local rings

For any ring R and any prime ideal $\mathfrak{p} \subset R$ we set

$$\mathrm{gr}_{\mathfrak{p}}(R) = \bigoplus_{n \geq 0} \mathfrak{p}^n / \mathfrak{p}^{n+1},$$

which is a graded algebra over $R/\mathfrak{p}$.

In what follows we assume that R is a noetherian regular local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. Moreover, assume that $R/\mathfrak{p}$ is regular. Then we have

$$\mathrm{gr}_\mathfrak{p}(R) = \mathrm{Sym}_{R/\mathfrak{p}}\big(\mathrm{gr}_\mathfrak{p}^1(R)\big), \quad \mathrm{gr}_\mathfrak{p}^1(R) = \mathfrak{p}/\mathfrak{p}^2.$$

where $\mathrm{Sym}_{R/\mathfrak{p}}(M)$ denotes the symmetric algebra of a free $R/\mathfrak{p}$-module M. Concretely, let $(y_1, \ldots, y_r, u_1, \ldots, u_e)$ be a system of regular parameters for R such that $\mathfrak{p} = (y_1, \ldots, y_r)$. Then $\mathrm{gr}_\mathfrak{p}(R)$ is identified with a polynomial ring over $R/\mathfrak{p}$:

$$\mathrm{gr}_\mathfrak{p}(R) = (R/\mathfrak{p})[Y_1, \ldots, Y_r] \quad \text{where } Y_i = y_i \mod \mathfrak{p}^2 \ (1 \leq i \leq r).$$

Fix an ideal $J \subset R$. In case $J \subset \mathfrak{p}$ we set

$$\mathrm{gr}_\mathfrak{p}(R/J) = \bigoplus_{n \geq 0} (\mathfrak{p}/J)^n/(\mathfrak{p}/J)^{n+1},$$

and define an ideal $In_\mathfrak{p}(J) \subset \mathrm{gr}_\mathfrak{p}(R)$ by the exact sequence

$$0 \to In_\mathfrak{p}(J) \to \mathrm{gr}_\mathfrak{p}(R) \to \mathrm{gr}_\mathfrak{p}(R/J) \to 0.$$

Note

$$In_\mathfrak{p}(J) = \bigoplus_{n \geq 0} (J \cap \mathfrak{p}^n + \mathfrak{p}^{n+1})/\mathfrak{p}^{n+1}.$$

For $f \in R$ and a prime ideal $\mathfrak{p} \subset R$ put

(1.2)
$$v_\mathfrak{p}(f) = \begin{cases} \max\{\nu \mid f \in \mathfrak{p}^\nu\} & , \quad f \neq 0, \\ \infty & , \quad f = 0 \end{cases}$$

called the *order* of f at $\mathfrak{p}$. For prime ideals $\mathfrak{p} \subset \mathfrak{q} \subset R$, we have the following semi-continuity result (cf. [H1] Ch.III §3):

(1.3)
$$v_\mathfrak{p}(f) \leq v_\mathfrak{q}(f) \quad \text{for } \forall f \in R.$$

Define the *initial form* of f at $\mathfrak{p}$ as

$$in_\mathfrak{p}(f) := f \mod \mathfrak{p}^{v_\mathfrak{p}(f)+1} \in \mathfrak{p}^{v_\mathfrak{p}(f)}/\mathfrak{p}^{v_\mathfrak{p}(f)+1} \in \mathrm{gr}_\mathfrak{p}(R).$$

In case $J \subset \mathfrak{p}$ we have

$$In_\mathfrak{p}(J) = \{in_\mathfrak{p}(f) \mid f \in J\}.$$

Definition 1.17 *(1) A system $(f_1, \ldots, f_m)$ of elements in J is a standard base of J, if $(in_\mathfrak{m}(f_1), \ldots, in_\mathfrak{m}(f_m))$ is a standard base of $In_\mathfrak{m}(J)$ in the polynomial ring $\mathrm{gr}_\mathfrak{m}(R)$.*

(2) We define $\nu^*(J, R)$ as the ν^*-invariant (cf. Definition 1.1) for $In_\mathfrak{m}(J) \subset \mathrm{gr}_\mathfrak{m}(R)$.

(2) The absolute ν^*-invariant $\nu^*(\mathcal{O})$ of a noetherian local ring $\mathcal{O}$ with maximal ideal $\mathfrak{n}$ is defined as the absolute ν^*-invariant (cf. Remark 1.9 (b)) $\nu^*(gr_\mathfrak{n}\mathcal{O})$.

It is shown in [H3] (2.21.d) that a standard base $(f_1, \ldots, f_m)$ of J generates J.

Next we define the directrix $\mathrm{Dir}(\mathcal{O})$ of any noetherian local ring $\mathcal{O}$. First we introduce some basic notations.

Let $\mathfrak{n}$ be the maximal ideal of $\mathcal{O}$, let x be the corresponding closed point of $\mathrm{Spec}(\mathcal{O})$, and let $k(x) = \mathcal{O}/\mathfrak{n}$ be the residue field at x. Define

$T(\mathcal{O}) = \mathrm{Spec}(\mathrm{Sym}_k(\mathfrak{n}/\mathfrak{n}^2))$: the Zariski tangent space of $\mathrm{Spec}(\mathcal{O})$ at x,

$C(\mathcal{O}) = \mathrm{Spec}(\mathrm{gr}_\mathfrak{n}(\mathcal{O})) = C(\mathrm{gr}_\mathfrak{n}(\mathcal{O}))$: the *(tangent) cone* of $\mathrm{Spec}(\mathcal{O})$ at x.

We note that $\dim C(\mathcal{O}) = \dim \mathcal{O}$ and that the map

$$\mathrm{Sym}_k(\mathfrak{n}/\mathfrak{n}^2) \twoheadrightarrow \mathrm{gr}_\mathfrak{n}(\mathcal{O})$$

gives rise to a closed immersion $C(\mathcal{O}) \hookrightarrow T(\mathcal{O})$.

Definition 1.18 *Let $K/k(x)$ be a field extension. Then the directrix of $\mathcal{O}$ over K,*

$$\mathrm{Dir}_K(\mathcal{O}) \subseteq C(\mathcal{O}) \times_{k(x)} K \subseteq T(\mathcal{O}) \times_{k(x)} K$$

is defined as the directrix $\mathrm{Dir}_K(\mathrm{gr}_\mathfrak{n}(\mathcal{O})) \subseteq C(\mathrm{gr}_\mathfrak{n}(\mathcal{O}))$ of $\mathrm{gr}_\mathfrak{n}(\mathcal{O})$ over K (cf. Remark 1.9 (a)). We set

$$e(\mathcal{O})_K = \dim(\mathrm{Dir}_K(\mathcal{O}))$$

and simply write $e(\mathcal{O})$ for $e(\mathcal{O})_K$ with $K = k(x)$.

Remark 1.19 *By definition, for R regular as above and an ideal $J \subset \mathfrak{m}$ we have*

$$\mathrm{Dir}_K(R/J) = \mathrm{Spec}(\mathrm{gr}_\mathfrak{m}(R)_K/\mathcal{T}(J,K)\mathrm{gr}_\mathfrak{m}(R)_K) \subseteq \mathrm{Spec}(\mathrm{gr}_\mathfrak{m}(R)_K/In_\mathfrak{m}(J)_K) = C(R/J)_K,$$

where $\mathcal{T}(J,K) = \mathcal{T}(In_\mathfrak{m}(J),K) \subseteq \mathrm{gr}_\mathfrak{m}^1(R)_K$ is the smallest K-sub vector space such that

$$(In_\mathfrak{m}(J)_K \cap K[\mathcal{T}(J,K)] \cdot \mathrm{gr}_\mathfrak{m}(R)_K = In_\mathfrak{m}(J)_K,$$

i.e., such that $In_\mathfrak{m}(J)_K$ is generated by elements in $K[\mathcal{T}(J,K)]$. Moreover

$$\mathrm{Dir}_K(R/J) \cong \mathrm{Sym}_K(\mathrm{gr}_\mathfrak{m}^1(R)_K/\mathcal{T}(J,K)) \subseteq T(R)_K.$$

For $K = k$ we simply write $\mathcal{T}(J) = \mathcal{T}(J,k)$.

Lemma 1.20 *Let the assumptions be as above.*

(1) $e(R/J)_K \leq \dim(R/J)$.

(2) For field extensions $k \subset K \subset L$, we have $e(R/J)_K \leq e(R/J)_L$. The equality holds if one of the following conditions holds:

   (i) L/K is separable (not necessarily algebraic).
   (ii) $e(R/J)_K = \dim(R/J)$.

Proof This follows from Lemma 1.10, because $\dim(R/J) = \dim C(R/J)$. $\square$

Definition 1.21 *We define $\overline{e}(R/J) = e(R/J)_{\overline{k}}$ for an algebraic closure $\overline{k}$ of k. By Lemma 1.20 we have $e(R/J)_K \leq \overline{e}(R/J) \leq \dim(R/J)$ for any extension K/k.*

For later use we note the following immediate consequence of the construction of a standard base below Corollary 1.4.

Lemma 1.22 *Let $\mathcal{T} = \mathcal{T}(J)$ be as in Lemma 1.7. There exists a standard base $(f_1, \ldots, f_m)$ of J such that $in_{\mathfrak{m}}(f_i) \in k[\mathcal{T}]$ for for all i.*

Finally, we define the Hilbert functions of a noetherian local ring $\mathcal{O}$ with maximal ideal $\mathfrak{m}$ and residue field F as those of the associated graded ring:

$$H_{\mathcal{O}}^{(t)}(n) = H^{(t)}(\mathrm{gr}_{\mathfrak{m}}(\mathcal{O}))\,.$$

Explicitly, the Hilbert function is the element of $\mathbb{N}^{\mathbb{N}}$ defined by

$$H_{\mathcal{O}}^{(0)}(n) = \dim_F(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \quad (n \in \mathbb{N}).$$

For an integer $t \geq 1$ we define $H_{\mathcal{O}}^{(t)}$ inductively by:

$$H_{\mathcal{O}}^{(t)}(n) = \sum_{i=0}^{n} H_{\mathcal{O}}^{(t-1)}(i).$$

In particular, $H_{\mathcal{O}}^{(1)}(n)$ is the length of the $\mathcal{O}$-module $\mathcal{O}/\mathfrak{m}^{n+1}$ and $H_{\mathcal{O}}^{(1)}$ is called the Hilbert-Samuel function of $\mathcal{O}$. We will sometimes call all functions of the form $H_{\mathcal{O}}^{(n)}$ Hilbert-Samuel functions. The Hilbert function measures how far away $\mathcal{O}$ is from being a regular ring:

Lemma 1.23 *Let $\mathcal{O}$ be a noetherian local ring of dimension d, and define $\Phi^{(t)}$ as in Definition 1.13. Then $H_{\mathcal{O}}^{(0)} \geq \Phi^{(d)}$, and equality holds if and only if $\mathcal{O}$ is regular.*

Proof (see also [Be] Theorem (2) and [Si2], property (4) on p. 46) Since $\dim(\mathcal{O}) = \dim(gr_{\mathfrak{m}}\mathcal{O})$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$, and since $\mathcal{O}$ is regular if and only if $gr_{\mathfrak{m}}\mathcal{O} \cong k[X_1, \ldots, X_d]$ where $k = \mathcal{O}/\mathfrak{m}$, this follows from Lemma 1.14.

For later purpose, we note the following facts.

Lemma 1.24 *Let the assumptions be as above and let $\overline{\mathcal{O}} = \mathcal{O}/\mathfrak{a}$ be a quotient ring of $\mathcal{O}$. Then $H_{\mathcal{O}}^{(t)} \geq H_{\overline{\mathcal{O}}}^{(t)}$ and the equality holds if and only if $\mathcal{O} = \overline{\mathcal{O}}$.*

Proof Let $\overline{\mathfrak{m}}$ be the maximal ideal of $\overline{\mathcal{O}}$. The inequality holds since the natural maps $\mathfrak{m}^{n+1} \to \overline{\mathfrak{m}}^{n+1}$ are surjective. Assume $H_{\mathcal{O}}^{(s)} = H_{\overline{\mathcal{O}}}^{(s)}$ for some $s \geq 0$. By Definition 1.11 it implies $H_{\mathcal{O}}^{(t)} = H_{\overline{\mathcal{O}}}^{(t)}$ for all $t \geq 0$, in particular for $t = 1$. This implies that the natural maps $\pi_n : \mathcal{O}/\mathfrak{m}^{n+1} \to \overline{\mathcal{O}}/\overline{\mathfrak{m}}^{n+1}$ are isomorphisms for all $n \geq 0$. Noting $\mathrm{Ker}(\pi_n) \simeq \mathfrak{a}/\mathfrak{a} \cap \mathfrak{m}^{n+1}$, we get $\mathfrak{a} \subset \bigcap_{n \geq 0} \mathfrak{m}^{n+1} = (0)$ and hence $\mathcal{O} = \overline{\mathcal{O}}$. $\square$

Lemma 1.25 *Let $\mathcal{O}$ and $\mathcal{O}'$ be noetherian local rings.*

(a) For all non-negative integers a and e one has

$$\dim \mathcal{O} \geq e \quad \Longleftrightarrow \quad H_{\mathcal{O}}^{(a)} \geq \Phi^{(e+a)}.$$

(b) For all non-negative integers a and b one has

$$H_{\mathcal{O}}^{(a)} \geq H_{\mathcal{O}'}^{(b)} \quad \Longrightarrow \quad \dim \mathcal{O} + a \geq \dim \mathcal{O}' + b.$$

Proof Let $d = \dim \mathcal{O}$. (a): If $d \geq e$, then we get $H_{\mathcal{O}}^{(a)} \geq \Phi^{(d+a)} \geq \Phi^{(e+a)}$, by Lemma 1.23. Conversely, assume $H_{\mathcal{O}}^{(a)} \geq \Phi^{(e+a)}$. Then form Lemma 1.14 we get

$$m\Phi^{(d+a)} \geq H_{\mathcal{O}}^{(a)} \geq \Phi^{(e+a)}$$

for some integer $m \geq 1$. If $d < e$ this is a contradiction, because of the asymptotic behavior of $\Phi^{(t)}$. Hence $d \geq e$.
For (b) let $d' = \dim \mathcal{O}'$. If $H_{\mathcal{O}}^{(a)} \geq H_{\mathcal{O}'}^{(b)}$ then

$$H_{\mathcal{O}}^{(a)} \geq H_{\mathcal{O}'}^{(b)} \geq \Phi^{(d'+b)}$$

and by (a) we have $d \geq d' + b - a$. (Note: If $d' + b - a < 0$, the statement is empty; if $e = d' + b - a \geq 0$, then we can apply (a).)

1.3 Invariants for schemes

Let X be a locally noetherian scheme.

Definition 1.26 *For any point $x \in X$ define*

$$\nu_x^*(X) = \nu^*(\mathcal{O}_{X,x}) \quad and \quad \mathrm{Dir}_x(X) = \mathrm{Dir}(\mathcal{O}_{X,x})$$

and

$$e_x(X) = e(\mathcal{O}_{X,x}) = \dim_{k(x)}(\mathrm{Dir}_x(X)), \quad \bar{e}_x(X) = \bar{e}(\mathcal{O}_{X,x}), \quad e_x(X)_K = e(\mathcal{O}_{X,x})_K,$$

where $K/k(x)$ is a field extension. If $X \subseteq Z$ is a closed subscheme of a (fixed) regular excellent scheme Z, define

$$\nu_x^*(X, Z) = \nu^*(J_x, \mathcal{O}_{Z,x}),$$

where $\mathcal{O}_{X,x} = \mathcal{O}_{Z,x}/J_x$. We also define

$$I\mathrm{Dir}_x(X) \subset \mathrm{gr}_{\mathfrak{m}_x}(\mathcal{O}_{Z,x})$$

to be the ideal defining $\mathrm{Dir}_x(X) \subset T_x(Z) = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}_x}(\mathcal{O}_{Z,x}))$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{Z,x}$.

We note that always

$$\mathrm{Dir}_x(X) \subseteq C_x(X) \subseteq T_x(X) \ (\subseteq T_x(Z) \text{ for } X \subset Z \text{ as above }),$$

where $C_x(X) = C(\mathcal{O}_{X,x})$ is called the tangent cone of X at x and $T_x(X) = T(\mathcal{O}_{X,x})$ is called the Zariski tangent space of X at x (similarly for Z).

Lemma 1.27 *Let X be a locally noetherian scheme.*

(1) *Let $\pi : X' \to X$ be a morphism with X' locally noetherian, and let $x' \in X'$ be a point lying over $x \in X$. Assume that π is quasi-étale at x' in the sense of Bennett [Be] (1.4), i.e., that $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is flat and $\mathfrak{m}_x \mathcal{O}_{X',x'} = \mathfrak{m}_{x'}$ where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ and $\mathfrak{m}_{x'} \subset \mathcal{O}_{X',x'}$ are the respective maximal ideals. (In particular, this holds if π is étale.) Then there is a canonical isomorphism*

(1.4)
$$C_{x'}(X') \cong C_x(X) \times_{k(x)} k(x')$$

so that $\nu_{x'}^*(X') = \nu_x^*(X)$. If $k(x')/k(x)$ is separable, then

$$\mathrm{Dir}_{x'}(X') \cong \mathrm{Dir}_x(X) \times_{k(x)} k(x'), \quad \text{and hence} \quad e_{x'}(X') = e_x(X).$$

Consider in addition that there is a cartesian diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\ i\ } & Z' \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\
X & \xrightarrow{\ i\ } & Z
\end{array}
$$

where i and i' are closed immersions, Z and Z' are regular excellent schemes and π_Z is quasi-étale at x'. Then

$$T_{x'}(Z) \cong T_x(Z) \times_{k(x)} k(x') \quad \text{and} \quad \nu_{x'}^*(X', Z') = \nu_x^*(X, Z).$$

(2) *Let $\pi : X' \to X$ be a morphism and let $x \in X$ and $x' \in X$ with $\pi(x') = x$. Assume that X' is locally noetherian, that π is flat and that the fibre $X_x' = X' \times_X x$ of π over x is regular at x' (e.g., assume that π is smooth around x'). Then there are non-canonical isomorphisms*

(1.5)
$$C_{x'}(X') \cong C_x(X) \times_{k(x)} C_{x'}(X_x') \cong C_x(X) \times_{k(x)} \mathbb{A}_{k(x)}^d,$$

where $d = \dim(\mathcal{O}_{X_x',x'}) = \mathrm{codim}_{X_x'}(x')$. Hence

$$\mathrm{Dir}_{x'}(X') \cong \mathrm{Dir}_x(X) \times_{k(x)} \mathbb{A}_{k(x')}^d, \quad e_{x'}(X') = e_x(X) + d \quad \text{and} \quad \nu_{x'}^*(X') = \nu_x^*(X).$$

(3) *Let X_0 be an excellent scheme, and let $f : X \to X_0$ be a morphism with X locally noetherian. Let $x_0 \in X_0$ and*

$$\hat{X}_0 = \mathrm{Spec}(\hat{\mathcal{O}}_{X_0,x_0}), \quad \hat{X} = X \times_{X_0} \hat{X}_0,$$

where $\hat{\mathcal{O}}_{X_0,x_0}$ is the completion of $\mathcal{O}_{X_0,x_0}$. Then for any $x \in X$ and any $\hat{x} \in \hat{X}$ lying over x, there are non-canonical isomorphisms

(1.6)
$$C_{\hat{x}}(\hat{X}) \cong C_x(X) \times_{k(x)} C_{\hat{x}}(\hat{X}_x) \cong C_x(X) \times_{k(x)} \mathbb{A}_{k(\hat{x})}^d,$$

where $\hat{X}_x = \hat{X} \times_X x$ is the fibre over x for the morphism $\pi : \hat{X} \to X$ and $d = \dim(\mathcal{O}_{\hat{X}_x,\hat{x}}) = \mathrm{codim}_{\hat{X}_x}(\hat{x})$. Hence

$$\mathrm{Dir}_{\hat{x}}(\hat{X}) \cong \mathrm{Dir}_x(X) \times_{k(x)} \mathbb{A}_{k(\hat{x})}^d, \quad e_{\hat{x}}(\hat{x}) = e_x(X) + d \quad \text{and} \quad \nu_{\hat{x}}^*(\hat{X}) = \nu_x^*(X).$$

19

Assume further that there is a commutative diagram

(1.7)
$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & Z \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
X_0 & \xrightarrow{\ i_0\ } & Z_0
\end{array}
$$

where Z_0 is a regular excellent scheme, Z is a regular locally noetherian scheme, and i_0 and i are closed immersions. Denote

$$
\hat{Z}_0 = \mathrm{Spec}(\hat{\mathcal{O}}_{Z_0,x_0}), \quad \hat{Z} = Z \times_{Z_0} \hat{Z}_0,
$$

where $\hat{\mathcal{O}}_{Z_0,x_0}$ is the completion of $\mathcal{O}_{Z_0,x_0}$, so that $\hat{X} = X \times_{Z_0} \hat{Z} = X \times_Z \hat{Z}$ can be regarded as a closed subscheme of $\hat{Z}$. Then $\hat{Z}$ is regular, and

$$
\nu_{\hat{x}}^*(\hat{X}, \hat{Z}) = \nu_x^*(X, Z).
$$

Proof (1): It suffices to show (1.4). Let $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ be a flat local morphism of local noetherian rings, with $\mathfrak{m}_A B = \mathfrak{m}_B$. Then we have isomorphisms

(1.8)
$$
\mathfrak{m}_A^n \otimes_A B \xrightarrow{\ \sim\ } \mathfrak{m}_B^n
$$

for all $n \geq 0$. In fact, this holds for $n = 0$, and, by induction and flatness of B over A, the injection $\mathfrak{m}_A^n \hookrightarrow \mathfrak{m}_A^{n-1}$ induces an injection

$$
\mathfrak{m}_A^n \otimes_A B \hookrightarrow \mathfrak{m}_A^{n-1} \otimes_A B \xrightarrow{\sim} \mathfrak{m}_B^{n-1},
$$

whose image is $\mathfrak{m}_A^n B = \mathfrak{m}_B^n$. From (1.8) we now deduce isomorphisms for all n

$$
(\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}) \otimes_{k_A} k_B \cong (\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}) \otimes_A B \xrightarrow{\ \sim\ } \mathfrak{m}_B^n/\mathfrak{m}_B^{n+1},
$$

where $k_A = A/\mathfrak{m}_A$ and $k_B = B/\mathfrak{m}_B$, and hence the claim (1.4).

(2): As for (1.6), consider the local rings $A = \mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$, and $A' = \mathcal{O}_{X',x'}$, with maximal ideal $\mathfrak{m}'$ and residue field $k' = A'/\mathfrak{m}'$. By assumption, π is flat and has regular fibers, and hence the same is true for the local morphism $A \to A'$ since it is obtained from π by localization in X and X'. $A \otimes_{\mathcal{O}_{X,x}} \hat{\mathcal{O}}_{X,x}$. Hence, by [Si2], Lemma (2.2), the closed subscheme $\mathrm{Spec}(A'/\mathfrak{m}A') \hookrightarrow \mathrm{Spec}(A')$ is permissible, i.e. it is regular and $\mathrm{gr}_{\mathfrak{m}A'}(A')$ is flat over $A'/\mathfrak{m}A'$. By [H1] Ch. II, p. 184, Proposition 1, we get a non-canonical isomorphism

$$
\mathrm{gr}_{\mathfrak{m}'}(A') \cong (\mathrm{gr}_{\mathfrak{m}A'}(A') \otimes_{A'} k') \otimes_{k'} \mathrm{gr}_{\mathfrak{m}'/\mathfrak{m}A'}(A'/\mathfrak{m}A').
$$

On the other hand, by flatness of A' over A we get canonical isomorphisms

$$
\mathfrak{m}^n A'/\mathfrak{m}^{n+1} A' \cong (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_A A'
$$

for all n. Hence we get an isomorphism $\mathrm{gr}_{\mathfrak{m}A'}(A') \otimes_{A'} k' \cong \mathrm{gr}_{\mathfrak{m}}(A) \otimes_A A'$ and the above isomorphism becomes

$$
\mathrm{gr}_{\mathfrak{m}'}(A') \cong \mathrm{gr}_{\mathfrak{m}}(A) \otimes_k \mathrm{gr}_{\mathfrak{m}'/\mathfrak{m}A'}(A'/\mathfrak{m}A'),
$$

which is exactly the first isomorphism in (1.6). Since $A'/\mathfrak{m}A' = \mathcal{O}_{X'_x,x'}$ is regular of dimension d, we have an ismorphism $\mathrm{gr}_{\mathfrak{m}'/\mathfrak{m}A'}(A'/\mathfrak{m}A') \cong k'[T_1, \ldots, T_d]$, which gives the second isomorphism in (1.6).

The statements for the directrix and the ν^*-invariant of X' at x' now follow from Remark 1.9 (c).

(3): The claims not involving Z_0 and Z follow by applying (2) to the morphism

$$\tilde{\pi} : \hat{X} = X \times_{X_0} \mathrm{Spec}(\mathcal{O}_{X_0,x_0}) \times_{\mathrm{Spec}(\mathcal{O}_{X,x})} \hat{X}_0 \to X \times_{X_0} \mathrm{Spec}(\mathcal{O}_{X_0,x_0}).$$

In fact, since X_0 is excellent, the morphism $\pi_0 : \mathcal{O}_{X_0,x_0} \to \hat{\mathcal{O}}_{X_0,x_0}$ is flat, with geometrically regular fibers, and so the same is true for $\tilde{\pi}$, which is a base change of π_0. Now consider the diagram (1.7). The above, applied to $Z \to Z_0$, gives isomorphisms

$$C_{\hat{x}}(\hat{Z}) \cong C_x(Z) \times_{k(x)} C_{\hat{x}}(\hat{Z}_x) \cong C_x(Z) \times_{k(x)} \mathbb{A}^d_{k(\hat{x})} \cong \mathbb{A}^{N+d}_{k(\hat{x})}$$

where $N = \dim(\mathcal{O}_{Z,x})$, because Z is regular and $\hat{Z}_x \cong \hat{X}_x$. By Lemmas 1.23 and 1.14 this implies that $\hat{Z}$ is regular at x. The final equality in (3) follows from the isomorphism $T_{\hat{x}}(\hat{Z}) \cong C_{\hat{x}}(\hat{Z})$ and Remark 1.9 (c). $\square$

Next we introduce Hilbert-Samuel functions for excellent schemes. Recall that an excellent scheme X is catenary so that for any irreducible closed subschemes $Y \subset Z$ of X, all maximal chains of irreducible closed subschemes

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_r = Z$$

have the same finite length r denoted by $\mathrm{codim}_Z(Y)$. For any irreducible closed subschemes $Y \subset Z \subset W$ of X we have

$$\mathrm{codim}_W(Y) = \mathrm{codim}_W(Z) + \mathrm{codim}_Z(Y).$$

Definition 1.28 *Let X be a locally noetherian catenary scheme (e.g., an excellent scheme), and fix an integer $N \geq \dim X$. (Recall that all schemes are assumed to be finite dimensional.)*

(1) For $x \in X$ let $I(x)$ be the set of irreducible components Z of X with $x \in Z$.

(2) Define the function $\phi_X := \phi_X^N : X \to \mathbb{N}$ by $\phi_X(x) = N - \psi_X(x)$, where

$$\psi_X(x) = min \left\{ \mathrm{codim}_Z(x) \mid Z \in I(x) \right\}.$$

(3) Define $H_X := H_X^N : X \to \mathbb{N}^{\mathbb{N}}$ as follows. For $x \in X$ let

$$H_X(x) = H_{\mathcal{O}_{X,x}}^{(\phi_X(x))} \in \mathbb{N}^{\mathbb{N}}.$$

(4) For $\nu \in \mathbb{N}^{\mathbb{N}}$ we define

$$X(\geq \nu) := \{x \in X \mid H_X(x) \geq \nu\} \quad and \quad X(\nu) := \{x \in X \mid H_X(x) = \nu\}.$$

The set $X(\nu)$ is called the Hilbert-Samuel stratum for ν.

By sending Z to its generic point η, the set $I(x)$ can be identified with the set of generic points (and hence the set of irreducible components) of the local ring $\mathcal{O}_{X,x}$. Therefore $\psi_X(x)$ depends only on $\mathcal{O}_{X,x}$, and $\phi_X(x)$ and $H_X(x)$ depend only on N and $\mathcal{O}_{X,x}$.

Remark 1.29 *(a) The choice of N does not matter. For sake of definiteness, we could have taken $N = \dim X$, and the readers are invited to do so, if they prefer. But when dealing with two different schemes X and X', the difference $\dim X - \dim X'$ would always appear. Note that we can even have $\dim U < \dim X$ for an open subscheme $U \subseteq X$. In the applications, there will usually be a common bound for the dimensions of the schemes considered, which we can take for N.*
(b) A more sophisticated way would be to consider the whole array

$$\underline{H}_X := (H_X^0, H_X^1, H_X^2, \ldots)$$

where, for a function $\varphi : \mathbb{N} \to \mathbb{N}$ and possibly negative $m \in \mathbb{Z}$, $\varphi^{(m)} : \mathbb{N} \to \mathbb{N}$ is defined inductively by the formulae

$$\varphi^{(0)} = \varphi \quad , \quad \varphi^{(m+1)}(n) = \sum_{i=0}^{n} \varphi^{(m)}(i) \quad , \quad \varphi^{(m-1)}(n) = \varphi^{(m)}(n) - \varphi^{(m)}(n-1) \,.$$

Then $(\varphi^{(m_1)})^{(m_2)} = \varphi^{(m_1+m_2)}$ for $m_1, m_2 \in \mathbb{Z}$, and for a second function $\psi : \mathbb{N} \to \mathbb{N}$ one has

$$\varphi \le \psi \implies \varphi^{(1)} \le \psi^{(1)} \,,$$

(in the product order (1.1)), but the converse does not hold in general. With these definitions we could use the ordering

$$\varphi \preceq \psi :\Longleftrightarrow \varphi^{(m)} \le \psi^{(m)} \quad \text{for all } m \gg 0$$

(not to be confused with the ordering $\varphi(n) \le \psi(n)$ for $n \gg 0$) in all applications below, and no choice of N appears. Note that $\varphi = \psi \iff \varphi^{(m)} = \psi^{(m)}$, for all $m \in \mathbb{Z}$.

In the rest of the paper, a choice of N will be assumed and often suppressed in the notations. We shall need the following semi-continuity property of ϕ_X.

Lemma 1.30 *(1) For $x, y \in X$ with $x \in \overline{\{y\}}$ one has $I(y) \subseteq I(x)$ and*

$$\phi_X(y) \le \phi_X(x) + \operatorname{codim}_{\overline{\{y\}}}(x) \,.$$

(2) For $y \in X$, there is a non-empty open subset $U \subseteq \overline{\{y\}}$ such that

$$I(x) = I(y) \quad \text{and} \quad \phi_X(y) = \phi_X(x) + \operatorname{codim}_{\overline{\{y\}}}(x)$$

for all $x \in U$.

Proof (1): The inclusion $I(y) \subseteq I(x)$ is clear, and for $Z \in I(y)$ one has

(1.9) $$\operatorname{codim}_Z(x) = \operatorname{codim}_Z(y) + \operatorname{codim}_{\overline{\{y\}}}(x) \,.$$

Thus $\psi_X(x) \le \psi_X(y) + \operatorname{codim}_{\overline{\{y\}}}(x)$, and the result follows.

(2): Let $Z_1, \ldots, Z_r$ be the irreducible components of X which do not contain y. Then we may take $U = \overline{\{y\}} \cap (X \setminus \bigcup_{i=1}^r Z_i)$. In fact, if $x \in U$, then $I(y) = I(x)$, and from (1.9) we get

$$\psi_X(x) = \psi_X(y) + \mathrm{codim}_{\overline{\{y\}}}(x).$$

Now we study the Hilbert-Samuel function H_X. The analogue of Lemma 1.14 (a) and Lemma 1.23 is the following, where N is as in 1.28.

Lemma 1.31 *For $x \in X$ one has $H_X(x) \geq \Phi^{(N)}$, and equality holds if and only if x is a regular point.*

Proof We have

$$H_X(x) = H_{\mathcal{O}_{X,x}}^{(N - \psi_X(x))} \geq \Phi^{(N - \psi_X(x) + \mathrm{codim}_X(x))} \geq \Phi^{(N)}.$$

Here the first inequality follows from Lemma 1.23, and the second inequality holds because $\mathrm{codim}_X(x) \geq \psi_X(x)$. If $H_X(x) = \Phi^{(N)}$, then all inequalities are equalities, and hence, again by Lemma 1.23, x is a regular point. Conversely, if x is regular, then there is only one irreducible component of X on which x lies, and hence $\mathrm{codim}_X(x) = \psi_X(x)$, so that the second inequality is an equality. Moreover, by Lemma 1.23, the first inequality is an equality.

Remark 1.32 *In particular, X is regular if and only if $X(\nu) = \emptyset$ for all $\nu \in \mathbb{N}^{\mathbb{N}}$ except for $\nu = \nu_X^{reg}$, where $\nu_X^{reg} = H_X(x)$ for a regular point x of X which is independent of the choice of a regular point, viz., equal to $\Phi^{(N)}$.*

We have the following important upper semi-continuity of the Hilbert-Samuel function.

Theorem 1.33 *Let X be a locally noetherian catenary scheme.*

(1) If $x \in X$ is a specialization of $y \in X$, i.e., $x \in \overline{\{y\}}$, then $H_X(x) \geq H_X(y)$.

(2) For any $y \in X$, there is a dense open subset U of $\overline{\{y\}}$ such that $H_X(x) = H_X(y)$ for all $x \in U$.

(3) The function H_X is upper semi-continuous, i.e., for any $\nu \in \mathbb{N}^{\mathbb{N}}$, $X(\geq \nu)$ is closed in X.

Proof (1): We have

$$H_X(x) = H_{\mathcal{O}_{X,x}}^{(\phi_X(x))} \geq H_{\mathcal{O}_{X,y}}^{(\phi_X(x) + \mathrm{codim}_{\overline{\{y\}}}(x))} \geq H_{\mathcal{O}_{X,y}}^{(\phi_X(y))} = H_X(y).$$

Here the first inequality holds by results of Bennett ([Be], Theorem (2)), as improved by Singh ([Si1], see p. 202, remark after Theorem 1), and the second holds by Lemma 1.30 (1).

(2): First of all, there is a non-empty open set $U_1 \subseteq \overline{\{y\}}$ such that $\overline{\{y\}} \subseteq X$ is permissible (see 2.1) at each $x \in U_1$ ([Be] Ch. 0, p. 41, (5.2)). Then

$$H_{\mathcal{O}_{X,x}}^{(0)} = H_{\mathcal{O}_{X,y}}^{(\mathrm{codim}_{\overline{\{y\}}}(x))}$$

for all $x \in U_1$, see [Be] Ch. 0, p. 33, (2.1.2). On the other hand, by Lemma 1.30 (2) there is a non-empty open subset $U_2 \subseteq \overline{\{y\}}$ such that $\phi_X(y) = \phi_X(x) + \operatorname{codim}_{\overline{\{y\}}}(x)$ for $x \in U_2$. Thus, for $x \in U = U_1 \cap U_2$ we have

$$H_X(x) = H_{\mathcal{O}_{X,x}}^{(\phi_X(x))} = H_{\mathcal{O}_{X,y}}^{(\phi_X(x)+\operatorname{codim}_{\overline{\{y\}}}(x))} = H_{\mathcal{O}_{X,y}}^{(\phi_X(y))} = H_X(y) \,.$$

By the following lemma, (3) is equivalent to the conjunction of (1) and (2).

Let X be a locally noetherian topological space which is Zariski, i.e., in which every closed irreducible subset admits a generic point. (For example, let X be a locally noetherian scheme.) Recall that a map $H : X \longrightarrow G$ into an ordered abelian group $(G, \leq)$ is called upper semi-continuous if the set

$$X_{\geq \nu} := X_{\geq \nu}^H := \{x \in X \mid H(x) \geq \nu\}$$

is closed for all $\nu \in G$. We note that this property is compatible with restriction to any topological subspace. In particular, if X is a scheme, H is also upper semi-continuous after restricting it to a subscheme or an arbitrary localization.

Lemma 1.34 *(a) The map H is upper semi-continuous if and only if the following holds.*

(1) If $x, y \in X$ with $x \in \overline{\{y\}}$, then $H(x) \geq H(y)$.

(2) For all $y \in X$ there is a dense open subset $U \subset \overline{\{y\}}$ such that $H(x) = H(y)$ for all $x \in U$.

(b) Assume H is upper semi-continuous. Then the set $X_\nu = \{x \in X \mid H(x) = \nu\}$ is locally closed for each $\nu \in G$, and its closure is contained in $X_{\geq \nu}$. In particular, X_ν is closed if ν is a maximal element in G. Moreover the set $X_{max} = \{x \in X \mid x \in X_\nu \text{ for some maximal } \nu \in G\}$ is closed.

(c) If X is noetherian, then H takes only finitely many values.

Proof We may restrict to the case where X is noetherian by taking an open covering (U_α) by noetherian subspaces. In fact, a subset of X is closed (resp. locally closed) if and only if this holds for the intersection with each U_α. Moreover, in (1), if $x \in U_\alpha$, then $y \in U_\alpha$. In (2) we may take $U \subset U_\alpha$ where $y \in U_\alpha$. If X is noetherian, then, by [Be], Ch. III, Lemma (1.1), $X_{\geq \nu}$ is closed if and only if the following conditions hold.

(1') If $y \in X_{\geq \nu}$, then every $x \in \overline{\{y\}}$ is in $X_{\geq \nu}$.

(2') If $y \notin X_{\geq \nu}$, then $(X - X_{\geq \nu}) \cap \overline{\{y\}}$ is open in $\overline{\{y\}}$.

Obviously, (1) above is equivalent to (1') for all ν. Assuming (1), and (2') for all ν, we get (2) by taking the following set for $\mu = H(y)$:

$$U = X(\mu) \cap \overline{\{y\}} = \cup_{\nu > \mu} (X - X_{\geq \nu}) \cap \overline{\{y\}} \,.$$

Conversely assume (1) and (2), and let $\nu \in G$. By noetherian induction we show that $X_{\geq \nu}$ is closed. Assume $\emptyset \neq Y \subset X$ is a minimal closed subset on which this is wrong. Since Y has only finitely many irreducible components, Y must be irreducible. Let η be

the generic point of Y, and let $H(\eta) = \mu$. By (2) there is a dense open subset $U \subset Z$ such that $H(x) = \mu$ for all $x \in U$. If $\nu = \mu$, then $Y_{\geq \nu} = Y$ by (1), so it is closed. If $\nu \neq \mu$, then $Y_{\geq \nu} \subset A$, hence is closed in A (minimality of Y), hence in Y – contradiction!

Similarly, we show that H only takes finitely many values on (noetherian) X, which shows (c). Assume that $Y \neq \emptyset$ is a minimal closed subset on which this is wrong; again Y is necessarily irreducible. Let η be the generic point of Y, and let U be as in (1), for η. By minimality of Y, H takes only finitely many values on the closed set $A = Y - U$ which is a contradiction. The first claim in (b) follows from the equality

$$X_\nu = X_{\geq \nu} - \cup_{\mu > \nu} X_\mu = X_{\geq \nu} - \cup_{\mu > \nu} X_{\geq \mu},$$

and the other claims follow easily (recall we may assume (c)).

Definition 1.35 *For X locally noetherian catenary, let $\Sigma_X := \{H_X(x)| \ x \in X\} \subset \mathbb{N}^{\mathbb{N}}$, and let Σ_X^{max} be the set of the maximal elements in Σ_X. The set*

$$(1.10) \qquad\qquad X_{max} = \underset{\nu \in \Sigma_X^{max}}{\cup} X(\nu)$$

is called the Hilbert-Samuel locus of X (note that (1.10) is a disjoint union).

By definition $X(\nu) \neq \emptyset$ if and only if $\nu \in \Sigma_X$. By Lemma 1.34 we have

Lemma 1.36 *Let X be a locally noetherian catenary scheme.*

(a) *For each $\nu \in \mathbb{N}^{\mathbb{N}}$ the set $X(\nu)$ is locally closed in X and its closure in X is contained in the closed set*

$$X(\geq \nu) = \underset{\mu \geq \nu}{\cup} X(\mu).$$

In particular $X(\nu)$ is closed in X if $\nu \in \Sigma_X^{max}$, and X_{max} is closed.

(b) *If X is noetherian, then Σ_X is finite.*

In the following we will regard the sets $X(\nu)$, $X(\geq \nu)$ and X_{max} as (locally closed) subschemes of X, endowed with the reduced subscheme structure.

Lemma 1.37 *Let X be a locally noetherian scheme.*

(1) *Let $\pi : X' \to X$ be a morphism with X' locally noetherian and let $x \in X$ and $x' \in X'$ with $\pi(x') = x$. Assume that π is flat and that the fibre $X'_x = X' \times_X x$ of π above x is regular at x' (e.g., assume that π is smooth around x', or that π is quasi-étale at x' in the sense of Lemma 1.27 (1)). Then one has*

$$(1.11) \qquad\qquad H^{(0)}_{\mathcal{O}_{X',x'}} = H^{(d)}_{\mathcal{O}_{X,x}} \quad and \quad \psi_{X'}(x') = \psi_X(x) + d,$$

where $d = \dim(\mathcal{O}_{X'_x, x'}) = \mathrm{codim}_{X'_x}(x')$. Hence, if X and X' are catenary, then $H_{X'}(x') = H_X(x)$. In particular, if π is flat with regular fibers (e.g., assume that π is étale or smooth), then

$$(1.12) \qquad \pi^{-1}(X(\nu)) = X'(\nu) \cong X(\nu) \times_X X' \qquad for \ all \ \nu \in \mathbb{N}^{\mathbb{N}}.$$

Moreover, X is regular at x if and only if X' is regular at x'.

(2) Let X_0 be an excellent scheme, and let $f : X \to X_0$ be a morphism which is locally of finite type. Let $x_0 \in X_0$ and

$$\hat{X}_0 = \mathrm{Spec}(\hat{\mathcal{O}}_{X_0,x_0}), \quad \pi : \hat{X} = X \times_{X_0} \hat{X}_0 \to X ,$$

where $\hat{\mathcal{O}}_{X_0,x_0}$ is the completion of $\mathcal{O}_{X_0,x_0}$. Then for any $x \in X$ and any $\hat{x} \in \hat{X}$ lying over x we have

(1.13)
$$H^{(0)}_{\mathcal{O}_{\hat{X},\hat{x}}} = H^{(d)}_{\mathcal{O}_{X,x}} \quad and \quad \psi_{\hat{X}}(\hat{x}) = \psi_X(x) + d ,$$

where
$$d = \mathrm{codim}_{\hat{X}_x}(\hat{x}), \quad \hat{X}_x = \hat{X} \times_X y.$$

In particular, $H_{\hat{X}}(\hat{x}) = H_X(x)$, and

(1.14)
$$\pi^{-1}(X(\nu)) = \hat{X}(\nu) \cong X(\nu) \times_X \hat{X} \quad for \ all \ \nu \in \mathbb{N}^{\mathbb{N}} .$$

For the proof we shall use the following two lemmas.

Lemma 1.38 Let $f : W \to Z$ be a flat morphism of schemes and let $w \in W$ and $z = f(w) \in Z$. Then we have

$$\mathrm{codim}_W(w) = \mathrm{codim}_Z(z) + \dim \mathcal{O}_{W_z,w}, \quad W_z = W \times_Z z.$$

In particular w is a generic point of W if and only if $f(w)$ is a generic point of Z and $\mathrm{codim}_{W_z}(w) = 0$.

Proof See [EGA IV], 2, (6.1.2).

Lemma 1.39 $f : W \to Z$ be a quasi-finite morphism of excellent schemes and let $w \in W$ and $z = f(w) \in Z$. Assume that W and Z are irreducible. Then we have

$$\mathrm{codim}_W(w) = \mathrm{codim}_Z(z).$$

Proof See [EGA IV], 2, (5.6.5).

Proof of Lemma 1.37 (1): The first equality in (1.11) follows from Lemma 1.27 (1.6). We show the second equality. We may assume that X is reduced. Since $\pi : X' \to X$ is flat, Lemma 1.38 implies that if η' is a generic point of X' with $x' \in \overline{\{\eta'\}}$, then $\eta = \pi(\eta')$ is a generic point of X with $x \in \overline{\{\eta\}}$. Moreover, if ξ is a generic point of X with $x \in Z := \overline{\{\xi\}}$, then there exists a generic point ξ' of X' such that $x' \in \overline{\{\xi'\}}$. Indeed one can take a generic point ξ' of $\pi^{-1}(Z)$ such that $x' \in \overline{\{\xi'\}}$. Then Lemma 1.38 applied to the flat morphism $\pi_Z : X' \times_X Z \to Z$ implies that $\pi(\xi') = \xi$ and ξ' is of codimension 0 in $\pi_Z^{-1}(\xi) = \pi^{-1}(\xi)$ and hence ξ' is a generic point of X'. This shows that we may consider the case where X is integral, and it suffices show the following.

Claim 1.40 Assume X is integral. Let W be an irreducible component of X' containing x'. Then
$$\mathrm{codim}_W(x') = \mathrm{codim}_X(x) + d, \quad d = \mathrm{codim}_{X'_x}(x').$$

Since the question is local at x, we may assume $X = \mathrm{Spec}(A)$ for the local noetherian ring $A = \mathcal{O}_{X,x}$. If A is normal, then by [EGA IV], 2, (6.5.4) the local ring $B = \mathcal{O}_{X',x'}$ is normal as well, because the fibers of π are regular and hence normal. Therefore both A and B are integral, we have $\psi_X(x) = \dim(\mathcal{O}_{X,x})$ and $\psi_{X'}(x') = \dim(\mathcal{O}_{X',x'})$, and the claim follows from Lemma 1.38.

In general let $\tilde{A}$ be the normalization of A, let $\tilde{X} = \mathrm{Spec}(\tilde{A})$ (the normalization of X) and consider the cartesian diagram

$$
\begin{array}{ccc}
\tilde{X}' & \xrightarrow{\ g'\ } & X' \\
{\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\
\tilde{X} & \xrightarrow{\ g\ } & X
\end{array}
$$

in which the vertical morphisms are flat. Since X is excellent, the morphism g is finite and surjective, and so is its base change g'. We claim that a point $\xi \in \tilde{X}'$ is generic if and only if $\eta' = g'(\xi)$ is a generic point of X'. Indeed, by Lemma 1.38, ξ is generic if and only if it maps to the generic point $\tilde{\eta}$ of $\tilde{X}$ and is of codimension zero in the fibre $\tilde{\pi}^{-1}(\tilde{\eta})$. Since the fibres over $\tilde{\eta}$ and the generic point η of X are the same, this is the case if and only if η' satisfies the corresponding properties for π, i.e., if η' is a generic point of X'. Now let W be an irreducible component of X' containing x'. By the last claim together with the surjectivity of g', there is an irreducible component $\tilde{W}$ of $\tilde{X}'$ dominating W. Since g' is finite, it is a closed map, and so we have even $g'(\tilde{W}) = W$. Thus there is a point $z' \in \tilde{W}$ with $g'(z') = x'$. If $z = \tilde{\pi}(z')$, then $g(z) = y$ and

$$
(1.15) \qquad\qquad \mathrm{codim}_{\tilde{X}'_z}(z') = \mathrm{codim}_{X'_x}(x') = d \,,
$$

where $\tilde{X}'_z = X'_x \times_x z$. We now conclude

$$
\mathrm{codim}_W(x') = \mathrm{codim}_{W'}(z') = \mathrm{codim}_{\tilde{X}}(z) + d = \mathrm{codim}_X(x) + d \,,
$$

where the first (resp. the third) equality follows from Lemma 1.39 applied to the finite morphism $\tilde{W} \to W$ (resp. $\tilde{X} \to X$) and the second equality follows from (1.15) and the first case of the proof noting that $\tilde{X}$ is normal. This shows Claim 1.40 and completes the proof of (1.11). The next claims, including the first equality in (1.12), are now obvious. To see the isomorphism of schemes in (1.12), note the following. For any morphism $f : S' \to S$ of schemes and any subscheme $T \subset S$, $T \times_S S'$ is identified with a subscheme of S', whose underlying topological space is homeomorphic with $f^{-1}(T)$. So we have to show that $X(\nu) \times_X X'$ is reduced. But this follows from [EGA IV], 2, (3.3.5) and the flatness of π. The last statement in (1) now follows from Remark 1.32. (See also [EGA IV], 2, (6.5.2) for another proof.)

Finally, (2) follows from applying (1) to the morphism $\hat{X}' \to X' \times_X \mathrm{Spec}(\mathcal{O}_{X,x})$ (compare the proof of Lemma 1.27, deduction of (3) from (2)).

Remark 1.41 *Bennett [Be] defined global Hilbert-Samuel functions by $H_{X,x}^{(i)} = H_{\mathcal{O}_{X,x}}^{(i+d(x))}$, where $d(x) = \dim(\overline{\{x\}})$, and showed that these functions have good properties for so-called weakly biequidimensional excellent schemes. By looking at the generic points and the closed points one easily sees that this function coincides with our function $H_X(x)$ (for $N = \dim X$) if and only if X is biequidimensional.*

2 Permissible blow-ups

We discuss some fundamental results on the behavior of the ν^*-invariants, the e-invariants, and the Hilbert-Samuel functions under permissible blow-ups.

Let X be a locally noetherian scheme and let $D \subset X$ be a closed reduced subscheme. Let $I_D \subset \mathcal{O}_X$ the ideal sheaf of D in X and $\mathcal{O}_D = \mathcal{O}_X/I_D$. Put

$$\mathrm{gr}_{I_D}(\mathcal{O}_X) = \bigoplus_{t \geq 0} I_D^t/I_D^{t+1}.$$

Definition 2.1 *(1) X is normally flat along D at $x \in D$ if the stalk $\mathrm{gr}_{I_D}(\mathcal{O}_X)_x$ of $\mathrm{gr}_{I_D}(\mathcal{O}_X)$ at x is a flat $\mathcal{O}_{D,x}$-module. X is normally flat along D if X is normally flat along D at all points of D, i.e., if $\mathrm{gr}_{I_D}(\mathcal{O}_X)$ is a flat $\mathcal{O}_D$-module.*

(2) $D \subset X$ is permissible at $x \in D$ if D is regular at x, if X is normally flat along D at x, and if D contains no irreducible component of X containing x. $D \subset X$ is permissible if D is permissible at all points of D.

(3) The blowup $\pi_D : B\ell_D(X) \to X$ in a permissible center $D \subset X$, is called a permissible blowup.

For a closed subscheme $D \subset X$, the normal cone of $D \subset X$ is defined as:

$$C_D(X) = \mathrm{Spec}(\mathrm{gr}_{I_D}(\mathcal{O}_X)) \to D.$$

Theorem 2.2 *(1) There is a dense open subset $U \subset D$ such that X is normally flat along D at all $x \in U$.*

(2) The following conditions are equivalent:

(i) X is normally flat along D at $x \in D$.

(ii) $T_x(D) \subset \mathrm{Dir}_x(X)$ and the natural map $C_x(X) \to C_D(X) \times_D x$ induces an isomorphism $C_x(X)/T_x(D) \overset{\sim}{\to} C_D(X) \times_D x$, where $T_x(D)$ acts on $C_x(X)$ by the addition in $T_x(X)$.

Assume that in addition that X is a closed subscheme of a regular locally noetherian scheme Z. Let $x \in D$ and set $R = \mathcal{O}_{Z,x}$ with the maximal ideal $\mathfrak{m}$ and $k = R/\mathfrak{m}$. Let $J \subset R$ (resp. $\mathfrak{p} \subset R$) be the ideal defining $X \subset Z$ (resp. $D \subset Z$). Then the following conditions are equivalent to the conditions (i) and (ii) above.

(iii) Let $u : \mathrm{gr}_{\mathfrak{p}}(R) \otimes_{R/\mathfrak{p}} k \to \mathrm{gr}_{\mathfrak{m}}(R)$ be the natural map. Then $In_{\mathfrak{m}}(J)$ is generated in $\mathrm{gr}_{\mathfrak{m}}(R)$ by $u(In_{\mathfrak{p}}(J))$.

(iv) There exists a standard base $f = (f_1, \ldots, f_m)$ of J such that $v_{\mathfrak{m}}(f_i) = v_{\mathfrak{p}}(f_i)$ for all $i = 1, \ldots, m$ (cf. (1.2)) .

(3) Let $\pi : X' \to X$ be a morphism, with X' locally noetherian, and let $D' = X' \times_X D$, regarded as a closed subscheme of X'. Let $x' \in D'$ and $x = \pi(x') \in D$, and assume that π is flat at x' and that the fiber X'_x over x is regular at x'. Then X' is normally flat along D' at x' if and only if X is normally flat along D at x. Moreover, D' is regular (resp. permissible) at x' if and only D is regular (resp. permissible) at x.

Proof (1) follows from [H1] Ch. I Theorem 1 on page 188 and (2) from [Gi1], II §2, Theorem 2.2 and 2.2.3 on page II-13 to II-15. (3): Since $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is flat, one has

$$\mathrm{gr}_{D'}(\mathcal{O}_{X'})_{x'} \cong \mathrm{gr}_D(\mathcal{O}_X)_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'} \cong \mathrm{gr}_D(\mathcal{O}_X)_x \otimes_{\mathcal{O}_{D,x}} \mathcal{O}_{D',x'}.$$

Thus the first claim follows from the following general fact: If $A \to B$ is a flat morphism of local rings (hence faithfully flat), and M is an A-module, then M is flat over A if and only if $M \otimes_A B$ is flat over B. Since $D' \to D$ is flat and its fiber over x is the same as for π, hence regular, the next claim (on regularity) follows from 1.37 (1) (or [EGA IV], 2, (6.5.2)). The last claim (on nowhere density) follows from the flatness of π by the arguments used at the beginning of the proof of Lemma 1.37: If η' is a generic point of X' with $x' \in \overline{\{\eta'\}} \subset D' = \pi^{-1}(D)$, then $\eta = \pi(\eta')$ is a generic point of X with $x \in \overline{\{\eta\}} \subset D$. Conversely, if start with η as in the latter situation, there is a generic point η' of X' mapping to η with $x' \in \overline{\{\eta'\}}$. But then $\eta' \in D'$.

There is a numerical criterion for normal flatness due to Bennett, which we carry over to our setting: Let the assumption be as in the beginning of this section. Assume in addition that X is catenary (e.g., let X be excellent) and let $H_X(x)$ be as in Definition 1.28.

Theorem 2.3 *Assume that D is regular. Let $x \in D$, and let y be the generic point of the component of D which contains x. Then the following conditions are equivalent.*

(1) X is normally flat along D at x.

(2) $H^{(0)}_{\mathcal{O}_{X,x}} = H^{(\mathrm{codim}_Y(x))}_{\mathcal{O}_{X,y}}$, where $Y = \overline{\{y\}}$, the closure of y in X.

(3) $H_X(x) = H_X(y)$.

Proof The equivalence of (1) and (2) was proved by Bennett ([Be], Theorem (3)). The rest is a special case of the following lemma.

Lemma 2.4 *Let $x, y \in X$ with $x \in \overline{\{y\}}$. Then the following are equivalent.*

(1) $H^{(0)}_{\mathcal{O}_{X,x}} = H^{(\mathrm{codim}_{\overline{\{y\}}}(x))}_{\mathcal{O}_{X,y}}$.

(2) $H_X(x) = H_X(y)$.

If these conditions hold, then $I(x) = I(y)$ and $\psi_X(x) = \psi_X(y) + \mathrm{codim}_Y(x)$.

Proof By the definition of the considered functions, for the equivalence of (1) and (2) it suffices to show that either of (1) and (2) implies

(2.1) $$\phi_X(y) = \phi_X(x) + \mathrm{codim}_Y(x),$$

where $Y = \overline{\{y\}}$. Assume (2). By Lemma 1.25 we have

$$\dim \mathcal{O}_{X,x} + \phi_X(x) = \dim \mathcal{O}_{X,y} + \phi_X(y).$$

On the other hand,

$$\dim \mathcal{O}_{X,x} = \mathrm{codim}_X(x) \geq \mathrm{codim}_Y(x) + \mathrm{codim}_X(y) = \mathrm{codim}_Y(x) + \dim \mathcal{O}_{X,y}.$$

29

Thus we get

$$\phi_X(y) = \dim \mathcal{O}_{X,x} + \phi_X(x) - \dim \mathcal{O}_{X,y} \geq \phi_X(x) + \operatorname{codim}_Y(x),$$

which implies (2.1) by Lemma 1.30.

Next assume (1). Let $\mathcal{O} = \mathcal{O}_{X,x}$ and let $\mathfrak{p} \subset \mathcal{O}$ be the prime ideal corresponding to y. It suffices to show that $\mathfrak{p}$ contains all minimal prime ideals of $\mathcal{O}$. In fact, this means that y is contained in all irreducible components of X which contain x, i.e., that $I(y) = I(x)$. Since, for any irreducible $Z \in I(y)$ we have

$$\operatorname{codim}_Z(x) = \operatorname{codim}_Z(y) + \operatorname{codim}_Y(x),$$

we deduce the equality $\psi_X(x) = \psi_X(y) + \operatorname{codim}_Y(x)$ and hence (2.1). At the same time we have proved the last claims of the lemma. As for the claim on $\mathcal{O}$ and $\mathfrak{p}$ let

$$(0) = \mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_r$$

be a reduced primary decomposition of the zero ideal of $\mathcal{O}$, and let $\mathfrak{p}_i = \operatorname{Rad}(\mathfrak{P}_i)$ be the prime ideal associated to $\mathfrak{P}_i$. Then the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ contains all minimal prime ideals of $\mathcal{O}$, and it suffices to show that $\mathfrak{p}$ contains all ideals $\mathfrak{P}_i$ (Note that an ideal $\mathfrak{a}$ is contained in $\mathfrak{p}$ if and only if $\operatorname{Rad}(\mathfrak{a})$ is contained in $\mathfrak{p}$). Assume the contrary. We may assume $\mathfrak{P}_1 \not\subseteq \mathfrak{p}$. Put $\mathcal{O}' = \mathcal{O}/\mathfrak{Q}$ with $\mathfrak{Q} = \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_r$ and let $\mathcal{O}_{\mathfrak{p}}$ (resp. $\mathcal{O}'_{\mathfrak{p}}$) be the localization of $\mathcal{O}$ (resp. $\mathcal{O}'$) at $\mathfrak{p}$ (resp. $\mathfrak{p}\mathcal{O}'$). Then $\mathcal{O}_{\mathfrak{p}} = \mathcal{O}'_{\mathfrak{p}}$ and $\mathcal{O}/\mathfrak{p} = \mathcal{O}'/\mathfrak{p}\mathcal{O}'$ and

$$H_{\mathcal{O}}^{(0)} \geq H_{\mathcal{O}'}^{(0)} \geq H_{\mathcal{O}'_{\mathfrak{p}}}^{(d)} = H_{\mathcal{O}_{\mathfrak{p}}}^{(d)},$$

where $d = \operatorname{codim}_Y(x) = \dim \mathcal{O}/\mathfrak{p} = \dim \mathcal{O}'/\mathfrak{p}\mathcal{O}'$. The first inequality follows from Lemma 1.24 and the second inequality follows from [Be], Theorem (2). Hence (2) implies $H_{\mathcal{O}}^{(1)} = H_{\mathcal{O}'}^{(1)}$. By Lemma 1.24 this implies $\mathcal{O} = \mathcal{O}'$ so that $\mathfrak{Q} = 0$, contradicting the assumption that the primary decomposition is reduced. $\square$

The above criterion is complemented by the following observation.

Lemma 2.5 *Let X be connected. If there is an irreducible component $Z \subseteq X$ such that $H_X(x) = H_X(y)$ for all $x, y \in Z$ (i.e., $Z \subseteq X(\nu)$ for some $\nu \in \mathbb{N}^{\mathbb{N}}$), then $Z = X$.*

Proof We have to show that X is irreducible. Assume not. Then there exists an $x \in X$ which is contained in two different irreducible components. Let $\mathcal{O} = \mathcal{O}_{X,x}$ be the local ring of X at x, let

(2.2) $$< 0 > = \mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_r$$

be a reduced primary decomposition of the zero ideal of $\mathcal{O}$, and let $\mathfrak{p}_i = \operatorname{Rad}(\mathfrak{P}_i)$ be the prime ideal associated to $\mathfrak{P}_i$. By assumption we have $r \neq 1$. Then there is an i such that the trace of Z in $\mathcal{O}$ is given by $\mathfrak{p}_i$. Let $\eta_i \in \operatorname{Spec}(\mathcal{O}/\mathfrak{P}_i) \subset \operatorname{Spec}(\mathcal{O}) \subset X$ be the generic point of Z (corresponding to $\mathfrak{p}_i$). By assumption we have

$$H_{\mathcal{O}_{X,\eta_i}}^{(N)} = H_X(\eta_i) = H_X(x) = H_{\mathcal{O}}^{(N-\psi_X(x))}$$

where N is as in Definition 1.28, because $\psi_X(\eta_i) = 0$. On the other hand, we have

$$(2.3) \qquad H^{(N)}_{(\mathcal{O}/\mathfrak{P}_i)_{\mathfrak{p}_i}} \leq H^{(N-c)}_{\mathcal{O}/\mathfrak{P}_i} \leq H^{(N-c)}_{\mathcal{O}} \leq H^{(N-\psi_X(x))}_{\mathcal{O}},$$

where $c := \dim(\mathcal{O}/\mathfrak{P}_i) = \dim(\mathcal{O}/\mathfrak{p}_i) = \operatorname{codim}_{\overline{\{\eta_i\}}}(x) = \operatorname{codim}_Z(x)$. Here the first inequality holds by the results of Bennett/Singh recalled in the proof of Theorem 1.33 (1), the second inequality follows from Lemma 1.24, and the last inequality holds since $\psi_X(x) \leq c$. Now, since $\mathfrak{p}_i$ is a minimal prime ideal and (2.2) is reduced, we have an isomorphism

$$\mathcal{O}_{X,\eta_i} = \mathcal{O}_{\mathfrak{p}_i} \cong (\mathcal{O}/\mathfrak{P}_i)_{\mathfrak{p}_i}.$$

Therefore we have equalities in (2.3). By the other direction of Lemma 1.24 we conclude that $\mathcal{O} = \mathcal{O}/\mathfrak{P}_i$, i.e., $\mathfrak{P}_i = 0$, which is a contradiction if $r \neq 1$.$\square$

We now prove a semi-continuity property for $e_x(X)$, the dimension of the directrix at x.

Theorem 2.6 *Let X be an excellent scheme, or a scheme embeddable into a regular scheme, and let $x, y \in X$ such that $D = \overline{\{y\}}$ is permissible and $x \in D$. Then*

$$e_y(X) \leq e_x(X) - \dim(\mathcal{O}_{D,y}).$$

The question only depends on the local ring $\mathcal{O}_{X,x}$, and by Lemma 1.27 (3) we may assume that we consider the spectrum of a complete local ring and the closed point x in it. Moreover, by the Cohen structure theory of complete local noetherian rings (see [EGA IV], 1, (0.19.8.8)) every such ring is a quotient of a (complete) regular local ring. Therefore Theorem 2.6 is implied by the following result.

Theorem 2.7 *Let R be a regular local ring with maximal ideal $\mathfrak{m}$. Let $J \subset R$ be an ideal. Let $\mathfrak{p} \subset R$ be a prime ideal such that $J \subset \mathfrak{p}$ and that $\operatorname{Spec}(R/\mathfrak{p}) \subset \operatorname{Spec}(R/J)$ is permissible. Let $R_{\mathfrak{p}}$ be the localization of R at $\mathfrak{p}$ and $J_{\mathfrak{p}} = JR_{\mathfrak{p}}$. Then*

$$e(R_{\mathfrak{p}}/J_{\mathfrak{p}}) \leq e(R/J) - \dim(R/\mathfrak{p}).$$

Proof Set $A = R/\mathfrak{p}$ and let K be the quotient field of A. Set $M_K = M \otimes_A K$ for an A-module M. By definition there exists a K-subspace $V = I\operatorname{Dir}(R_{\mathfrak{p}}/J_{\mathfrak{p}}) \subset \operatorname{gr}^1_{\mathfrak{p}}(R)_K$ of dimension s such that $\dim_K(V) = \dim(R_{\mathfrak{p}}) - e(R_{\mathfrak{p}}/J_{\mathfrak{p}})$ and

$$(2.4) \qquad In_{\mathfrak{p}}(J)_K = (K[V] \cap In_{\mathfrak{p}}(J)_K) \cdot \operatorname{gr}_{\mathfrak{p}}(R)_K.$$

Lemma 2.8 *Assume that there exist free A-modules $T, S \subset \operatorname{gr}^1_{\mathfrak{p}}(R)$ such that $\operatorname{gr}^1_{\mathfrak{p}}(R) = T \oplus S$ and $V = T_K$. Let $u : \operatorname{gr}_{\mathfrak{p}}(R) \otimes_{R/\mathfrak{p}} k \to \operatorname{gr}_{\mathfrak{m}}(R)$ be the natural map. Then $u(T) \supset I\operatorname{Dir}(R/J) \cap \operatorname{gr}^1_{\mathfrak{m}}(R)$.*

Note that the assumption of the lemma is satisfied if $\dim(A) = 1$, by the theory of elementary divisors. Theorem 2.6 is a consequence of the conclusion of Lemma 2.8 by noting

$$\dim_k(u(T)) = \dim_k(T \otimes_A k) = \dim_K(V) = \dim(R_{\mathfrak{p}}) - e(R_{\mathfrak{p}}/J_{\mathfrak{p}})$$

so that the lemma finishes the proof of the theorem in case $\dim(R/\mathfrak{p}) = 1$.

We show Lemma 2.8. The assumption of the lemma implies

$$
(2.5) \qquad \mathrm{gr}_{\mathfrak{p}}(R) = \mathrm{Sym}_A(\mathrm{gr}_{\mathfrak{p}}^1(R)) = A[T] \otimes_A A[S],
$$

where $A[T] = \mathrm{Sym}_A(T)$ (resp. $A[S] = \mathrm{Sym}_A(S)$) is the sub A-algebra of $\mathrm{gr}_{\mathfrak{p}}(R)$ generated by T (resp. S).

Claim 2.9

$$
In_{\mathfrak{p}}(J) = (A[T] \cap In_{\mathfrak{p}}(J)) \cdot \mathrm{gr}_{\mathfrak{p}}(R).
$$

By Theorem 2.2(2)(iii) the claim implies that $In_{\mathfrak{m}}(J)$ is generated by $u(A[T]) \cap In_{\mathfrak{m}}(J)$, which implies Lemma 2.8. Thus it suffices to show the claim. Note

$$
(2.6) \qquad (K[V] \cap In_{\mathfrak{p}}(J)_K) \cap \mathrm{gr}_{\mathfrak{p}}(R) = (A[T] \cap In_{\mathfrak{p}}(J))_K \cap \mathrm{gr}_{\mathfrak{p}}(R) = A[T] \cap In_{\mathfrak{p}}(J).
$$

Indeed (2.5) implies $K[V] \cap \mathrm{gr}_{\mathfrak{p}}(R) = A[T]$ and the flatness of $\mathrm{gr}_{\mathfrak{p}}(R/J) = \mathrm{gr}_{\mathfrak{p}}(R)/In_{\mathfrak{p}}(J)$ implies $In_{\mathfrak{p}}(J)_K \cap \mathrm{gr}_{\mathfrak{p}}(R) = In_{\mathfrak{p}}(J)$.

Take any $\phi \in In_{\mathfrak{p}}(J)$. By (2.4) and (2.6) there exists $c \in A$ such that

$$
c\phi = \sum_{1 \le i \le m} a_i \psi_i, \quad \psi_i \in A[T] \cap In_{\mathfrak{p}}(J), \ a_i \in \mathrm{gr}_{\mathfrak{p}}(R),
$$

Choosing a basis $Z_1, \ldots, Z_r$ of the A-module S, (2.5) allows us to identify $\mathrm{gr}_{\mathfrak{p}}(R)$ with the polynomial ring $A[T][Z] = A[T][Z_1, \ldots, Z_r]$ over $A[T]$. Then, expanding

$$
\phi = \sum_{B \in \mathbb{Z}_{\ge 0}^r} Z^B \phi_B, \quad a_i = \sum_{B \in \mathbb{Z}_{\ge 0}^r} Z^B a_{i,B}, \quad \text{with } \phi_B \in A[T], \ a_{i,B} \in A[T],
$$

we get

$$
c\phi_B = \sum_{1 \le i \le m} a_{i,B} \psi_i \quad \text{for any } B \in \mathbb{Z}_{\ge 0}^r.
$$

Since $\sum_{1 \le i \le m} a_{i,B} \psi_i \in A[T] \cap In_{\mathfrak{p}}(J)$, this implies $\phi_B \in A[T] \cap In_{\mathfrak{p}}(J)$ by (2.6) so that $\phi \in (A[T] \cap In_{\mathfrak{p}}(J)) \cdot \mathrm{gr}_{\mathfrak{p}}(R)$. This completes the proof of the claim.

To finish the proof of Theorem 2.6, it suffices to reduce it to the case $\dim(R/\mathfrak{p}) = 1$ as remarked below Lemma 2.8. Assume $\dim(R/\mathfrak{p}) > 1$ and take a prime ideal $\mathfrak{q} \supset \mathfrak{p}$ such that $R/\mathfrak{q}$ is regular of dimension 1. Let $R_{\mathfrak{q}}$ be the localization of R at $\mathfrak{q}$ and $J_{\mathfrak{q}} = JR_{\mathfrak{q}}$. Noting

$$
\mathrm{gr}_{\mathfrak{p}R_{\mathfrak{q}}}(R_{\mathfrak{q}}/J_{\mathfrak{q}}) \simeq \mathrm{gr}_{\mathfrak{p}}(R/J) \otimes_{R/\mathfrak{p}} R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}},
$$

the assumption implies that $\mathrm{Spec}(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) \subset \mathrm{Spec}(R_{\mathfrak{q}}/JR_{\mathfrak{q}})$ is permissible. By the induction on $\dim(R)$, we have

$$
e(R_{\mathfrak{p}}/J_{\mathfrak{p}}) \le e(R_{\mathfrak{q}}/J_{\mathfrak{q}}) - \dim(R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}}) = e(R_{\mathfrak{q}}/J_{\mathfrak{q}}) - (\dim(R/\mathfrak{p}) - 1)
$$

(Note that any regular local ring is catenary [EGA IV], 1, (0.16.5.12)). Hence we are reduced to show

$$
e(R_{\mathfrak{q}}/J_{\mathfrak{q}}) \le e(R/J) - \dim(R/\mathfrak{q}) = e(R_{\mathfrak{q}}/J_{\mathfrak{q}}) \le e(R/J) - 1.
$$

This completes the proof of Theorem 2.6.

Bennett and Hironaka proved results about the behavior of Hilbert-Samuel functions in permissible blow-ups which are fundamental in resolution of singularities. We recall these results (as well as some improvements by Singh) and carry them over to our setting.

Theorem 2.10 *Let X be an excellent scheme, or a scheme which is embeddable in a regular scheme. Let $D \subset X$ be a permissible closed subscheme, and let*

$$\pi_X : X' = B\ell_D(X) \to X \quad \text{and}$$

be the blowup with center D. Take any points $x \in D$ and $x' \in \pi_X^{-1}(x)$ and let $\delta = \delta_{x'/x} := \operatorname{trdeg}_{k(x)}(k(x'))$. Then:

(1) $H^{(\delta)}_{\mathcal{O}_{X',x'}} \leq H^{(0)}_{\mathcal{O}_{X,x}}$ *and* $\phi_{X'}(x') \leq \phi_X(x) + \delta$ *and* $H_{X'}(x') \leq H_X(x)$.

(2) $H_{X'}(x') = H_X(x) \quad \Leftrightarrow \quad H^{(\delta)}_{\mathcal{O}_{X',x'}} = H^{(0)}_{\mathcal{O}_{X,x}}$.

(3) *If the equalities in (2) hold, then $\phi_{X'}(x') = \phi_X(x) + \delta$, the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$ is injective, and $I(x') = \{Z' \mid Z \in I(x)\}$ where Z' denotes the strict transform of $Z \in I(x)$.*

(4) *If the equalities in (2) hold, then, for any field extension $K/k(x')$ one has*

$$e_{x'}(X')_K \leq e_x(X)_K - \delta_{x'/x}.$$

Assume in addition that $X \hookrightarrow Z$ is a closed immersion into a regular scheme Z, and let

$$\pi_Z : Z' = B\ell_D(Z) \to Z$$

be the blowup with center D. Then:

(5) $\nu^*_{x'}(X', Z') \leq \nu^*_x(X, Z)$ *in the lexicographic order.*

(6) $H^{(\delta)}_{\mathcal{O}_{X',x'}} = H^{(0)}_{\mathcal{O}_{X,x}} \quad \Leftrightarrow \quad \nu^*_{x'}(X', Z') = \nu^*_x(X, Z)$.

Proof In a slightly weaker form, viz., $H^{(1+\delta)}_{\mathcal{O}_{X',x'}} \leq H^{(1)}_{\mathcal{O}_{X,x}}$, the first inequality in (1) was proved by Bennett ([Be] Theorem (2)), and Hironaka gave a simplified proof ([H4] Theorem I). In the stronger form above it was proved by Singh ([Si1], Remark after Theorem 1). For the second inequality, since $\dim(X) = \dim(X')$, it suffices to show

(2.7) $\psi_X(x) \leq \psi_{X'}(x') + \delta$.

Note that X is universally catenary by assumption (because excellent schemes and closed subschemes of regular schemes are universally catenary (by definition and by [EGA IV], 2, (5.6.4), respectively). Let $Y_1, \dots, Y_r$ be the irreducible components of X and let Y'_i be the strict transform of Y_i under π_X. Then $Y'_1, \dots, Y'_r$ are the irreducible components of X'. If $x' \in Y'_i$, then $x \in Y_i$ and [EGA IV], 2, (5.6.1) implies (note that $\mathcal{O}_{Y_i,x}$ is universally catenary since X is)

(2.8) $\operatorname{codim}_{Y_i}(x) = \operatorname{codim}_{Y'_i}(x') + \delta$.

(2.7) follows immediately from this.

The last inequality in (1) now follows:

$$(2.9) \qquad H_{X'}(x') = H_{\mathcal{O}_{X',x'}}^{(\phi_{X'}(x'))} \leq H_{\mathcal{O}_{X',x'}}^{(\phi_X(x)+\delta)} \leq H_{\mathcal{O}_{X,x}}^{(\phi_X(x))} = H_X(x).$$

Claims (5) and (6) were proved by Hironaka in [H4] Theorems II and III, hence it remains to show (2), (3) and (4).

As for (2), assume $H_{X'}(x') = H_X(x)$, i.e., that equality holds everywhere in (2.9). To show $H_{\mathcal{O}_{X',x'}}^{(\delta)} = H_{\mathcal{O}_{X,x}}^{(0)}$, it suffices to show that

$$(2.10) \qquad \phi_{X'}(x') = \phi_X(x) + \delta.$$

Let $d = \dim(\mathcal{O}_{X,x})$ and $d' = \dim(\mathcal{O}_{X',x'})$. By Lemma 1.25(b) the assumption implies

$$(2.11) \qquad d' + \phi_{X'}(x') = d + \phi_X(x).$$

On the other hand, by Lemma 1.25(b), the inequality $H_{\mathcal{O}_{X',x'}}^{(\delta)} \leq H_{\mathcal{O}_{X,x}}^{(0)}$ from (1) implies

$$(2.12) \qquad d' + \delta \leq d.$$

From (2.11) and (2.12) we deduce $\phi_X(x) + \delta \leq \phi_{X'}(x')$, which implies (2.10), in view of (1).

Conversely assume $H_{\mathcal{O}_{X',x'}}^{(\delta)} = H_{\mathcal{O}_{X,x}}^{(0)}$. To show $H_{X'}(x') = H_X(x)$, it again suffices to show (2.10). We have to show

$$(2.13) \qquad \psi_X(x) = \psi_{X'}(x') + \delta.$$

In view of (2.8) and with the notations there, it suffices to show that $x' \in Y_i'$ if $x \in Y_i$, i.e., the third claim of (3). For this, by the lemma below, it suffices to show the injectivity of $\mathcal{O} = \mathcal{O}_{X,x} \to \mathcal{O}_{X',x'} = \mathcal{O}'$, i.e., the second claim of (3).

Lemma 2.11 *Let $f : A \to B$ be a homomorphism of noetherian rings. Assume that there is a minimal prime ideal of A, which does not lie in the image of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Then f is not injective.*

Proof Let $(0) = \mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_r$ be a primary decomposition of the zero ideal in B, and let $\mathfrak{p}_i = \mathrm{Rad}(\mathfrak{P}_i)$, which is a prime ideal. Assume $A \to B$ is injective. Then, with $\mathfrak{Q}_i = \mathfrak{P}_i \cap A$, $(0) = \mathfrak{Q}_1, \cap \ldots \cap \mathfrak{Q}_r$ is a primary decomposition in A. Therefore $\{\mathfrak{q}, \ldots, \mathfrak{q}_r\}$, with $\mathfrak{q}_i = \mathrm{Rad}(\mathfrak{Q}_i) = \mathfrak{p}_i \cap A$, is the set of all associated prime ideals of A and hence contains all minimal prime ideals of A, see [Ku] VI Theorems 2.18 and 2.9. This contradicts the assumption.

Before we go on, we note the following consequence of Lemmas 1.27 and 1.37: For all claims of Theorem 2.10, we may, via base change, assume that $X = \mathrm{Spec}(\mathcal{O}_{X,x})$, and that $\mathcal{O}_{X,x}$ is a quotient of a regular local ring R. In fact, the last property holds either by assumption, or X is excellent and we may base change with the completion $\hat{\mathcal{O}}$ of $\mathcal{O} = \mathcal{O}_{X,x}$ which then is a quotient of a regular ring by Cohen's structure theorem. It also suffices to check the injectivity of $\mathcal{O} \to \mathcal{O}'$ after this base change, because $\mathcal{O} \to \hat{\mathcal{O}}$ is faithfully flat.

Now we use the results of Hironaka in [H4]. First consider the case $k(x') = k(x)$, where $\delta = 0$. Let $\mathfrak{p} \subset \mathcal{O} = \mathcal{O}_{X,x}$ be the prime ideal corresponding to D, let $A = gr_{\mathfrak{p}}(\mathcal{O}_{X,x}) \otimes_{\mathcal{O}/\mathfrak{p}} k(x)$, and let $F = \pi_X^{-1}(x) \subset X'$ be the scheme theoretic fibre of $\pi_X : X' \to X$ over x. Then $F = \mathrm{Proj}(A)$, and by [H4] (4.1) we have inequalities

$$(2.14) \qquad H_{\mathcal{O}_{X',x'}}^{(1+\delta)} \leq H_{\mathcal{O}_{F,x'}}^{(2+\delta+s)} \leq H_{C_{X,D,x}}^{(1+s)} = H_{\mathcal{O}_{X,x}}^{(1)} \,,$$

where $s = \dim \mathcal{O}_{D,x}$ and $C_{X,D,x} = \mathrm{Spec}(A)$. By our assumption, these are all equalities. Moreover, if $\mathcal{O} = R/J$ for a regular local ring R with maximal ideal $\mathfrak{m}$, then there is a system of regular parameters $(x_0, \ldots, x_r, y_1, \ldots, y_s)$ of R such that the ideal P of D in R is generated by $x_0, \ldots, x_r$. If $X_0, \ldots, X_r, Y_1, \ldots, Y_s$ are the initial forms of $x_0, \ldots, x_r, y_1, \ldots, y_s$ (with respect to $\mathfrak{m}$), then the fiber E over x in the blowup Z' of $Z = \mathrm{Spec}(R)$ in the center D is isomorphic to $\mathrm{Proj}(k[X.])$ for $k = k(x)$ and the polynomial ring $k[X.] = k[X_0, \ldots, X_r]$, and there is a homogeneous ideal $I \subset k[X.]$ such that $A = k[X.]/I$ and $F \subset E$ identifies with the canonical immersion $\mathrm{Proj}(k[X.]/I) \subset \mathrm{Proj}(k[X.])$. We may assume that $x' \in F \subset E$ lies in the standard open subset $D_+(X_0) \subset \mathrm{Proj}(k[X.]) = E$. If furthermore $k(x') = k(x) = k$ as we assume, then $\delta = 0$, and by equality in the middle of (2.14) and the proof of lemma 8 in [H4] there is a graded k-algebra B and an isomorphism of graded $k[X_0]$-algebras $A \cong B[X_0]$.

On the other hand, let $\mathfrak{n}$ be the maximal ideal of $\mathcal{O}$, and let $z_1, \ldots, z_s \in \mathfrak{n}$ be elements whose images $Z_1, \ldots, Z_s \in \mathfrak{n}/\mathfrak{n}^2$ form a basis of $\mathfrak{n}/\mathfrak{p} + \mathfrak{n}^2$. Then one has an isomorphism of graded algebras

$$A[Z_1, \ldots, Z_s] \cong gr_{\mathfrak{n}}(\mathcal{O})$$

induced by the canonical map $A \to gr_{\mathfrak{n}}(\mathcal{O})$, because $\mathfrak{p} \subset \mathcal{O}$ is permissible ([H1] II 1. Proposition 1). This shows that the image of X_0 in $\mathfrak{n}/\mathfrak{n}^2$, is not zero and not a zero divisor in $gr_{\mathfrak{n}}(\mathcal{O})$. Therefore the image of $x_0 \in R$ in $\mathcal{O}$ is not zero and not a zero divisor in $\mathcal{O}$.

Now we claim that every element in the kernel of $\mathcal{O} \to \mathcal{O}'$ is annihilated by a power of x_0, which then gives a contradiction if this kernel is non-zero. Let $R' = \mathcal{O}_{Z',x'}$ be the monoidal transform of R with center P corresponding to $x' \in Z'$, where $\mathfrak{p} = P/J$, and let $J' \subset R'$ be the strict transform of J, so that $\mathcal{O}' := \mathcal{O}_{X',x'} = R'/J'$. Then, since

$$R' = R[\tfrac{x_1}{x_0}, \ldots, \tfrac{x_r}{x_0}]_{<\tfrac{x_1}{x_0}, \ldots, \tfrac{x_r}{x_0}>} \quad \text{and} \quad PR' = x_0 R' \,,$$

it follows from [H1] III Lemma 6, p. 216 that there are generators $f_1, \ldots, f_m$ of J and natural numbers $n_1, \ldots, n_m$ such that J' is generated by $f_1/x_0^{n_1}, \ldots, f_m/x_0^{n_m}$. Evidently this implies that every element in the kernel of $\mathcal{O} \to \mathcal{O}'$ is annihilated by a power of x_0.

Now consider the case that the residue field extension $k(x')/k(x)$ is arbitrary. We reduce to the residually rational case $(k(x') = k(x))$ by the same technique as in [H4]. As there, one may replace X by $\mathrm{Spec}\mathcal{O}_{X,x}$, and consider a cartesian diagram

$$
\begin{array}{ccccc}
x' & X' & \xleftarrow{\ i'\ } & \tilde{X}' & \tilde{x}' \\
\downarrow & \pi_X \downarrow & & \downarrow \pi_{\tilde{X}} & \downarrow \\
x & X & \xleftarrow{\ i\ } & \tilde{X} & \tilde{x} \,,
\end{array}
$$

where i is a faithfully flat monogenic map which is either finite or the projection $\tilde{X} = \mathbb{A}^1_X \to X$, and $\tilde{f}$ is the blow-up of $\tilde{D} = i^{-1}(D)$, which is again permissible. Moreover, $\tilde{x} \in \tilde{X}$ is the generic point of $i^{-1}(x)$ such that $k(\tilde{x})$ is a monogenic field extension of $k(x)$, and there is a point $\tilde{x}' \in \tilde{X}'$ which maps to $x' \in X'$ and $\tilde{x} \in \tilde{X}$ and satisfies $k(\tilde{x}') = k(x')$. Furthermore one has the inequalities

$$H^{(1+\delta)}_{\mathcal{O}_{X',x'}} \leq H^{(1+\tilde{\delta})}_{\mathcal{O}_{\tilde{X}',\tilde{x}'}} \leq H^{(1)}_{\mathcal{O}_{\tilde{X},\tilde{x}}} = H^{(1)}_{\mathcal{O}_{X,x}},$$

where $\tilde{\delta} = tr.deg(k(\tilde{x}')/k(\tilde{x}))$ ($= \delta$ if $k(\tilde{x})/k(x)$ is algebraic, and $\delta - 1$ otherwise). By our assumption all inequalities become in fact equalities, and by induction on the number of generators of $k(x')$ over $k(x)$ (starting with the residually rational case proved above), we may assume that $\mathcal{O}_{\tilde{X},\tilde{x}} \to \mathcal{O}_{\tilde{X}',\tilde{x}'}$ is injective. Since $\mathcal{O}_{X,x} \to \mathcal{O}_{\tilde{X},\tilde{x}}$ is injective, we obtained the injectivity of $\mathcal{O}_{X,x} \to \mathcal{O}_{X',x'}$. This completes the proof of (2), and while doing it, we also proved the claims in (3).

Finally we prove (4). Still under the assumption that X is embedded in a regular scheme Z, Hironaka proved in [H2] Theorem (1,A), that the equality $\nu^*_{x'}(X', Z') = \nu^*_x(X, Z)$ implies the inequality in (4). Together with (2) and (6) this implies (4) and finishes the proof of Theorem 2.10. $\square$

Corollary 2.12 *For $\nu \in \Sigma^{max}_X$ (cf. Definition 1.35), either $\nu \notin \Sigma_{X'}$ or $\nu \in \Sigma^{max}_{X'}$. We have*

$$X'(\nu) \subset \pi_X^{-1}(X(\nu)) \quad and \quad \pi_X^{-1}(X(\nu)) \subset \bigcup_{\mu \leq \nu} X'(\mu).$$

Definition 2.13 *Let the assumption be as in Theorem 2.10 and put $k' = k(x')$.*

  (1) $x' \in \pi_X^{-1}(x)$ is near to x if $H_{X'}(x') = H_X(x)$.

  (2) x' is very near to x if it is near to x and $e_{x'}(X') + \delta_{x'/x} = e_x(X)_{k'} = e_x(X)$.

We recall another result of Hironaka, as improved by Mizutani, which plays a crucial role in this paper. Let again X be an excellent scheme or a scheme embeddable in a regular scheme. Let $D \subset X$ be a permissible closed subscheme, and let

$$\pi_X : X' = B\ell_D(X) \to X \quad and$$

be the blowup with center D. Take any points $x \in D$ and $x' \in \pi_X^{-1}(x)$.

Theorem 2.14 *Assume that x' is near to x. Assume further that $\operatorname{char}(k(x)) = 0$, or $\operatorname{char}(k(x)) \geq \dim(X)/2 + 1$, where $k(x)$ is the residue field of x. Then*

$$x' \in \mathbb{P}(\operatorname{Dir}_x(X)/T_x(D)) \subset \pi_X^{-1}(x),$$

where $\mathbb{P}(V)$ is the projective space associated to a vector space V.

Proof First we note that the inclusion above is induced by the inclusion of cones (i.e., spectra of graded algebras) $\operatorname{Dir}_x(X)/T_x(D) \subset C_x(X)/T_x(D)$ and the isomorphism $C_x(X)/T_x(D) \cong C_D(X)_x$ of cones from Theorem 2.2 (2)(ii). More precisely, it is induced by applying the Proj-construction to the surjection of graded $k(x)$-algebras

$$A_D = gr_{\mathfrak{n}_x}(\mathcal{O}_{X,x})/\operatorname{Sym}(T_x(D)) \twoheadrightarrow \operatorname{Sym}(Dir_x(X))/\operatorname{Sym}(T_x(D)) = B_D$$

where we identify affine spaces with the associated vector spaces and note that $\mathrm{Proj}(A_D) = \pi_X^{-1}(x)$. Since the claim is local, we may pass to the local ring $\mathcal{O} = \mathcal{O}_{X,x}$ of X at x. Further, as in the proof of Theorem 2.10, we may assume that X is embedded into a regular scheme Z. If $\pi_Z : Z' = B\ell_D(Z) \to Z$ denotes the blowup of Z in D, we have a further inclusion

$$\mathbb{P}(C_x(X)/T_x(D)) = \pi_X^{-1}(x) \subset \mathbb{P}(T_x(Z)/T_x(D)) = \pi_Z^{-1}(x).$$

Therefore the claim that $x' \in \mathbb{P}(\mathrm{Dir}_x(X)/T_x(D))$ follows from [H4], Theorem IV, [H5], Theorem 2, and [Miz]. In fact, by the second reference there is a certain canonical subgroup scheme $B_{\mathbf{P},x'} \subset \mathbf{V} = T_x(Z)/T_x(D)$ just depending on $x' \in \mathbf{P} = \mathbb{P}(\mathbf{V})$, which has the following properties. It is defined by homogeneous equations in the coordinates of $\mathbf{V}$, hence is a subcone of $\mathbf{V}$, and the associated subspace $\mathbb{P}(B_{\mathbf{P},x'})$ contains x'. Moreover, by the first reference it is a vector subspace of $\mathbf{V}$ if $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) = p > 0$ with $p \geq \dim(B_{\mathbf{P},x'})$, and in [Miz] this was improved to the (sharp) bound $p \geq \dim(B_{\mathbf{P},x'})/2 + 1$. On the other hand, by the first reference, the action of $B_{\mathbf{P},x'}$ on $\mathbf{V}$ respects $C_x(X)/T_x(D)$ if x' is near to x. Since $0 \in C_x(X)$, we conclude that $B_{\mathbf{P},x'}$ is contained in $C_x(X)/T_x(D)$, and hence has dimension at most $d = \dim(X) - \dim(D) \leq \dim(X)$. Therefore, by the assumption $p \geq \dim(X)/2 + 1$, $B_{\mathbf{P},x'}$ is a vector subspace of $C_x(X)/T_x(D)$ and is thus contained in the biggest such subspace - which is $\mathrm{Dir}_x(X)/T_x(D)$. Therefore $x' \in \mathbb{P}(B_{\mathbf{P},x'}) \subset \mathbb{P}(\mathrm{Dir}_x(X)/T_x(D))$.

Lemma 2.15 *Consider $X(\nu)$ for $\nu \in \Sigma_X^{max}$. Let $\pi : X' = B\ell_D(X) \to X$ be the blowup with permissible center D contained in $X(\nu)$. Let $Y \subset X(\nu)$ be an irreducible closed subset which contains D as a proper subset. Then:*

(1) $Y' \subset X'(\nu)$, where $Y' \subset X'$ be the proper transform.

(2) Assume $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$. Then we have $e_x(X) \geq 1$ for any $x \in D$ and $e_{x'}(X') \geq 1$ for any $x' \in \pi^{-1}(D) \cap Y'$.

Proof Let η (resp. η') be the generic point of Y (resp. Y'). Take points $x \in D$ and $x' \in \pi^{-1}(x) \cap Y'$. Then we have

$$H_X(x) \geq H_{X'}(x') \geq H_{X'}(\eta') = H_X(\eta),$$

where the first inequality follows from Theorem 2.10(1), the second from theorem 1.33, and the last equality follow from the fact $\mathcal{O}_{X,\eta} \cong \mathcal{O}_{X',\eta'}$. Since $Y \subset X(\nu)$, we have $H_X(x) = H_X(\eta) = \nu$ so that the above inequalities are equalities. This implies $Y' \subset X'(\nu)$, which proves (1). Next we show (2). If $e_x(X) = 0$ for $x \in D$, then there is no point of $B\ell_x(X)$ which is near to x by Theorem 2.14. Thus (1) implies $e_x(X) \geq 1$. To show $e_{x'}(X') \geq 1$ for $x' \in \pi^{-1}(D) \cap Y'$, let $W \subset X'$ be the closure of x' in X'. By assumption W is a proper closed subset of Y'. Since $e_{x'}(X')$ is a local invariant of $\mathcal{O}_{X',x'}$, we may localize X' at x' to assume W is regular. Then, by Theorem 2.3, $W' \subset X'$ is permissible. Now the assertion follows from the previous assertion applied to $B\ell_W(X') \to X'$. $\square$

3 B-Permissible blow-ups - the embedded case

Let Z be a regular scheme and let $\mathbb{B} \subset Z$ be a simple normal crossing divisor on Z. For each $x \in Z$, let $\mathbb{B}(x)$ be the subdivisor of B which is the union of the irreducible components of $\mathbb{B}$ containing x.

Definition 3.1 *Let $D \subset Z$ be a regular subscheme and $x \in D$. We say D is normal crossing (n.c.) with $\mathbb{B}$ at x if there exists a system $z_1, \ldots, z_d$ of regular parameters of $R := \mathcal{O}_{Z,x}$ satisfying the following conditions:*

(1) $D \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle z_1, \ldots, z_r \rangle)$ for some $1 \le r \le d$.

(2) $\mathbb{B}(x) \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle \prod_{j \in J} z_j \rangle)$ for some $J \subset \{1, \ldots, d\}$.

We say D is transversal with $\mathbb{B}$ at x if in addition the parameters can be chosen such that $J \subset \{r+1, \ldots, d\}$. We say D is n.c. (resp. transversal) with $\mathbb{B}$ if D is n.c. (resp. transversal) with $\mathbb{B}$ at any point $x \in D$.

Note that D is n.c. with $\mathbb{B}$ if and only if D is transversal with the intersection of any set of irreducible components of $\mathbb{B}$ which do not contain D.

Let $D \subset Z$ be n.c. with $\mathbb{B}$. Consider $Z' = B\ell_D(Z) \xrightarrow{\pi_Z} Z$. Let $\widetilde{\mathbb{B}} = B\ell_{D \times_Z \mathbb{B}}(\mathbb{B}) \subset Z'$ be the strict transform of $\mathbb{B}$ in Z', let $E := \pi_Z^{-1}(D)$ be the exceptional divisor, and let $\mathbb{B}' = \widetilde{\mathbb{B}} \cup E$ be the complete transform of $\mathbb{B}$ in Z'. We easily see the following:

Lemma 3.2 *$\widetilde{\mathbb{B}}$ and $\mathbb{B}'$ are simple normal crossing divisors on Z'.*

For the following, and for the comparison with the next section, it will be more convenient to consider the set $\mathcal{B} = C(\mathbb{B})$ of irreducible components of $\mathbb{B}$.

Definition 3.3 *A simple normal crossings boundary on Z is a set $\mathcal{B} = \{B_1, \ldots, B_n\}$ of regular divisors on Z such that the associated divisor $\mathrm{div}(\mathcal{B}) = B_1 \cup \ldots \cup B_n$ is a (simple) normal crossings divisor. For $x \in Z$ let $\mathcal{B}(x) = \{B \in \mathcal{B} \mid x \in B\}$. Often the elements of $\mathcal{B}$ are also called components of $\mathcal{B}$.*

An equivalent condition is that the B_i intersect transversally, i.e., that for each subset $\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ the intersection $B_{i_1} \times_Z \ldots \times_Z B_{i_r}$ is regular of pure codimension r in Z. The associations

(3.1)
$$ \mathbb{B} \mapsto \mathcal{B} = C(\mathbb{B}) \quad , \quad \mathcal{B} \mapsto \mathbb{B} = \mathrm{div}(\mathcal{B}) $$

give mutual inverse bijections between the set of simple normal crossing (s.n.c.) divisors on Z and the set of simple normal crossing (s.n.c.) boundaries on Z, and we will now use the second language. Via (3.1) the above definitions correspond to the following in the setting of boundaries.

Definition 3.4 *(a) A regular subscheme $D \subset Z$ is transversal with a s.n.c. boundary $\mathcal{B}$ at x if for $\mathcal{B}(x) = \{B_1, \ldots, B_r\}$ it intersects all multiple intersections $B_{i_1} \times_Z \ldots \times_Z B_{i_s}$ transversally at x, and D is normal crossing (n.c.) with $\mathcal{B}$ at x if it is transversal with $\mathcal{B}(x) - \mathcal{B}(x, D)$ at x, where $\mathcal{B}(x, D) = \{B \in \mathcal{B}(x) \mid D \subset B\}$. D is transversal (resp. n.c.) with $\mathcal{B}$, if it is transversal (resp. n.c.) with $\mathcal{B}$ at all points $x \in D$.*

(b) If D is n.c. with $\mathcal{B}$, and $Z' = B\ell_D(Z) \xrightarrow{\pi_Z} Z$ is the blow-up of D, then the strict and the complete transform of $\mathcal{B}$ are defined as

$$\widetilde{\mathcal{B}} := \{ \tilde{B} \mid B \in \mathcal{B} \} \quad , \quad \mathcal{B}' := \widetilde{\mathcal{B}} \cup \{E\}$$

respectively, where $\tilde{B} = B\ell_{D \times_Z B}(B)$ is the strict transform of B in Z, and $E = \pi_Z^{-1}(D)$ is the exceptional divisor. (Note that $\widetilde{\mathcal{B}}$ and $\mathcal{B}'$ are s.n.c. boundaries on Z' by Lemma 3.2.)

In the following, consider a regular scheme Z and a simple normal crossing boundary $\mathcal{B}$ on Z. Moreover let $X \subset Z$ be a closed subscheme.

Definition 3.5 *Let $D \subset X$ be a regular closed subscheme and $x \in D$. We say D is $\mathcal{B}$-permissible at x if $D \subset X$ is permissible at x and D is n.c. with $\mathcal{B}$ at x. We say $D \subset X$ is $\mathcal{B}$-permissible if $D \subset X$ is permissible at all $x \in D$.*

Definition 3.6 *A history function for $\mathcal{B}$ on X is a function*

(3.2) $$O : X \to \{ \text{subsets of } \mathcal{B} \} \; ; \; x \to O(x),$$

which satisfies the following conditions:

(O1) For any $x \in X$, $O(x) \subset \mathcal{B}(x)$.

(O2) For any $x, y \in X$ such that $x \in \overline{\{y\}}$ and $H_X(x) = H_X(y)$, we have $O(y) \subset O(x)$.

(O3) For any $y \in X$, there exists a non-empty open subset $U \subset \overline{\{y\}}$ such that $O(x) = O(y)$ for all $x \in U$ such that $H_X(x) = H_X(y)$.

For such a function, we put for $x \in X$,

$$N(x) = \mathcal{B}(x) - O(x).$$

A component of $\mathcal{B}$ is called old (resp. new) for x if it is a component of $O(x)$ (resp. $N(x)$).

A basic example of a history function for B on X is given by the following:

Lemma 3.7 *The function $O(x) = \mathcal{B}(x)$ $(x \in X)$, is a history function for $\mathcal{B}$ on X. In fact it satisfies 3.6 (O2) and (O3) without the condition $H_X(x) = H_X(y)$.*

Proof Left to the readers.

Define the Hilbert-Samuel function of (X, O) as:

$$H_X^O : X \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \; ; \; x \to (H_X(x), \mid O(x) \mid),$$

where $\mid O(x) \mid$ is the cardinality of $O(x)$. We endow $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ with the lexicographic order:

$$(\nu, \mu) \geq (\nu', \mu') \Leftrightarrow \nu > \nu' \text{ or } \nu = \nu' \text{ and } \mu \geq \mu'.$$

Theorem 1.33 and the conditions in 3.6 immediately imply the following:

Theorem 3.8 *Let the assumption be as above.*

(1) *If $x \in X$ is a specialization of $y \in X$, then $H^O_X(x) \geq H^O_X(y)$.*

(2) *For any $y \in X$, there is a dense open subset U of $\overline{\{y\}}$ such that $H^O_X(y) = H^O_X(x)$ for any $x \in U$.*

In other words (see Lemma 1.34 (a)), the function H^O_X is upper semi-continuous on X. Let
$$\Sigma^O_X := \{H^O_X(x) \mid x \in X\} \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N},$$
and let $\Sigma^{O,max}_X$ be the set of of the maximal elements in Σ^O_X.

Definition 3.9 *(1) For $\widetilde{\nu} \in \Sigma^O_X$ we define*
$$X(\widetilde{\nu}) = X^O(\widetilde{\nu}) = \{x \in X \mid H^O_X(x) = \widetilde{\nu}\}$$
and
$$X(\geq \widetilde{\nu}) = X^O(\geq \widetilde{\nu}) = \{x \in X \mid H^O_X(x) \geq \widetilde{\nu}\},$$
and we call

(3.3)
$$X^O_{max} = \bigcup_{\widetilde{\nu} \in \Sigma^{O,max}_X} X(\widetilde{\nu}).$$

the O-Hilbert-Samuel locus of X.

(2) We define
$$\mathrm{Dir}^O_x(X) := \mathrm{Dir}_x(X) \cap \bigcap_{B \in O(x)} T_x(B) \subset T_x(Z).$$
$$e^O_x(X) = \dim_{k(x)}(\mathrm{Dir}^O_x(X)).$$

By Lemma 1.34, $X(\widetilde{\nu})$ is locally closed, with closure contained in $X(\geq \widetilde{\nu})$. Moreover, $X(\widetilde{\nu})$ is closed for $\widetilde{\nu} \in \Sigma^{O,max}_X$, the union in (3.3) is disjoint, X^O_{max} is a closed subset of X, and Σ^O_X is finite if X is noetherian. Theorems 2.2 and 2.3 imply the following:

Theorem 3.10 *Let $D \subset X$ be a regular closed subscheme and $x \in D$. Then the following conditions are equivalent:*

(1) *$D \subset X$ is permissible at x and there is an open neighborhood U of x in Z such that $D \cap U \subset B$ for every $B \in O(x)$.*

(2) *$H^O_X(x) = H^O_X(y)$ for any $y \in D$ such that x is a specialization of y.*

Under the above conditions, we have

(3.4)
$$T_x(D) \subset \mathrm{Dir}^O_x(X).$$

Definition 3.11 *A closed subscheme $D \subset X$ is called O-permissible at x, if it satisfies the equivalent conditions in Theorem 3.10, and it is called O-permissible, if it is O-permissible at all $x \in D$.*

Remark 3.12 *Note that $D \subset X$ is $\mathcal{B}$-permissible at x if and only if D is O-permissible at x and n.c with $N(x)$ at x.*

Theorem 3.13 *Let the assumption be as in Theorem 3.10 and assume that D is irreducible. Assume:*

(1) $D \subset X$ is O-permissible at x.

(2) $e_\eta(X) = e_x(X) - \dim(\mathcal{O}_{D,x})$ (cf. Theorem 2.6),

where η is the generic point of D. Then we have

$$e_\eta^O(X) \leq e_x^O(X) - \dim(\mathcal{O}_{D,x}).$$

Proof First we claim that (1) and (2) hold after replacing x by any point $y \in D$ such that $x \in \overline{\{y\}}$ and $\overline{\{y\}}$ is regular at x. Indeed the claim follows from the inequalities:

$$H_X^O(\eta) \leq H_X^O(y) \leq H_X^O(x),$$
$$e_\eta(X) \leq e_y(X) - \dim(\mathcal{O}_{D,y})$$
$$\leq \left(e_x(X) - \dim(\mathcal{O}_{\overline{\{y\}},x})\right) - \dim(\mathcal{O}_{D,y}) = e_x(X) - \dim(\mathcal{O}_{D,x}),$$

which follows from Theorems 3.8 and 2.6. By the claim we are reduced to the case where $\dim(\mathcal{O}_{D,x}) = 1$ by the same argument as in the proof of Theorem 2.6. Let $R = \mathcal{O}_{Z,x}$ and let $J \subset \mathfrak{p} \subset R$ be the ideals defining $X \subset Z$ and $D \subset Z$, respectively. Let $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$ and $J_\mathfrak{p} = JR_\mathfrak{p}$. By Lemma 2.8 (2) implies that there exists a part of a system of regular parameters $y = (y_1, \ldots, y_r)$ of R such that $y \subset \mathfrak{p}$ and that

$$I\mathrm{Dir}(R_\mathfrak{p}/J_\mathfrak{p}) = \langle in_\mathfrak{p}(y_1), \ldots, in_\mathfrak{p}(y_r) \rangle \subset \mathrm{gr}_\mathfrak{p}(R_\mathfrak{p}),$$
$$I\mathrm{Dir}(R/J) = \langle in_\mathfrak{m}(y_1), \ldots, in_\mathfrak{m}(y_r) \rangle \subset \mathrm{gr}_\mathfrak{m}(R),$$

We can take $\theta_1, \ldots, \theta_s \in R$ such that $(y_1, \ldots, y_r, \theta_1, \ldots, \theta_s)$ is a part of a system of regular parameters of R and that there exists an irreducible component B_i of $O(x)$ for each $i = 1, \ldots, s$ such that $B_i \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle \theta_i \rangle)$ and

$$I\mathrm{Dir}^O(R/J) = \langle in_\mathfrak{m}(y_1), \ldots, in_\mathfrak{m}(y_r), in_\mathfrak{m}(\theta_1), \ldots, in_\mathfrak{m}(\theta_s) \rangle,$$

where $I\mathrm{Dir}^O(R/J) \subset \mathrm{gr}_\mathfrak{m}(R)$ is the ideal defining $\mathrm{Dir}_x^O(X) \subset T_x(Z)$. Now (1) implies $O(x) = O(\eta)$ so that $D \subset B_i$ and $\theta_i \in \mathfrak{p}$ for all $i = 1, \ldots, s$. Hence $(y_1, \ldots, y_r, \theta_1, \ldots, \theta_s)$ is a part of a system of regular parameters of $R_\mathfrak{p}$ and

$$I\mathrm{Dir}^O(R_\mathfrak{p}/J_\mathfrak{p}) \supset \langle in_\mathfrak{p}(y_1), \ldots, in_\mathfrak{p}(y_r), in_\mathfrak{p}(\theta_1), \ldots, in_\mathfrak{p}(\theta_s) \rangle,$$

which implies the conclusion of Theorem 3.13. $\square$

Let $D \subset X$ be a $\mathcal{B}$-permissible closed subscheme. Consider the diagram

(3.5)
$$
\begin{array}{ccc}
X' = B\ell_D(X) & \xrightarrow{\;i'\;} & B\ell_D(Z) = Z' \\
{\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\
X & \xrightarrow{\;i\;} & Z
\end{array}
,
$$

and let $\mathcal{B}'$ and $\widetilde{\mathcal{B}}$ be the complete and strict transform of $\mathcal{B}$ in Z', respectively. For a given history function $O(x)$ $(x \in X)$, we define functions $O', \widetilde{O} : X' \to \{\text{subsets of } \mathcal{B}'\}$ as follows: Let $x' \in X'$ and $x = \pi_X(x') \in X$. Then define

$$(3.6) \qquad O'(x') = \begin{cases} \widetilde{O(x)} \cap \mathcal{B}'(x') & \text{if } H_{X'}(x') = H_X(x) \\ \mathcal{B}'(x') & \text{otherwise,} \end{cases}$$

where $\widetilde{O(x)}$ is the strict transform of $O(x)$ in Z', and

$$(3.7) \qquad \widetilde{O}(x') = \begin{cases} \widetilde{O(x)} \cap \mathcal{B}'(x') & \text{if } H_{X'}(x') = H_X(x) \\ \widetilde{\mathcal{B}}(x') & \text{otherwise.} \end{cases}$$

Note that $O'(x') = \widetilde{O}(x') = \widetilde{O(x)} \cap \widetilde{\mathcal{B}}(x')$ if x' is near to x.

Lemma 3.14 *The functions $x' \to O'(x')$, $x' \to \widetilde{O}(x')$ are history functions for $\mathcal{B}'$ on X.*

The proof of Lemma 3.14 will be given later.

Definition 3.15 *We call $(\mathcal{B}', O')$ and $(\widetilde{\mathcal{B}}, \widetilde{O})$ the complete and strict transform of $(\mathcal{B}, O)$ in Z', respectively.*

Theorem 3.16 *Take points $x \in D$ and $x' \in \pi_X^{-1}(x)$. Then $H_{X'}^{\widetilde{O}}(x') \leq H_{X'}^{O'}(x') \leq H_X^O(x)$. In particular we have*

$$\pi_X^{-1}(X^{O'}(\widetilde{\nu})) \subset \bigcup_{\widetilde{\mu} \leq \widetilde{\nu}} X'^{O'}(\widetilde{\mu}) \quad \text{for } \widetilde{\nu} \in \Sigma_X^{max},$$

and the same holds for $\widetilde{O}$ in place of O'.

Proof This follows immediately from Theorem 2.10, (3.6) and (3.7).

In the following we mostly use the complete transform $(\mathcal{B}', O')$ and, for ease of notation, we often write $H_{X'}^O(x')$ and $\Sigma_{X'}^O$ instead of $H_{X'}^{O'}(x')$ and $\Sigma_{X'}^{O'}$, similarly for $\Sigma_{X'}^{O,max}$ etc., because everything just depends on O.

Definition 3.17 *We say that $x' \in \pi_X^{-1}(x)$ is O-near to x if the following equivalent conditions hold:*

(1) $H_{X'}^O(x') = H_X^O(x)$ $\qquad$ ($\Leftrightarrow H_{X'}^{O'}(x') = H_X^O(x) \Leftrightarrow H_{X'}^{\widetilde{O}}(x') = H_X^O(x)$).

(2) x' is near to x and contained in the strict transforms of all $B \in O(x)$.

Call x' very O-near to x if x' is O-near and very near to x and $e_{x'}^O(X') = e_x^O(X) - \delta_{x'/x}$.

The following result is an immediate consequence of Theorem 2.14 and Definition 3.3 (2).

Theorem 3.18 *Assume that $x' \in X'$ is O-near to $x = \pi_Z(x') \in X$. Assume further that $\operatorname{char}(k(x)) = 0$, or $\operatorname{char}(k(x)) \geq \dim(X)/2 + 1$, where $k(x)$ is the residue field of x. Then*

$$x' \in \mathbb{P}(\operatorname{Dir}_x^O(X)/T_x(D)) \subset \mathbb{P}(T_x(Z)/T_x(D)) = \pi_Z^{-1}(x)$$

Proof of Lemma 3.14: Take $y', x' \in X'$ such that $x' \in \overline{\{y'\}}$ and $H_{X'}(x') = H_{X'}(y')$. We want to show $O'(y') \subset O'(x')$. Put $x = \pi_X(x'), y = \pi_X(y') \in X$. We have $x \in \overline{\{y\}}$. By Theorems 1.33 and 2.10

$$H_{X'}(y') \leq H_X(y) \leq H_X(x) \geq H_{X'}(x') = H_{X'}(y').$$

First assume $H_X(x) \geq H_{X'}(x')$, which implies $H_{X'}(y') = H_X(y) = H_X(x)$. By 3.6 (O2) and (3.6) we get

$$O'(y') = \widetilde{O(y)} \cap \mathcal{B}'(y') \subset \widetilde{O(x)} \cap \mathcal{B}'(x') = O'(x').$$

Next assume $H_X(x) > H_{X'}(x')$. Then, by Lemma 3.7 and (3.6), we get

$$O'(y') \subset \mathcal{B}'(y') \subset \mathcal{B}'(x') = O'(x').$$

Next we show that for $y' \in X'$, there exists a non-empty open subset $U' \subset \overline{\{y'\}}$ such that $H_{X'}(x') = H_{X'}(y')$ and $O'(x') = O'(y')$ for all $x' \in U'$. Put $y = \pi_X(y')$. By Lemma 3.7 and 3.6 (O3), there exists a non-empty open subset $U \subset \overline{\{y\}}$ such that $H_X(x) = H_X(y)$, $\mathcal{B}(x) = \mathcal{B}(y)$ and $O(x) = O(y)$ for all $x \in U$. By Theorem 1.33(3) and Lemma 3.7, there exists a non-empty open subset $U' \subset \overline{\{y'\}} \cap \pi_X^{-1}(U)$ such that $H_{X'}(x') = H_{X'}(y')$ and $\mathcal{B}'(x') = \mathcal{B}'(y')$ for all $x' \in U'$. We now show U' satisfies the desired property. Take $x' \in U'$ and put $x = \pi_X(x')$. By the assumption we have $H_{X'}(x') = H_{X'}(y')$ and $H_X(x) = H_X(y)$.

First assume $H_{X'}(x') = H_X(x)$, which implies $H_{X'}(y') = H_X(y)$. By (3.6) we get

$$O'(y') = \widetilde{O(y)} \cap \mathcal{B}'(y') = \widetilde{O(x)} \cap \mathcal{B}'(x') = O'(x').$$

Next assume $H_{X'}(x') < H_X(x)$, which implies $H_{X'}(y') < H_X(y)$. By (3.6) we get

$$O'(y') = \mathcal{B}'(y') = \mathcal{B}'(x') = O'(x').$$

This completes the proof of Lemma 3.14 for $(\mathcal{B}', O')$. The proof for $(\widetilde{\mathcal{B}}, \widetilde{O})$ is similar. $\square$

For $x' \in X'$ let $N'(x') = \mathcal{B}'(x') - O'(x')$ be the set of the new components of $(\mathcal{B}', O')$ for x'. If x' is near to $x = \pi_X(x') \in D$, i.e., $H_{X'}(x') = H_X(x)$, then

$$(3.8) \qquad N'(x') = (\widetilde{N(x)} \cap \mathcal{B}'(x')) \cup \{E\} \quad \text{with} \quad E = \pi_Z^{-1}(D)$$

where $\widetilde{N(x)}$ is the strict transform of $N(x)$ in X'. If x' is not near to x, then $N'(x') = \emptyset$. Similarly, $\widetilde{N}(x') = \widetilde{\mathcal{B}}(x') - \widetilde{O}(x') = \widetilde{N(x)} \cap \widetilde{\mathcal{B}}(x') \subset N'(x')$ if x' is near to x, and $\widetilde{N}(x') = \emptyset$, otherwise. We study the transversality of $N'(x')$ with a certain regular subscheme of E.

Definition 3.19 *For a $k(x)$-linear subspace $T \subset T_x(Z)$, we say that T is transversal with $N(x)$ (notation: $T \pitchfork N(x)$) if*

$$\dim_{k(x)} \left(T \cap \bigcap_{B \in N(x)} T_x(B) \right) = \dim_{k(x)}(T) - \mid N(x) \mid.$$

Lemma 3.20 *Let $\pi_Z : Z' = B\ell_D(Z) \to Z$ be as in (3.5). Assume $D = x$ and $T \pitchfork N(x)$. Then the closed subscheme $\mathbb{P}(T) \subset E = \mathbb{P}(T_x(Z))$ is n.c. with $N'(x')$ and $\widetilde{N}(x')$ at each $x' \in \pi_X^{-1}(x)$.*

Proof Let $R = \mathcal{O}_{Z,x}$ with the maximal ideal $\mathfrak{m}$. For each $B \in N(x)$, take $h_B \in R$ such that $B \times_Z \operatorname{Spec}(R) = \operatorname{Spec}(R/\langle h_B \rangle)$. Put $H_B = in_{\mathfrak{m}}(h_B) \in \operatorname{gr}_{\mathfrak{m}}^1(R)$. In view of (3.8) the lemma follows from the following facts: The ideal $\langle H_B \rangle \subset \operatorname{gr}_{\mathfrak{m}}(R)$ defines the subschemes

$$T_x(B) \subset T_x(Z) = \operatorname{Spec}(\operatorname{gr}_{\mathfrak{m}}(R)) \quad \text{and} \quad E \times_{Z'} \tilde{B} \subset E = \operatorname{Proj}(\operatorname{gr}_{\mathfrak{m}}(R)),$$

where $\tilde{B}$ is the strict transform of B in Z'. $\square$

Lemma 3.21 *Let* $\pi_Z : Z' = B\ell_D(Z) \to Z$ *be as in (3.5). Assume* $T \pitchfork N(x)$ *and* $T_x(D) \subset T$ *and* $\dim_{k(x)}(T/T_x(D)) = 1$. *Consider*

$$\{x'\} = \mathbb{P}(T/T_x(D)) \subset \mathbb{P}(T_x(Z)/T_x(D)) = \pi_Z^{-1}(x).$$

Let $D' \subset E$ *be any closed subscheme such that* $x' \in D'$ *and* π_Z *induces an isomorphism* $D' \simeq D$. *Then* D' *is n.c. with* $N'(x')$ *and* $\widetilde{N}(x')$ *at* x'.

Proof It suffices to consider $N'(x')$. For $B \in \mathcal{B}(x, D)$ we have $T_x(D) \subset T_x(B)$ so that the assumptions of the lemma imply $T_x(B) \cap T = T_x(D)$. By the argument of the last part of the proof of Lemma 3.20, this implies $\{x'\} = \mathbb{P}(T/T_x(D)) \not\subset \tilde{B}$. Thus we are reduced to showing D' is n.c. with $N'(x') \cap (\mathcal{B}(x) - \mathcal{B}(x, D))'(x')$, which follows from:

Lemma 3.22 *Let* Z *be a regular scheme and* $D, W \subset Z$ *be regular closed subschemes such that* D *and* W *intersect transversally. Let* $\pi : Z' = B\ell_D(Z) \to Z$ *and let* $\widetilde{W}$ *be the strict transform of* W *in* Z'. *Suppose* $D' \subset E := \pi^{-1}(D)$ *is a closed subscheme such that* π *induces an isomorphism* $D' \xrightarrow{\sim} D$. *Then* D' *and* $\widetilde{W}$ *intersect transversally.*

Proof By definition, $\widetilde{W} = B\ell_{D \cap W}(W)$. The transversality of D and W implies $B\ell_{D \cap W}(W) \simeq B\ell_D(Z) \times_Z W = Z' \times_Z W$. Thus

$$E \times_{Z'} \widetilde{W} \simeq E \times_{Z'} (Z' \times_Z W) = E \times_Z W = E \times_D (D \times_Z W).$$

Hence we get

$$D' \times_{Z'} \widetilde{W} = D' \times_E (E \times_{Z'} \widetilde{W}) \simeq D' \times_E (E \times_D (D \times_Z W)) = D' \times_D (D \times_Z W) \simeq W \times_Z D,$$

where the last isomorphism follows from the assumption $D' \xrightarrow{\sim} D$. This completes the proof of the lemma. $\square$

Theorem 3.23 *Let* $\pi_Z : Z' = B\ell_D(Z) \to Z$ *be as in (3.5). Take* $x \in X$ *and* $x' \in \pi_X^{-1}(x)$. *Assume* $\operatorname{char}(k(x)) = 0$, *or* $\operatorname{char}(k(x)) \geq \dim(X)/2 + 1$.

(1) *If* x' *is* O-*near and very near to* x, *then* $e_{x'}^O(X') \leq e_x^O(X) - \delta_{x'/x}$.

(2) *Assume* x' *is very* O-*near and* $N(x) \pitchfork \operatorname{Dir}_x^O(X)$. *Then* $N'(x') \pitchfork \operatorname{Dir}_{x'}^O(X')$.

Proof We first show (1). Assume that x' is O-near and very near to x. For the sake of the later proof of (2), we also assume $N(x) \pitchfork \operatorname{Dir}_x^O(X)$. By doing this, we do not lose generality for the proof of (1) since we may take $N(x) = \emptyset$. Put $R = \mathcal{O}_{Z,x}$ (resp. $R' = \mathcal{O}_{Z',x'}$) with

the maximal ideal $\mathfrak{m}$ (resp. $\mathfrak{m}'$) and $k = k(x) = R/\mathfrak{m}$ (resp. $k' = k(x') = R'/\mathfrak{m}'$). By the assumptions and Theorem 3.18 there exists a system of regular parameters of R

$$(y_1, \ldots, y_r, \theta_1, \ldots, \theta_q, u_1, \ldots, u_a, u_{a+1}, \ldots, u_{a+b}, v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+t})$$

satisfying the following conditions: Fixing the identification:

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[Y, \Theta, U, V] = k[Y_1, \ldots, Y_r, \Theta_1, \ldots, \Theta_q, U_1, \ldots, U_{a+b}, V_1, \ldots, V_{s+t}],$$

$$(Y_i = in_{\mathfrak{m}}(y_i), \ \Theta_i = in_{\mathfrak{m}}(\theta_i), \ U_i = in_{\mathfrak{m}}(u_i), \ V_i = in_{\mathfrak{m}}(v_i) \ \in \mathrm{gr}_{\mathfrak{m}}^1(R))$$

we have

(i) $D \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle y_1, \ldots, y_r,, u_1, \ldots, u_{a+b}\rangle)$.

(ii) $I\mathrm{Dir}_x(X) = \langle Y_1, \ldots, Y_r \rangle \subset \mathrm{gr}_{\mathfrak{m}}(R)$ (cf. Definition 1.26),

(iii) For $1 \le i \le q$, there exists $B \in O(x)$ such that $B \times_Z \mathrm{Spec}(R) = B_\theta^{(i)}$, where $B_\theta^{(i)} := \mathrm{Spec}(R/\langle \theta_i \rangle)$, and we have

$$\mathrm{Dir}_x^O(X) = \mathrm{Dir}_x(X) \cap \bigcap_{1 \le i \le q} T_x(B_\theta^{(i)}).$$

(iv) $N(x) \times_Z \mathrm{Spec}(R) = \bigcup_{1 \le i \le b} B_u^{(a+i)} \cup \bigcup_{1 \le j \le t} B_v^{(s+j)}$.

Here $B_u^{(i)} = \mathrm{Spec}(R/\langle u_i \rangle)$, $B_v^{(j)} = \mathrm{Spec}(R/\langle v_j \rangle)$. Let $B_\theta'^{(i)}$ (resp. $B_u'^{(i)}$, resp. $B_v'^{(i)}$) be the strict transform of $B_\theta^{(i)}$ (resp. $B_u^{(i)}$, resp. $B_v^{(i)}$) in $\mathrm{Spec}(R')$. Let

$$\Xi = \{i \in [a+1, a+b] \mid x' \in B_u'^{(i)}\}.$$

By Theorem 3.16 there exists $i_0 \in [1, a+b] - \Xi$ such that

$$(y_1', \ldots, y_r', \theta_1', \ldots, \theta_q', w, u_j' \ (j \in \Xi), v_1, \ldots, v_{s+t}),$$

where $w = u_{i_0}$, $y_i' = y_i/w$, $\theta_i' = \theta/w$, $u_j' = u_j/w$, is a part of a system of regular parameters of R' so that the polynomial ring:

$$k'[Y_1', \ldots, Y_r', \Theta_1', \ldots, \Theta_q', W, U_j' \ (j \in \Xi), V_1, \ldots, V_{s+t}],$$

where $Y_i' = in_{\mathfrak{m}'}(y_i')$, $\Theta_i' = in_{\mathfrak{m}'}(\theta_i')$, $W = in_{\mathfrak{m}'}(w)$, $U_j' = in_{\mathfrak{m}'}(u_j')$, is a subring of $\mathrm{gr}_{\mathfrak{m}'}(R')$. It also implies

$$N'(x') := \bigcup_{i \in N'(x')} B'^{(i)} = E \cup \bigcup_{i \in \Xi} B_u'^{(i)} \cup \bigcup_{1 \le j \le t} B_v'^{(i+s)},$$

where $E = \pi_Z^{-1}(D)$ and $E \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle w \rangle)$. Note

$$T_{x'}(E) \subset T_{x'}(Z') = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}'}(R')) \text{ is defined by } \langle W \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R').$$

Moreover

$$T_{x'}(B_\theta'^{(j)}) \subset T_{x'}(Z') \text{ is defined by } \langle \Theta_j \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R').$$
$$T_{x'}(B_u'^{(i)}) \subset T_{x'}(Z') \text{ is defined by } \langle U_i' \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R') \text{ for } i \in \Xi.$$
$$T_{x'}(B_v'^{(j)}) \subset T_{x'}(Z') \text{ is defined by } \langle V_j \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R').$$

On the other hand, by Theorem 8.3 the assumption that x' is very near to x implies that there exist $\lambda_1, \ldots, \lambda_r \in k'$ such that

$$I\mathrm{Dir}_{x'}(X') = \langle Y_1' + \lambda_1 W, \ldots, Y_r' + \lambda_r W \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R').$$

so that

$$(3.9) \qquad I\mathrm{Dir}_{x'}^O(X') \supset \langle Y_1' + \lambda_1 W, \ldots, Y_r' + \lambda_r W, \Theta_1, \ldots, \Theta_q \rangle.$$

This clearly implies the assertion of (1). If x' is very O-near, the inclusion in (3.9) is equality and then it implies $N'(x') \pitchfork \mathrm{Dir}_{x'}^O(X')$. Thus the proof of Theorem 3.23 is complete. $\square$

Corollary 3.24 *Let $\pi_Z : Z' = B\ell_D(Z) \to Z$ be as in (3.5) and take closed points $x \in D$ and $x' \in \pi_X^{-1}(x)$ such that x' is O-near to x. Assume $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$. Assume further that there is an integer $e \geq 0$ for which the following hold:*

(1) $e_x(X)_{k(x')} \leq e$, and either $e \leq 2$ or $k(x')$ is separable over $k(x)$.

(2) $N(x) \pitchfork \mathrm{Dir}_x^O(X)$ or $e_x^O(X) \leq e - 1$.

Then $N'(x') \pitchfork \mathrm{Dir}_{x'}^O(X')$ or $e_{x'}^O(X') \leq e - 1$.

Proof We claim

If $e_{x'}^O(X') \geq e$, then $e_x(X) = e = e_{x'}(X')$ and $e_x^O(X) = e = e_{x'}^O(X')$, so that x' is very O-near to x.

First we show the first equality which implies x' is very near to x. Indeed the assumption implies by Lemma 1.20 and Theorem 2.10

$$e \leq e_{x'}^O(X') \leq e_{x'}(X') \leq e_x(X)_{k(x')} \leq e.$$

Hence $e_{x'}(X') = e_x(X)_{k(x')} = e$. It remains to show $e_x(X) = e$. If $k(x')$ is separable over $k(x)$, this follows from Lemma 1.20 (2). Assume $e \leq 2$ and $e_x(X) < e_x(X)_{k(x')} = 2$. Then Theorem 2.14 implies that $k(x') = k(x)$ so that $e_x(X) = e_x(X)_{k(x')}$, which is a contradiction. Since x' is very near to x, Theorem 3.23 (1) implies

$$e \leq e_{x'}^O(X) \leq e_x^O(X) \leq e_x(X) \leq e,$$

which shows the second equality and the claim is proved.

By the claim, if $e_x^O(X) \leq e - 1$, we must have $e_{x'}^O(X') \leq e - 1$. Hence it suffices to show $N'(x') \pitchfork \mathrm{Dir}_{x'}^O(X')$ assuming $N(x) \pitchfork \mathrm{Dir}_x^O(X)$ and $e_{x'}^O(X') \geq e$. By the claim the second assumption implies that x' is very O-near to x. Therefore the assertion follows from Theorem 3.23 (2). $\square$

Definition 3.25 *Call $(\mathcal{B}, O)$ admissible at $x \in X$, if $N(x) \pitchfork T_x(X)$, and call $(\mathcal{B}, O)$ admissible if it is admissible at all $x \in X$.*

We note that admissibility of $(\mathcal{B}, O)$ at x implies $\mathcal{B}_{in}(x) \subseteq O(x)$, where $\mathcal{B}_{in}(x)$ is defined as follows.

Definition 3.26 *Call $B \in \mathcal{B}$ inessential at $x \in X$, if it contains all irreducible components of X which contain x. Let*

$$\mathcal{B}_{in}(x) = \{B \in \mathcal{B} \mid Z \subseteq B \text{ for all } Z \in I(x)\}$$

be the set of inessential boundary components at x, where $I(x)$ is the set of the irreducible components of X containing x.

Definition 3.27 *Call $x \in X$ O-regular (or X O-regular at x), if*

(3.10) $$H_X^O(x) = (\nu_X^{reg}, |\mathcal{B}_{in}(x)|),$$

where ν_X^{reg} is as in Remark 1.32. Call X O-regular, if it is O-regular at all $x \in X$.

Lemma 3.28 *If $(\mathcal{B}, O)$ is admissible at x and X is O-regular at x, then X is regular and normal crossing with $\mathcal{B}$ at x.*

Proof The first claim follows from Lemma 1.31. Since x is regular, the assumption $N(x) \pitchfork T_x(X)$ means that $N(x)$ is transversal to X at x. On the other hand, for the (unique) connected component W on which x lies we have $W \subset B$ for all $B \in \mathcal{B}_{in}(x) = O(x)$ where the last equality holds by assumption. Thus X is n.c. with $\mathcal{B}$ at x. □

We have the following transition property.

Lemma 3.29 *Let $\pi_X : X' = B\ell_D(X) \to X$ be as in (3.5). If $(\mathcal{B}, O)$ admissible at $x \in X$, then $(\mathcal{B}', O')$ and $(\widetilde{\mathcal{B}}, \widetilde{O})$ are admissible at any $x' \in \pi_X^{-1}(x')$.*

Proof The proof is somewhat similar to that of Theorem 3.23: If x' is not near to x, then $N'(x')$ is empty by definition. Therefore we may consider the case where x' is near to x. Look at the surjection $R = \mathcal{O}_{Z,x} \to \mathcal{O}_{X,x}$ with kernel J, and let $R = \mathcal{O}_{Z,x} \to \mathcal{O}_{X,x}$ be the corresponding surjection for the local rings of the blowups at x', with kernel J'. Then there is a system of regular parameters for R

$$(f_1, \ldots, f_m, u_1, \ldots, u_{a+b}, v_1, \ldots, v_{r+s})$$

satisfying the following conditions:

(i) J has a standard basis $(f_1, \ldots, f_m, f_{m+1}, \ldots, f_n)$ with $f_1, \ldots, f_m \in \mathfrak{n} - \mathfrak{n}^2$ and $f_{m+1}, \ldots, f_n \in \mathfrak{n}^2$ for the maximal ideal $\mathfrak{n} \subset R$, so that the initial forms of $f_1, \ldots, f_m$ define $T_x(X)$ inside $T_x(Z)$.

(ii) $D \times_Z \text{Spec}(R) = \text{Spec}(R/\langle f_1, \ldots, f_m, u_1, \ldots, u_{a+b}\rangle)$.

(iii) $N(x) \times_Z \text{Spec}(R)$ is given by $\text{div}(u_{a+1}), \ldots, \text{div}(u_{a+b}), \text{div}(v_{s+1}), \ldots, \text{div}(v_{s+t})$.

Let

$$\Xi = \{i \in [a+1, a+b] \mid x' \in \text{div}(u_i)'\}.$$

Then there exists $i_0 \in [1, a+b] - \Xi$ such that

$$(f_1', \ldots, f_m', w, u_j' \ (j \in \Xi), v_1, \ldots, v_{s+t}),$$

where $w = u_{i_0}$, $f_i' = f_i/w$, $u_j' = u_j/w$, is a part of a system of regular parameters of R'. Since x' is near to x, we have $H^{(\delta)}_{\mathcal{O}_{X',x'}} = H^{(0)}_{\mathcal{O}_{X,x}}$ by Theorem 2.10, where $\delta = \mathrm{trdeg}_{k(x)}(k(x'))$. Evaluating at 1, we get $\dim T_{x'}(X') + \delta = \dim T_x(X)$. Similarly we get $\dim T_{x'}(Z') + \delta = \dim T_x(Z)$, and hence $\dim T_{x'}(Z') - \dim T_{x'}(X') = \dim T_x(Z) - \dim T_x(X)$. It follows that the initial forms of $f_1', \ldots, f_m'$ already define $T_{x'}(X')$ inside $T_{x'}(Z')$. This shows that $N'(x') \pitchfork T_{x'}(X')$, because $N'(x')$ is defined by w, u_j' ($j \in \Xi$), $v_{s+1}, \ldots, v_{s+t}$. $\square$

Finally we give the following functoriality for the objects we have introduced above.

Lemma 3.30 *Consider a cartesian diagram*

$$
\begin{array}{ccc}
X^* & \xrightarrow{\ i^*\ } & Z^* \\
\varphi \downarrow & & \downarrow \phi \\
X & \xrightarrow{\ i\ } & Z \, ,
\end{array}
$$

in which Z is a regular scheme, i (and hence i^*) is a closed immersion, and ϕ (and hence φ) is a flat morphism with regular fibers. Let $\mathbb{B} \subset Z$ be a simple normal crossing divisor, let O be a history function for $\mathbb{B}$ (i.e., for the associated boundary $\mathcal{B}$), and let $\mathbb{B}^* = \mathbb{B} \times_Z Z^*$.

(1) *Z^* is regular, and $\mathbb{B}^*$ is a simple normal crossing divisor on Z^*.*

(2) *Let $\mathcal{B}^*$ be the boundary associated to $\mathbb{B}^*$, let $x^* \in X^*$ and $x = \varphi(x^*)$. The map $\phi_x^{x^*} : \mathcal{B}(x) \to \mathcal{B}^*(x^*)$ which maps $B \in \mathcal{B}(x)$ to $\phi^{-1}(B) \cap \mathbb{B}^*(x^*)$, the unique component of $\mathcal{B}^*$ containing x^*, is a bijection; its inverse maps $B^* \in \mathcal{B}^*(x^*)$ to $\phi(B^*)$ (with the reduced subscheme structure).*

(3) *The function*

$$
O^* := \phi^{-1}O : X^* \to \{\, \text{subsets of}\, \mathcal{B}^*\,\} \quad ; \quad x^* \mapsto \phi_x^{x^*}(O(\varphi(x^*)))
$$

is a history function for $\mathcal{B}^*$ on X^*, and one has

(3.11)
$$
H^{O^*}_{X^*}(x^*) = H^O_X(x)
$$

for any $x^* \in X^*$ and $x = \varphi(x^*) \in X$. In particular, for any $\widetilde{\nu} \in \Sigma^O_X$ we have

(3.12)
$$
\phi^{-1}(X(\widetilde{\nu})) = X^*(\widetilde{\nu}) \cong X(\widetilde{\nu}) \times_X X^* \, .
$$

(4) *Let D be a closed subscheme of X, and let $D^* = D \times_X X^* = D \times_Z Z^*$, regarded as a closed subscheme of X^*. Let $x^* \in D^*$ and $x = \varphi(x^*) \in D$. Then D^* is transversal with $\mathcal{B}^*$ (resp. normal crossing with $\mathcal{B}^*$, resp. $\mathcal{B}^*$-permissible, resp. O^*-permissible) at x^* if and only if D is transversal with $\mathcal{B}$ (resp. normal crossing with $\mathcal{B}$, resp. $\mathcal{B}$-permissible, resp. O-permissible) at x.*

(5) There are unique morphisms φ' and ϕ' making the diagrams

$$
\begin{array}{ccc}
(X^*)' = B\ell_{D^*}(X^*) \xrightarrow{\varphi'} B\ell_D(X) = X' & \qquad & (Z^*)' = B\ell_{D^*}(Z^*) \xrightarrow{\phi'} B\ell_D(Z) = Z' \\
\pi_{X^*} \downarrow \qquad\qquad\qquad \downarrow \pi_X & & \pi_{Z^*} \downarrow \qquad\qquad\qquad \downarrow \pi_Z \\
X^* \xrightarrow{\quad\varphi\quad} X & & Z^* \xrightarrow{\quad\phi\quad} Z
\end{array}
$$

commutative. Moreover, the diagrams are cartesian, and the morphism $i : X \hookrightarrow Z$ induces a morphism between the diagrams.

(6) The diagrams in (5) identify $(\mathbb{B}^*)^\sim$ with $(\mathbb{B}^\sim) \times_{Z'} (Z^*)'$ and $(\mathbb{B}^*)'$ with $\mathbb{B}' \times_{Z'} (Z^*)'$, as well as $(O^*)^\sim$ with $(O^\sim)^* := (\phi')^{-1}(O^\sim)$ and $(O^*)'$ with $(O')^* := (\phi')^{-1}(O')$.

Proof (1): In view of the remarks after Definition 3.3, it suffices to show: If $Y \subset Z$ is a regular closed subscheme which is of pure codimension c, then $Y^* := Y \times_Z Z^* \subset Z^*$ is regular and of pure codimension c as well. But the first property is clear from Lemma 1.37 (1), and the second one follows from Lemma 1.38: If η^* is a generic point of Y^*, then its image η in Z is a generic point of Y, and the codimension of η^* in the fiber over η is zero. The latter fiber is the same for $Z^* \to Z$ and $Y^* \to Z$. Thus Lemma 1.38 implies that $\mathrm{codim}_{Z^*}(\eta^*) = \mathrm{codim}_Z(\eta)$.

(2): In particular, for $B \in \mathcal{B}(x)$, its preimage $\phi^{-1}(B) = B \times_Z Z^*$ is regular. Since $\mathbb{B}^* \to \mathbb{B}$ is flat, it follows as well that each generic point of $\mathbb{B}^*$ maps to a generic point of $\mathbb{B}$. Thus $\phi^{-1}(B)$ is the disjoint union of those irreducible components of $\mathbb{B}^*$ whose generic points lie above B. There is exactly one component which contains x^*; this is $\phi_x^{x^*}(B)$.

(3): Condition (O1) of Definition 3.6 holds by construction. As for (O2) let $x^*, y^* \in X^*$ with $x^* \in \overline{\{y^*\}}$ and $H_{X^*}(x^*) = H_{X^*}(y^*)$. Let $x = \varphi(x^*)$ and $y = \varphi(y^*)$. Then $x \in \overline{\{y\}}$, and $H_X(x) = H_X(y)$ by Lemma 1.37 (1). So we have $O(y) \subset O(x)$ by property (O2) for O. If $B^* \in O^*(y^*)$, then $y^* \in B^*$ and $\phi(B^*) \in O(y) \subset O(x)$. Hence $x^* \in B^*$ and $B^* \in O^*(x^*)$, which shows (O2) for O^*. Now we show (O3) for O^*. If $y^* \in X^*$, and $y = \varphi(y^*)$, there exists an open subset $U \subset \overline{\{y\}}$ such that $O(x) = O(y)$ for all $x \in U$ with $H_X(x) = H_X(y)$. Now φ maps $\overline{\{y^*\}}$ into $\overline{\{y\}}$, and we let U^* be the preimage of U in $\overline{\{y^*\}}$. Now let $x^* \in U^*$ with $H_{X^*}(x^*) = H_{X^*}(y^*)$. Then $x = \varphi(x^*) \in U$, and $H_X(x) = H_X(y)$ so that $O(x) = O(y)$. By the above we already know that $O^*(y^*) \subset O^*(x^*)$. On the other hand, if $B^* \in O^*(x^*)$, and $B = \phi(B^*)$, then $\phi^{-1}(B)$ is a disjoint union of irreducible components, and B^* is the unique component containing x^*. Since $x \in U$, we have $O(x) = O(y)$, so that $y \in B$. Hence there is a unique component B^{**} of $\pi^{-1}(B)$ containing y^*. Since $x^* \in \overline{\{y^*\}}$, it must equal to B^*. Hence $B^* \in O^*(y^*)$, and we have shown $O^*(x^*) = O^*(y^*)$ and hence (O3) for O^*. The equation (3.11) follows from Lemma 1.37 (1) and the bijection between $O^*(x^*)$ and $O(x)$, and the equality in (3.12) follows from this. The isomorphism in (3.12) follows as in the proof of (1.12).

(4): This follows via the same arguments as used for (1).

(5): The first claim for X follows from the universal property of blow-ups. In fact, if $I \subset \mathcal{O}_X$ is the ideal sheaf of $D \subset X$, then $\varphi^*(I)$ is the ideal sheaf of $D^* \subset X^*$, and this coincides with the image ideal sheaf $\varphi^{-1}\mathcal{O}_{X^*}$ by the flatness of φ. On the other hand, since the ideal sheaf of D^* is $\varphi^{-1}\mathcal{O}_{X^*}$, it follows from [EGA II] (3.5.3) that the left diagram is cartesian. Letting $X = Z$ we get the same for ϕ'. Finally, the universal property of blow-ups for the closed immersions $X \hookrightarrow Z$ and $X^* \hookrightarrow Z^*$ give the last claim.

(6): The first claim follows by by applying (5) to $\mathbb{B}^* \to \mathbb{B}$ and $D \times_Z \mathbb{B} \subset \mathbb{B}$ (for $(\mathbb{B})^\sim =$ $B\ell^{D \times_\mathbb{B}}(\mathbb{B})$), and by taking the base change of the first diagram in (5) with $D \hookrightarrow X$ (to treat the exceptional divisor in $(\mathbb{B})'$). For the remaining claim let $y \in (X^*)'$ with images x^*, x' and x in X^*, X' and X, respectively. Then $H_{X^*}^{O^*}(x^*) = H_X^O(x)$ and $H_{(X')^*}^{(O')^*}(y) = H_{X'}(x')$ by (3), and ϕ and ϕ' induce bijections $O^*(x^*) \cong O(x)$ and $(O')^*(y) \cong O'(x')$. The second claim follows from this and the first claim, which also gives a bijection $\widetilde{O^*(x^*)} \cong \widetilde{O(x)}^*$.

4 B-Permissible blow-ups - the non-embedded case

Let X be a locally noetherian scheme. We start with the following definition.

Definition 4.1 *A boundary on X is a multiset $\mathcal{B} = \{\{B_1, \ldots, B_r\}\}$ of locally principal closed subschemes of X.*

Recall that multisets are 'sets with multiplicities'; more precisely a multiset of r elements is an r-tuple in which one forgets the ordering. One can also think of sets in which an element can appear several times. This then makes clear how one can define elements, cardinalities, inclusions, intersections and unions of multisets. Note also that the locally principal subschemes need not be divisors; e.g., they could be X itself. Both this and the use of multisets is convenient for questions of functoriality, see below.

In the following, let X be a locally noetherian scheme and let $\mathcal{B} = \{\{B_1, \ldots, B_n\}\}$ be a boundary on X. Sometimes, we also call the elements of $\mathcal{B}$ components of $\mathcal{B}$, although they are neither irreducible nor connected in general. For each $x \in X$, let $\mathcal{B}(x) \subset \mathcal{B}$ be the submultiset given by the components containing x. We note that this definition is compatible with arbitrary localization in X. For any morphism $f : Y \to X$ we have a pull-back

(4.1) $$\mathcal{B}_Y := \mathcal{B} \times_X Y := \{\{B_Y := B \times_X Y \mid B \in \mathcal{B}\}\}.$$

We also write $f^{-1}(\mathcal{B})$. Note that, even if we start with a true set of locally principal divisors on X, the pull-back will be a multiset if there are $B_i \neq B_j$ in $\mathcal{B}$ with $(B_i)_Y = (B_j)_Y$. Also, some $(B_i)_Y$ might not be a divisor. For $x \in X$ we let $\mathcal{B}_x = f^{-1}(\mathcal{B})$ with $f : \mathrm{Spec}(\mathcal{O}_{X,x}) \to X$, which is a boundary on $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$.

Definition 4.2 *Let $D \subset X$ be a regular subscheme and let $x \in D$. We say D is transversal with $\mathcal{B}$ at x if for each submultiset $\{\{B_{i_1}, \ldots, B_{i_r}\}\} \subseteq \mathcal{B}(x)$, the scheme-theoretic intersection $D \times_X B_{i_1} \times_X B_{i_2} \times_X \ldots \times_X B_{i_r}$ is regular and of codimension r in D at x. (So this can only hold if $\mathcal{B}(x)$ is a true set.) We say D is normal crossing (n.c.) with $\mathcal{B}$ at x if D is transversal with $\mathcal{B}(x) - \mathcal{B}(x, D)$ at x where*

$$\mathcal{B}(x, D) = \{B \in \mathcal{B}(x) \mid D \subset B\}.$$

We say that D is transversal (resp. normal crossing) with $\mathcal{B}$, if D is transversal (resp. n.c.) with $\mathcal{B}$ at every $x \in D$.

Remark 4.3 *Obviously, D is transversal (resp. normal crossing) with $\mathcal{B}$ at x if and only if the pull-back $\mathcal{B}(x)_D$ (resp. $(\mathcal{B}(x) - \mathcal{B}(x, D))_D$) is a true set and a simple normal crossings boundary on the regular scheme D (see Definition 3.3), i.e., defines a divisor with normal crossings on D.*

Definition 4.4 *Let $D \subset X$ be a regular closed subscheme and $x \in D$. We say $D \subset X$ is $\mathcal{B}$-permissible at x if $D \subset X$ is permissible at x and D is n.c. with $\mathcal{B}$ at x. We say $D \subset X$ is $\mathcal{B}$-permissible if $D \subset X$ is $\mathcal{B}$-permissible at all $x \in D$.*

Let $D \subset X$ be any locally integral closed subscheme and let B be a locally principal (closed) subscheme of X. We now define a canonical locally principal subscheme B' on $X' = B\ell_D(X) \xrightarrow{\pi_X} X$, the blow-up of X in D. Locally we have $X = \operatorname{Spec}(A)$ for a ring A, D is given by a prime ideal $\mathfrak{p}$, and B is given by a principal ideal fA, $f \in A$. In this situation, $B\ell_D(X) = \operatorname{Proj}(A(\mathfrak{p}))$ for the graded A-algebra $A(\mathfrak{p}) = \oplus_{n \geq 0} \mathfrak{p}^n$. Define the homogenous element

$$
(4.2) \qquad f^h = \begin{cases} f \in A(\mathfrak{p})_0 = A & \text{if } f \notin \mathfrak{p} \\ f \in A(\mathfrak{p})_1 = \mathfrak{p} & \text{if } f \in \mathfrak{p}, \end{cases}
$$

Then the graded principal ideal $A(\mathfrak{p}, B) := f^h A(\mathfrak{p})$ only depends on B and not on the equation f, because it does not change if f is multiplied by a unit. Thus

$$
(4.3) \qquad B' := \operatorname{Proj}(A(\mathfrak{p})/A(\mathfrak{p}, B)) \subset \operatorname{Proj}(A(\mathfrak{p})) = X'
$$

gives a well-defined locally principal subscheme, which is a divisor if B is.

To show that the definition glues on a general X and gives a well-defined locally principal subscheme B' on X', we have to show that the above is compatible with localization. So let $g \in A$ and consider $\mathfrak{p}_g \subset A_g$. If $f \in \mathfrak{p}$ then $f \in \mathfrak{p}_g$ and there is nothing to show. The same holds if $f \notin \mathfrak{p}_g$, so that $f \notin \mathfrak{p}$. So we have to consider the case where $f \notin \mathfrak{p}$ but $f \in \mathfrak{p}_g$. Then $f = a/g^m$ for some $a \in \mathfrak{p}$ and some $m > 0$, so that $g^n f \in \mathfrak{p}$ for some $n > 0$, and hence $g \in \mathfrak{p}$ since $f \notin \mathfrak{p}$ and $\mathfrak{p}$ is a prime ideal. Consider any $h \in \mathfrak{p}$ and the associated chart $D_+(h) \subset \operatorname{Proj}(A(\mathfrak{p}))$, which is the spectrum of the subring $A(\mathfrak{p})_{(h)} \subset A_h$ which is the union of the sets

$$
\frac{\mathfrak{p}^n}{h^n}
$$

for all $n \geq 0$. By definition (and the assumption $f \notin \mathfrak{p}$), the trace of B' on this chart is defined by the ideal given by the union of the sets

$$
\frac{f\mathfrak{p}^n}{h^n},
$$

and the pull-back to A_g is defined by the union of the sets

$$
(4.4) \qquad \frac{f\mathfrak{p}^n}{h^n g^s}
$$

for all $n, s \geq 0$. On the other hand, the pull-back of $D_+(h)$ to A_g is also the spectrum of the ring which is the union of the sets

$$
\frac{\mathfrak{p}_g^n}{h^n}
$$

for $n > 0$, and by definition (4.2) (and the assumption $f \in \mathfrak{p}_g$), the ideal of B' for this ring is the union of the sets

$$
(4.5) \qquad \frac{f\mathfrak{p}_g^{n-1}}{h^n} = \bigcup_{s \geq 0} \frac{f\mathfrak{p}^{n-1}}{h^n g^s} = \bigcup_{s \geq 0} \frac{fg\mathfrak{p}^{n-1}}{h^n g^{s+1}}
$$

for all $n \geq 0$. But the union of the sets (4.4) and (4.5) is the same, because $g \in \mathfrak{p}$.

Definition 4.5 *The locally principal subscheme B' defined above is called the principal strict transform of B in X'.*

In the following, we will always use these principal strict transforms and will call them simply transforms.

Remark 4.6 *(a) There is always a commutative diagram of natural proper morphisms*

where $\tilde{B} = B\ell_{D \times_X B}(B)$ is the scheme-theoretic strict transform of B in X'. All morphisms are isomorphisms over $B \backslash D$, and $i : \tilde{B} \to B'$ is a closed immersion. However, it is not in general an isomorphism, and $\tilde{B}$ need not be a locally principal subscheme. In fact, with the notations above, $\tilde{B}$ is locally given by the graded ideal

$$(4.6) \qquad \widetilde{A(\mathfrak{p}, B)} := \oplus_n fA \cap \mathfrak{p}^n \quad \supset \quad A(\mathfrak{p}, B) = \begin{cases} \oplus_n f\mathfrak{p}^n & \text{if } f \notin \mathfrak{p} \\ \oplus_n f\mathfrak{p}^{n-1} & \text{if } f \in \mathfrak{p}, \end{cases}$$

and the indicated inclusion need not be an equality (or give an isomorphism after taking Proj).

(b) If X and D are regular, and B is a regular divisor, then $\tilde{B} = B'$. In fact, with the notations above, $\mathfrak{p}$ is a regular prime ideal, and locally we have $fA \cap \mathfrak{p}^n = f\mathfrak{p}^{n-v_\mathfrak{p}(f)}$ for the discrete valuation $v_\mathfrak{p}$ associated to $\mathfrak{p}$, because $v_\mathfrak{p}(fa) = v_\mathfrak{p}(f) + v_\mathfrak{p}(a)$. Moreover, $v_\mathfrak{p}(f) \in \{0, 1\}$ by assumption.

(c) If $i : X \hookrightarrow Z$ is a closed immersion into a regular scheme Z and $\mathbb{B}$ is a simple normal crossings divisor on Z, then the set $\mathcal{B} = C(\mathbb{B}) = \{B_1, \ldots, B_r\}$ of irreducible components is a simple normal crossings boundary on Z. In particular, it is a boundary in the sense of Definition 4.1, and $\mathcal{B}_X = i^{-1}(\mathcal{B})$, its pull-back to X, is a boundary on X (which may be a multiset). This construction connects the present section with the previous one. (See also Lemma 4.21 below.)

Now let $\mathcal{B} = \{\{B_1, \ldots, B_n\}\}$ be a boundary on X. Let B_i' be the (principal strict) transform of B_i in X', $i = 1, \ldots, n$, and let $E = D \times_X X'$ be the exceptional divisor.

Definition 4.7 *Call $\widetilde{\mathcal{B}} = \{\{B_1', \ldots, B_n'\}\}$ the strict transform and $\mathcal{B}' = \{\{B_1', \ldots, B_n', E\}\}$ the complete transform of $\mathcal{B}$ in X'.*

We note that E is always a locally principal divisor, so that $\widetilde{\mathcal{B}}$ and $\mathcal{B}'$ are boundaries on X. Moreover, we note the following useful functorialities.

Lemma 4.8 *(a) Let $Y \hookrightarrow X$ be a closed immersion, and assume that $D \subset Y$ is a nowhere dense, locally integral closed subscheme. Then for the closed immersion $Y' = B\ell_D(Y) \hookrightarrow B\ell_D(X) = X'$ one has*

$$(\mathcal{B}_Y)' = (\mathcal{B}')_{Y'} \quad and \quad \widetilde{\mathcal{B}_Y} = (\widetilde{\mathcal{B}})_{Y'}.$$

(b) Let $\varphi : X^* \to X$ be a flat morphism with regular fibers, let $D \subset X$ be a closed subscheme, and let $D^* = D \times_X X^*$, regarded as a closed subscheme of X^*. Then D^* is regular if and only D is regular, and in this case one has

$$(\mathcal{B}_{X^*})' = (\mathcal{B}')_{(X^*)'} \quad and \quad \widetilde{\mathcal{B}_{X^*}} = (\widetilde{\mathcal{B}})_{(X^*)'},$$

where $(X^*)' = B\ell_{D^*}(X^*) \to B\ell_D(X) = X'$ is the canonical morphism.

Proof The question is local on X, so we may assume that $X = \mathrm{Spec}(A)$ is affine, and take up the notations of (4.2).

(a): Here $Y = \mathrm{Spec}(A/\mathfrak{b})$ for an ideal $\mathfrak{b} \subset \mathfrak{p} \subset A$, and for $B = \mathrm{Spec}(A/fA) \in \mathcal{B}$, $B \times_X Y \in \mathcal{B}_Y$ is given by $(fA + \mathfrak{b})/\mathfrak{b}$. Thus the case distinction in (4.2) is the same for f and $\bar{f} = f + \mathfrak{b}$, and the claim follows from the equality $\bar{f}(\mathfrak{p}/\mathfrak{b})^n = (f\mathfrak{p}^n + \mathfrak{b})/\mathfrak{b}$ in the first case and the equality $\bar{f}(\mathfrak{p}/\mathfrak{b})^{n-1} = (f\mathfrak{p}^{n-1} + \mathfrak{b})/\mathfrak{b}$ in the second case. For the exceptional divisor $E_X = D \times_X X'$ on X' one has $E_X \times_{X'} Y' = D \times_X Y' = D \times_Y Y' = E_Y$, the exceptional divisor on Y'.

(b): Since $D^* \to D$ is flat with regular fibers, the first claim follows from 1.37 (1). For the following we may assume that $X^* = \mathrm{Spec}(A^*)$ for a flat A-algebra A^*. Then for $B = \mathrm{Spec}(A/fA) \in \mathcal{B}$, B_{X^*} equals $\mathrm{Spec}(A^*/fA^*)$, and D^* is defined by the ideal $\mathfrak{p}A^*$. Now $A/\mathfrak{p}$ is integral and the map $A/\mathfrak{p} \to A^* \otimes_A A/\mathfrak{p} = A^*/\mathfrak{p}A^*$ is flat, hence this map is also injective. Thus the case distinction in (4.2) is the same for f and $\mathfrak{p}$ on the one hand, and for f and $\mathfrak{p}A^*$ on the other. Hence $A(\mathfrak{p}A^*, B_{X^*}) = A(\mathfrak{p}, B) \otimes_A A^*$, and hence $(B_{X^*})' = (B')_{(X^*)'}$ as claimed. $\square$

Definition 4.9 *(1) A history function for a boundary $\mathcal{B}$ on X is a function*

(4.7) $$O : X \to \{submultisets\ of\ \mathcal{B}\} \ ;\ x \to O(x),$$

which satisfies the following conditions:

(O1) For any $x \in X$, $O(x) \subset \mathcal{B}(x)$.

(O2) For any $x, y \in X$ such that $x \in \overline{\{y\}}$ and $H_X(x) = H_X(y)$, we have $O(y) \subset O(x)$.

(O3) For any $y \in X$, there exists a non-empty open subset $U \subset \overline{\{y\}}$ such that $O(x) = O(y)$ for all $x \in U$ such that $H_X(x) = H_X(y)$.

For such a function, we put for $x \in X$,

$$N(x) = \mathcal{B}(x) - O(x).$$

A divisor $B \in \mathcal{B}$ is called old (resp. new) for O at x if it is a component of $O(x)$ (resp. $N(x)$).

(2) A boundary with history on X is a pair $(\mathcal{B}, O)$, where $\mathcal{B}$ is a boundary on X and O is a history function for $\mathcal{B}$.

A basic example of a history function for B on X is given by the following:

Lemma 4.10 *The function $O(x) = \mathcal{B}(x)$ $(x \in X)$, is a history function for $\mathcal{B}$ on X. In fact it satisfies 4.9 (O2) and (O3) without the condition $H_X(x) = H_X(y)$.*

Proof Left to the readers. $\square$

Define a function:
$$H_X^O : X \to \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \; ; \; x \to (H_X(x), |\, O(x)\, |),$$

where $|\, O(x)\, |$ is the cardinality of $O(x)$. We endow $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}$ with the lexicographic order:

$$(\nu, \mu) \geq (\nu', \mu') \Leftrightarrow \nu > \nu' \text{ or } \nu = \nu' \text{ and } \mu \geq \mu'.$$

The conditions in 4.9 and Theorem 1.33 immediately imply the following:

Theorem 4.11 *Let the assumption be as above.*

(1) If $x \in X$ is a specialization of $y \in X$, then $H_X^O(x) \geq H_X^O(y)$.

(2) For any $y \in X$, there is a dense open subset U of $\overline{\{y\}}$ such that $H_X^O(y) = H_X^O(x)$ for any $x \in U$.

In other words (see Lemma 1.34), the function H_X^O is upper semi-continuous on X. By noetherian induction Theorem 4.11 implies

$$\Sigma_X^O := \{H_X^O(x)|\ x \in X\} \subset \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$$

is finite. We define $\Sigma_X^{O,max}$ to be the set of of the maximal elements in Σ_X^O.

Definition 4.12 *(1) For $\widetilde{\nu} \in \Sigma_X^O$ we define*

$$X(\widetilde{\nu}) = X^O(\widetilde{\nu}) = \{x \in X \mid H_X^O(x) = \widetilde{\nu}\}$$

and

$$X(\geq \widetilde{\nu}) = X^O(\geq \widetilde{\nu}) = \{x \in X \mid H_X^O(x) \geq \widetilde{\nu}\}\,,$$

and we call

(4.8)
$$X_{max}^O = \bigcup_{\widetilde{\nu} \in \Sigma_X^{O,max}} X(\widetilde{\nu})\,.$$

the O-Hilbert-Samuel locus of X.

(2) We define
$$\mathrm{Dir}_x^O(X) := \mathrm{Dir}_x(X) \cap \bigcap_{B \in O(x)} T_x(B)\ \subset T_x(Z).$$

$$e_x^O(X) = \dim_{k(x)}(\mathrm{Dir}_x^O(X)).$$

By Theorem 4.11 and Lemma 1.34, $X(\widetilde{\nu})$ is locally closed, with closure contained in $X(\geq \widetilde{\nu})$. In particular, $X(\widetilde{\nu})$ is closed for $\widetilde{\nu} \in \Sigma_X^{O,max}$, the union in Definition 4.8 is disjoint and X_{max}^O is a closed subset of X. Theorems 2.2 and 2.3 imply the following:

Theorem 4.13 *Let $D \subset X$ be a regular closed subscheme and $x \in D$. Then the following conditions are equivalent:*

(1) $D \subset X$ is permissible at x and there is an open neighborhood U of x in Z such that $D \cap U \subset B$ for every $B \in O(x)$.

(2) $H_X^O(x) = H_X^O(y)$ for any $y \in D$ such that x is a specialization of y.

Under the above conditions, we have

$$(4.9) \qquad\qquad T_x(D) \subset \mathrm{Dir}_x^O(X).$$

Definition 4.14 *A closed subscheme $D \subset X$ is called O-permissible at x, if it satisfies the equivalent conditions in Theorem 4.13, and it is called O-permissible, if it is O-permissible at every $x \in D$.*

Remark 4.15 *Note that $D \subset X$ is $\mathcal{B}$-permissible at x if and only if D is O-permissible and n.c with $N(x)$ at x.*

Let $D \subset X$ be a $\mathcal{B}$-permissible closed subscheme. Consider the blow-up

$$(4.10) \qquad\qquad X' = B\ell_D(X) \xrightarrow{\pi_X} X$$

of X in D, and let $\mathcal{B}'$ be the complete transform of $\mathcal{B}$ in X'. For a given history function O for $\mathcal{B}$ on X, we define functions O', $\widetilde{O} : X' \to \{$subsets of $\mathcal{B}'\}$ as follows: Let $x' \in X'$ and $x = \pi_X(x') \in X$. Then define

$$(4.11) \qquad\qquad O'(x') = \begin{cases} \widetilde{O(x)} \cap \mathcal{B}'(x') & \text{if } H_{X'}(x') = H_X(x) \\ \mathcal{B}'(x') & \text{otherwise,} \end{cases}$$

where $\widetilde{O(x)}$ is the strict transform of $O(x)$ in X' and

$$(4.12) \qquad\qquad \widetilde{O}(x') = \begin{cases} \widetilde{O(x)} \cap \mathcal{B}'(x') & \text{if } H_{X'}(x') = H_X(x) \\ \widetilde{\mathcal{B}}(x') & \text{otherwise.} \end{cases}$$

Note that $O'(x') = \widetilde{O}(x') = \widetilde{O(x)} \cap \widetilde{\mathcal{B}}(x')$ if x' is near to x.

The proof of the following lemma is identical with that of Lemma 3.14.

Lemma 4.16 *The function $X' \to \{$subsets of $\mathcal{B}'\}$; $x' \to O'(x')$ is a history function.*

Definition 4.17 *We call $(\mathcal{B}', O')$ and $(\widetilde{\mathcal{B}}, \widetilde{O})$ the complete and strict transform of $(\mathcal{B}, O)$ in X', respectively.*

Results from the previous section (embedded case) have their companions in the non-embedded situation. We start with the following theorem analogous to Theorem 3.16.

Theorem 4.18 *Take points $x \in D$ and $x' \in \pi_X^{-1}(x)$. Then $H_{X'}^{\widetilde{O}}(x') \leq H_{X'}^{O'}(x') \leq H_X^O(x)$. In particular we have*

$$\pi_X^{-1}(X^{O'}(\widetilde{\nu})) \subset \bigcup_{\widetilde{\mu} \leq \widetilde{\nu}} X'^{O'}(\widetilde{\mu}) \quad \text{for } \widetilde{\nu} \in \Sigma_X^{max},$$

and the same holds for $\widetilde{O}$ in place of O'.

Proof This follows immediately from Theorem 2.10, (4.11) and (4.12). $\square$

In the following we mostly use the complete transform $(\mathcal{B}', O')$ and, for ease of notation, we often write $H_{X'}^O(x')$ and $\Sigma_{X'}^O$ instead of $H_{X'}^{O'}(x')$ and $\Sigma_{X'}^{O'}$, similarly for $\Sigma_{X'}^{O,max}$ etc., because everything just depends on O.

Definition 4.19 *We say that $x' \in \pi_X^{-1}(x)$ is O-near to x if the following equivalent conditions hold:*

(1) $H_{X'}^O(x') = H_X^O(x)$ ($\Leftrightarrow$ $H_{X'}^{O'}(x') = H_X^O(x)$ $\Leftrightarrow$ $H_{X'}^{\widetilde{O}}(x') = H_X^O(x)$).

(2) x' is near to x and contained in the strict transforms of B for all $B \in O(x)$.

Call x' is very O-near to x if x' is O-near and very near to x and $e_{x'}^O(X') = e_x^O(X) - \delta_{x'/x}$.

The following result, the non-embedded analogue of Theorem 3.18, is an immediate consequence of Theorem 2.14 and Definition 4.12(2).

Theorem 4.20 *Assume that $x' \in X'$ is O-near to $x = \pi(x') \in X$. Assume further that $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$, where $k(x)$ is the residue field of x. Then*

$$x' \in \mathbb{P}(\mathrm{Dir}_x^O(X)/T_x(D)) \subset \mathbb{P}(C_x(X)/T_x(D)) = \pi_X^{-1}(x)$$

Next we show the compatibility of the notions for the non-embedded case in the present section with the corresponding notions for the embedded situation in the previous section.

Lemma 4.21 *Let $i : X \hookrightarrow Z$ be an embedding into a regular excellent scheme Z, let $\mathcal{B}$ be a simple normal crossings boundary on Z, and let $\mathcal{B}_X = i^{-1}(\mathcal{B})$ be its pull-back to X.*
(1) For a closed regular subscheme $D \subset X$ and $x \in D$, D is transversal (resp. normal crossing) with $\mathcal{B}$ at x in the sense of Definition 3.1 if and only if it is transversal (resp. normal crossing) with $\mathcal{B}_X$ in the sense of Definition 4.2 (which is an intrinsic condition on $(X, \mathcal{B}_X)$).
(2) Let O be a history function for $\mathcal{B}$ in the sense of Definition 3.6, and define the function

$$O_X : X \to \{submultisets \text{ of } \mathcal{B}_X\} \quad , \quad O_X(x) = \{\{B_X \mid B \in O(x)\}\}.$$

Then O_X is a history function for $\mathcal{B}_X$ in the sense of Definition 4.9, and one has

$$H_X^{O_X}(x) = H_X^O(x) \quad and \quad \mathrm{Dir}_x^{O_X}(X) = \mathrm{Dir}_x^O(X)$$

(and hence $e_x^{O_X}(X) = e_x^O(X)$) for all $x \in X$. Also, for a $k(x)$-linear subspace $T \subset T_x(X)$, the two notions for the transversality $T \pitchfork N(x)$ (Definition 3.19 for $\mathcal{B}$ and Definition 4.24 for $\mathcal{B}_X$) are equivalent.
(3) A regular closed subscheme $D \subset X$ is $\mathcal{B}_X$-permissible in the sense of Definition 4.4 if and only if it is $\mathcal{B}$-permissible in the sense of Definition 3.5. Moreover, it is O_X-permissible in the sense of Definition 4.14 if and only if it is O-permissible in the sense of Definition 3.11.
(4) Let $D \subset X$ be $\mathcal{B}$-permissible, let $\pi_X : X' = B\ell_D(X) \to X$ and $\pi_Z : Z' = B\ell_D(Z) \to Z$ be the respective blowups in D and $i' : X' \hookrightarrow Z'$ the closed immersion. Moreover let O a history function for $\mathcal{B}$. Then we have the equalities

$$((\mathcal{B}_X)', (O_X)') = ((\mathcal{B}')_{X'}, (O')_{X'}) \quad and \quad (\widetilde{\mathcal{B}_X}, \widetilde{O_X}) = ((\widetilde{\mathcal{B}})_{X'}, (\widetilde{O})_{X'})$$

for the complete transforms and strict transforms, respectively.

Proof The claims in (1), (2) and (3) easily follow from the definitions. For the claim on the directrix in (2) note that $T_x(B_X) = T_x(B) \cap T_x(X)$ (in $T_x(Z)$). The claim in (4) follows from Lemma 4.8. $\square$

Results in the non-embedded case which depend only on the local ring at a point (of the base scheme) can often be reduced to the embedded case. This relies on Lemma 4.21 and the following two observations.

Remark 4.22 *Let X be an excellent scheme, let $\mathcal{B}$ be a boundary on X, and let $x \in X$. Assume a property concerning $(X, \mathcal{B}, x)$ can be shown by passing to the local ring $\mathcal{O} = \mathcal{O}_{X,x}$, and its completion $\hat{\mathcal{O}}$. Then the following construction is useful. The ring $\hat{\mathcal{O}}$ is the quotient of a regular excellent ring R. Let $\mathcal{B}(x) = \{\{B_1, \ldots, B_r\}\}$, and let $f_1, \ldots, f_r$ be the local functions defining them in $\mathcal{O}_{X,x}$ (so we can have $f_i = f_j$ for $i \neq j$). Then we get a surjection*

$$R[X_1, \ldots, X_r] \twoheadrightarrow \mathcal{O}_{X,x}$$

mapping X_i to f_i, and the functions X_i define a simple normal crossings boundary on $Z = \mathrm{Spec}(R[X_1, \ldots, X_r])$, such that $\mathcal{B}(x)$ is its pull-back under $\mathrm{Spec}(\mathcal{O}_{X,x}) \hookrightarrow Z$. We may thus assume that X can be embedded in a regular excellent scheme Z with simple normal crossings boundary $\mathcal{B}_Z$, and that $\mathcal{B}$ is the pull-back of $\mathcal{B}_Z$ to X.

Now we apply Remark 4.22 and Lemma 4.21.

Theorem 4.23 *Let $D \subseteq X$ be an irreducible subscheme. Assume:*

(1) $D \subset X$ is O-permissible at x.

(2) $e_\eta(X) = e_x(X) - \dim(\mathcal{O}_{D,x})$ (cf. Theorem 2.6),

where η is the generic point of D. Then we have

$$e_\eta^O(X) \leq e_x^O(X) - \dim(\mathcal{O}_{D,x}).$$

Proof The question is local around x, and we may pass to $\mathcal{O}_{X,x}$ and then to its completion, since X is excellent. By 4.22 and 4.21 we may assume that we are in an embedded situation. Thus the claim follows from the corresponding result in the embedded case (Theorem 3.13). $\square$

Let $\pi_X : X' = Bl_D(X) \to X$ be as in (4.10). For $x' \in X'$ let $N'(x') = \mathcal{B}'(x') - O'(x')$ be the set of the new components of $(\mathcal{B}', O')$ for x'. If x' is near to $x = \pi_X(x') \in D$, i.e., $H_{X'}(x') = H_X(x)$, then

$$(4.13) \qquad N'(x') = (\widetilde{N(x)} \cap \mathcal{B}'(x')) \cup \{E\} \quad \text{with} \quad E = \pi_Z^{-1}(D)$$

where $\widetilde{N(x)}$ is the strict transform of $N(x)$ in X'. If x' is not near to x, then $N'(x') = \emptyset$. Similarly, $\tilde{N}(x') = \tilde{\mathcal{B}}(x') - \tilde{O}(x') = \widetilde{N(x)} \cap \tilde{\mathcal{B}}(x') \subset N'(x')$ if x' is near to x, and $\tilde{N}(x') = \emptyset$, otherwise. We study the transversality of $N'(x')$ with a certain regular subscheme of E.

Definition 4.24 *For a $k(x)$-linear subspace $T \subset T_x(X)$, we say that T is transversal with $N(x)$ (notation: $T \pitchfork N(x)$) if*

$$\dim_{k(x)} \left(T \cap \bigcap_{B \in N(x)} T_x(B) \right) = \dim_{k(x)}(T) - \mid N(x) \mid .$$

Lemma 4.25 *Assume $D = x$, $T \subset \mathrm{Dir}_x(X)$ and $T \pitchfork N(x)$. Then the closed subscheme $\mathbb{P}(T) \subset E_x = \mathbb{P}(C_x(X))$ is n.c. with $N'(x')$ (and hence also $\widetilde{N}(x')$) at x'.*

Proof In the same way as above, we this claim follows from the corresponding result in the embedded case (Lemma 3.20). $\square$

Lemma 4.26 *Let $\pi_X : X' = B\ell_D(X) \to X$ be as (4.10). Assume $T \pitchfork N(x)$ and $T_x(D) \subset T \subset \mathrm{Dir}_x(X)$ and $\dim_{k(x)}(T/T_x(D)) = 1$. Consider*

$$x' = \mathbb{P}(T/T_x(D)) \subset \mathbb{P}(C_x(X)/T_x(D)) = \pi_X^{-1}(x).$$

Let $D' \subset E = \pi_X^{-1}(D)$ be any closed subscheme such that $x' \in D'$ and π_X induces an isomorphism $D' \simeq D$. Then D' is n.c. with $N'(x')$ at x'.

Proof In the same way as above, this follows from the corresponding result in the embedded case (Lemma 3.21). $\square$

Theorem 4.27 *Let $\pi_X : X' = B\ell_D(X) \to X$ be as in (4.10). Take $x \in X$ and $x' \in \pi_X^{-1}(x)$. Assume $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$.*

   (1) If x' is O-near and very near to x, then $e_{x'}^O(X') \leq e_x^O(X) - \delta_{x'/x}$.

   (2) Assume x' is very O-near and $N(x) \pitchfork \mathrm{Dir}_x^O(X)$. Then $N'(x') \pitchfork \mathrm{Dir}_{x'}^O(X')$.

Proof Reduction to the embedded case (Theorem 3.23). $\square$

Corollary 4.28 *Let $\pi_X : X' = B\ell_D(X) \to X$ be as in (4.10) and take closed points $x \in D$ and $x' \in \pi_X^{-1}(x)$ such that x' is O-near to x. Assume $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$. Assume further that there is an integer $e \geq 0$ for which the following hold:*

   (1) $e_x(X)_{k(x')} \leq e$, and either $e \leq 2$ or $k(x')$ is separable over $k(x)$.

   (2) $N(x) \pitchfork \mathrm{Dir}_x^O(X)$ or $e_x^O(X) \leq e - 1$.

Then $N'(x') \pitchfork \mathrm{Dir}_{x'}^O(X')$ or $e_{x'}^O(X') \leq e - 1$.

This follows from Theorem 4.27 like Corollary 3.24 follows from Theorem 3.23.

Definition 4.29 *Call $(\mathcal{B}, O)$ admissible at $x \in X$, if $N(x) \pitchfork T_x(X)$, and call $(\mathcal{B}, O)$ admissible if it is admissible at all $x \in X$.*

We note that admissibility of $(\mathcal{B}, O)$ at x implies $\mathcal{B}_{in}(x) \subseteq O(x)$, where $\mathcal{B}_{in}(x)$ is defined as follows.

Definition 4.30 *Call $B \in \mathcal{B}$ inessential at $x \in X$, if it contains all irreducible components of X which contain x. Let*

$$\mathcal{B}_{in}(x) = \{B \in \mathcal{B} \mid Z \subseteq B \text{ for all } Z \in I(x)\}$$

be the set of inessential boundary components at x, where $I(x)$ is the set of the irreducible components of X containing x.

Definition 4.31 *Call $x \in X$ O-regular (or X O-regular at x), if*

(4.14) $$H_X^O(x) = (\nu_X^{reg}, |\mathcal{B}_{in}(x)|).$$

Call X O-regular, if it is O-regular at all $x \in X$.

Lemma 4.32 *If $(\mathcal{B}, O)$ is admissible at x and X is O-regular at x, then X is regular and normal crossing with $\mathcal{B}$ at x.*

Proof The proof is identical with that of Lemma 3.28. $\square$

Lemma 4.33 *Let $\pi : X' = B\ell_D(X) \to X$ be as (4.10). If $(\mathcal{B}, O)$ admissible at $x \in X$, then $(\mathcal{B}', O')$ and $(\widetilde{\mathcal{B}}, \widetilde{O})$ are admissible at any $x' \in X'$ with $x = \pi(x')$.*

Proof In the same way as in the proof of Lemma 4.23, this is reduced to the embedded case (Theorem 3.29). $\square$

Later we shall need the following comparison for a closed immersion.

Lemma 4.34 *Let $i : Y \hookrightarrow X$ be a closed immersion of excellent schemes, let $\mathcal{B}$ be a boundary on X, and let $\mathcal{B}_Y = i^{-1}(\mathcal{B})$ be its pull-back to Y.*
(1) For a closed regular subscheme $D \subset Y$ and $x \in D$, D is transversal (resp. normal crossing) with $\mathcal{B}$ at x if and only if it is transversal (resp. normal crossing) with $\mathcal{B}_Y$.
(2) Let O be a history function for $\mathcal{B}$, and define the function

$$O_Y : Y \to \{\text{submultisets of } \mathcal{B}_Y\} \quad , \quad O_Y(x) = \{\{B_Y \mid B \in O(x)\}\}.$$

Then O_Y is a history function for $\mathcal{B}_Y$. If $x \in Y$ and $(\mathcal{B}_Y, O_Y)$ is admissible at x, then $(\mathcal{B}, O)$ is admissible at x. (The converse does not always hold.)
(3) Let $D \subset Y$ be a regular closed subscheme which is permissible for Y and X. Then D is $\mathcal{B}_Y$-permissible if and only if it is $\mathcal{B}$-permissible. Moreover, it is O_Y-permissible if and only if it is O-permissible.
(4) Let $D \subset Y$ be $\mathcal{B}_Y$-permissible and $\mathcal{B}_X$-permissible, let $\pi_Y : Y' = B\ell_D(Y) \to Y$ and $\pi_X : X' = B\ell_D(X) \to X$ be the respective blowups in D and $i' : Y' \hookrightarrow X'$ the closed immersion. Moreover let O a history function for $\mathcal{B}$. Then we have the equality

$$((\mathcal{B}_Y)', (O_Y)') = ((\mathcal{B}')_{Y'}, (O')_{Y'}) \quad and \quad (\widetilde{\mathcal{B}_Y}, \widetilde{O_Y}) = ((\widetilde{\mathcal{B}})_{Y'}, (\widetilde{O})_{Y'})$$

for the complete and strict transforms, respectively.

The proofs are along the same lines as for Lemma 4.21. For (2) note that $T_x(Y) \subset T_x(X)$ and that for subspaces $T_1 \subset T_2 \subset T_x(X)$ one has $N(x) \pitchfork T_1 \Longrightarrow N(x) \pitchfork T_2$.

Remark 4.35 *Since a regular subscheme of a regular scheme is always permissible, Lemma 4.21 can be seen as a special case of Lemma 4.34.*

The following is the non-embedded analogue of Lemma 3.30.

Lemma 4.36 *Let $\varphi : X^* \to X$ be a flat morphism with regular fibers of locally noetherian schemes, let $\mathcal{B}$ be a boundary on X, and let O be a history function for $\mathcal{B}$ on X.*

(1) Let $\mathcal{B}^* = \varphi^{-1}(\mathcal{B}) = \mathcal{B}_{X^*}$ be the pull-back to X^*, let $x^* \in X^*$ and $x = \varphi(x^*)$. The map $\varphi_x^{x^*} : \mathcal{B}(x) \to \mathcal{B}^*(x^*)$ which maps $B \in \mathcal{B}(x)$ to $B^* = B \times_X X^*$ is a bijection.

(2) The function

$$O^* := \varphi^{-1}(O) : X^* \to \{\,\text{subsets of}\,\mathcal{B}^*\,\} \quad ; \quad x^* \mapsto \varphi_{\varphi(x^*)}^{x^*}(O(\varphi(x^*)))$$

is a history function for $\mathcal{B}^*$ on X^*, and one has

$$(4.15) \qquad\qquad\qquad\qquad H_{X^*}^{O^*}(x^*) = H_X^O(x)$$

for any $x^* \in X^*$ and $x = \varphi(x^*) \in X$. In particular, for any $\widetilde{\nu} \in \Sigma_X^O$ we have

$$(4.16) \qquad\qquad\qquad \varphi^{-1}(X(\widetilde{\nu})) = X^*(\widetilde{\nu}) \cong X(\widetilde{\nu}) \times_X X^* .$$

(3) Let D be a closed subscheme of X, and let $D^* = D \times_X X^* = D \times_Z Z^*$, regarded as a closed subscheme of X^*. Let $x^* \in D^*$ and $x = \varphi(x^*) \in D$. Then D^* is transversal with $\mathcal{B}^*$ (resp. normal crossing with $\mathcal{B}^*$, resp. $\mathcal{B}^*$-permissible, resp. O^*-permissible) at x^* if and only if D is transversal with $\mathcal{B}$ (resp. normal crossing with $\mathcal{B}$, resp. $\mathcal{B}$-permissible, resp. O-permissible) at x.

(4) There is a unique morphism φ' making the following diagram

$$
\begin{array}{ccc}
(X^*)' = B\ell_{D^*}(X^*) & \xrightarrow{\ \varphi'\ } & B\ell_D(X) = X' \\[4pt]
\pi_{X^*}\Big\downarrow & & \Big\downarrow \pi_X \\[4pt]
X^* & \xrightarrow{\ \ \varphi\ \ } & X
\end{array}
$$

commutative, where π_X and π_{X^*} is the structural morphisms. Moreover the diagram is cartesian, and it identifies $(\mathcal{B}^*)^\sim$ with $(\mathcal{B}^\sim)^* := (\varphi')^{-1}(\mathcal{B}^\sim)$ and $(\mathcal{B}^*)'$ with $(\mathcal{B}')^* := (\varphi')^{-1}(\mathcal{B}')$, as well as $(O^*)^\sim$ with $(O^\sim)^* := (\varphi')^{-1}(O^\sim)$ and $(O^*)'$ with $(O')^* := (\varphi')^{-1}(O')$.

Proof (1) is trivial, and in (2), the conditions for a history function are easily checked. Since we have a bijection between $O^*(x^*)$ and $O(x)$, equation (4.15) follows from Lemma 1.37 (1). By Remark 4.3, the first three cases in (3) follow from Lemma 3.30 (1). The last claim follows from (4.15).

In (4) the first two claims are shown like in the proof of Lemma 3.30 (5). The last four claims follow by applying the first two claims to every $B \in \mathcal{B}$.

5 Main theorems and strategy for their proofs

We will treat the following two situations in a parallel way:

(E) (*embedded case*) X is an excellent scheme, $i : X \hookrightarrow Z$ is a closed immersion into an excellent regular scheme Z, and $\mathcal{B}$ is a simple normal crossings boundary on Z (Definition 3.3).

(NE) (*non-embedded case*) X is an excellent scheme, and $\mathcal{B}$ is a boundary on X (Definition 4.1).

Definition 5.1 *(1) A point $x \in X$ is called $\mathcal{B}$-regular if X is regular at x (i.e. $\mathcal{O}_{X,x}$ is regular) and normal crossing with $\mathcal{B}$ at x. Call X $\mathcal{B}$-regular, if every $x \in X$ is $\mathcal{B}$-regular, i.e., if X is regular and $\mathcal{B}$ is normal crossing with X.*

(2) Call x strongly $\mathcal{B}$-regular if X is regular at x and for every $B \in \mathcal{B}(x)$, B contains the (unique) irreducible component on which x lies. (This amounts to the equation $\mathcal{B}(x) = \mathcal{B}_{in}(x)$ where $\mathcal{B}_{in}(x)$ is the (multi)set of inessential boundary components at x, see Definitions 3.26 (Case (E)) and 4.30 (Case (NE)).)

Denote by X_{reg} (resp. $X_{\mathcal{B}\mathrm{reg}}$, resp. $X_{\mathcal{B}\mathrm{sreg}}$) the set of the regular (resp. $\mathcal{B}$-regular, resp. strongly $\mathcal{B}$-regular) points of X. These are open subsets of X, and dense in X if X is reduced. Call $X_{\mathcal{B}\mathrm{sing}} = X - X_{\mathcal{B}\mathrm{reg}}$ the $\mathcal{B}$-singular locus of X.

We introduce the following definition for the case of non-reduced schemes.

Definition 5.2 *(1) Call $x \in X$ quasi-regular, if X_{red} is regular at x and X is normally flat along X_{red} at x. Call X quasi-regular if it is quasi-regular at all $x \in X$, i.e., if X_{red} is regular and X is normally flat along X_{red}. (Compare Definition 2.1, but we have reserved the name 'permissible' for subschemes not containing any irreducible component of X.)*

(2) Call $x \in X$ quasi-$\mathcal{B}$-regular, if X_{red} is $\mathcal{B}$-regular at x and X is normally flat along X_{red} at x. Call X quasi-$\mathcal{B}$-regular, if X is quasi-$\mathcal{B}$-regular at all $x \in X$, i.e., if X_{red} is $\mathcal{B}$-regular and X is normally flat along X_{red}. (Similar remark on comparison with $\mathcal{B}$-permissibility.)

(3) Call $x \in X$ strongly quasi-$\mathcal{B}$-regular, if X_{red} is strongly $\mathcal{B}$-regular at x and X is normally flat along X_{red} at x.

Note that X is regular if and only if X is quasi-regular and reduced. Similarly, X is $\mathcal{B}$-regular if and only if X is quasi-$\mathcal{B}$-regular and reduced. Finally, $x \in X$ is strongly $\mathcal{B}$-regular if and only if x is strongly quasi-$\mathcal{B}$-regular and $\mathcal{O}_{X,x}$ is reduced.

Denote by X_{qreg}, $X_{\mathcal{B}\mathrm{qreg}}$ and $X_{\mathcal{B}\mathrm{sqreg}}$ the sets of quasi-regular, quasi-$\mathcal{B}$-regular and strongly quasi-$\mathcal{B}$-regular points of X, respectively. By Theorem 2.2 these are dense open subsets of X. Moreover, we have inclusions

$$
\begin{array}{ccccccc}
X_{\mathrm{qreg}} & \supset & X_{\mathcal{B}\mathrm{qreg}} & \supset & X_{\mathcal{B}\mathrm{sqreg}} & \supset & X_{\mathrm{qreg}} \setminus (X_{\mathrm{qreg}} \cap \mathcal{B}) \\
\cup & & \cup & & \cup & & \cup \\
X_{\mathrm{reg}} & \supset & X_{\mathcal{B}\mathrm{reg}} & \supset & X_{\mathcal{B}\mathrm{sreg}} & \supset & X_{\mathrm{reg}} \setminus (X_{\mathrm{reg}} \cap \mathcal{B})
\end{array}
$$

where the last inclusions of both rows are equalities if no $B \in \mathcal{B}$ contains any irreducible component of X and the vertical inclusions are equalities if X is reduced.

Lemma 5.3 *Let X be a connected excellent scheme.*

(a) For $\nu \in \Sigma_X^{max}$ one has $X(\nu) \cap X_{qreg} \neq \emptyset$ if and only if $X = X(\nu)$. Thus H_X is not constant on X if and only if $X(\nu) \subset X - X_{qreg}$ for all $\nu \in \Sigma_X^{max}$.

(b) Let $(\mathcal{B}, O)$ be an admissible boundary with history on X. For $\widetilde{\nu} \in \Sigma_X^{O,max}$ one has $X^O(\widetilde{\nu}) \cap X_{\mathcal{B}sqreg} \neq \emptyset$ if and only if $X = X^O(\widetilde{\nu})$. Thus H_X^O is not constant on X if and only if $X^O(\widetilde{\nu}) \subseteq X - X_{\mathcal{B}sqreg}$ for all $\widetilde{\nu} \in \Sigma_X^{O,max}$.

Proof (a): Let $x \in X(\nu) \cap X_{qreg}$, let Z be an irreducible component of X containing x, and let η be the generic point of Z. Since X_{qreg} is open and dense in X, and is quasi-regular, η is contained in X_{qreg} and H_X is constant on X_{qreg} by Theorem 2.3. Therefore $\nu = H_X(x) = H_X(\eta)$, i.e., $\eta \in X(\nu)$. Since $\nu \in \Sigma_X^{max}$, $X(\nu)$ is closed, and we conclude that $Z = \overline{\{\eta\}} \subset X(\nu)$. By Lemma 2.5 we conclude that $X = X(\nu) = Z$. This proves the first claim (the other direction is trivial). The second claim is an obvious consequence. (b): For the non-trivial direction of the first claim let $\widetilde{\nu} = (\nu, m)$, with $\nu \in \mathbb{N}^{\mathbb{N}}$ and $m \geq 0$. Then $\nu \in \Sigma_X^{max}$ and $X^O(\widetilde{\nu}) \subseteq X(\nu)$. Consequently, if $X^O(\widetilde{\nu}) \cap X_{\mathcal{B}sqreg} \neq \emptyset$, then $X = X(\nu)$ by (a), and X is irreducible. If η is the generic point of X, then we conclude as above that $\eta \in X^O(\widetilde{\nu})$, and hence $X = X^O(\widetilde{\nu})$, since the latter set is closed. Again the second claim follows immediately.

Now we study blow-ups. Lemma 3.2 implies:

Lemma 5.4 *If $\pi : X' = Bl_D(X) \to X$ is the blow-up of X in a $\mathcal{B}$-permissible subscheme D, and $\mathcal{B}'$ is the complete transform of $\mathcal{B}$, then $\pi^{-1}(X_{\mathcal{B}sreg}) \subset X'_{\mathcal{B}'sreg}$ and $\pi^{-1}(X_{\mathcal{B}sqreg}) \subset X'_{\mathcal{B}'sqreg}$.*

We first consider the case (NE).

Definition 5.5 *(Case (NE)) A sequence of complete (resp. strict) $\mathcal{B}$-permissible blowups over X is a diagram*

(5.1)
$$
\begin{array}{ccccccccc}
\mathcal{B} = & \mathcal{B}_0 & & \mathcal{B}_1 & & \mathcal{B}_2 & & \mathcal{B}_{n-1} & & \mathcal{B}_n & \cdots \\[4pt]
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{n-1} & \xleftarrow{\pi_n} & X_n & \leftarrow \cdots
\end{array}
$$

where for any $n \geq 0$, $\mathcal{B}_n$ is a boundary on X_n, and

$$
X_{n+1} = Bl_{D_n}(X_n) \xrightarrow{\pi_{n+1}} X_n
$$

is the blow-up in a $\mathcal{B}_n$-permissible center $D_n \subset X_n$, and $\mathcal{B}_{n+1} = \mathcal{B}'_n$ is the complete transform of $\mathcal{B}_n$ (resp. $\mathcal{B}_{n+1} = \widetilde{\mathcal{B}}_n$ is the strict transform of $\mathcal{B}_n$).
Call a sequence as in (5.1) contracted if none of the morphisms π_n is an isomorphism. For a given sequence of $\mathcal{B}$-permissible blowups, define the associated contracted sequence by suppressing all isomorphisms in the sequence and renumbering as in (5.1) again.
We abbreviate (5.1) as $(X, \mathcal{B}) = (X_0, \mathcal{B}_0) \xleftarrow{\pi_1} (X_1, \mathcal{B}_1) \leftarrow \ldots$, for short.

We will prove Theorem 0.1 in the following, more general form.

Theorem 5.6 *Let $(X, \mathcal{B})$ be as in (NE), with X dimension at most two.*
(a) There is a canonical finite contracted sequence $S(X, \mathcal{B})$ of complete $\mathcal{B}$-permissible blow-ups over X

$$(X, \mathcal{B}) = (X_0, \mathcal{B}_0) \xleftarrow{\pi_1} (X_1, \mathcal{B}_1) \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_n} (X_n, \mathcal{B}_n)$$

such that π_{i+1} is an isomorphism over $(X_i)_{\mathcal{B}_i \text{sqreg}}$, $0 \le i < n$, and X_n is quasi-$\mathcal{B}$-regular. In particular, the morphism $X_n \to X$ is an isomorphism over $X_{\mathcal{B}\text{sqreg}}$.

Moreover the following functoriality holds:

(F1) (equivariance) The action of the automorphism group of $(X, \mathcal{B})$ extends to the sequence in a unique way.

(F2) (localization) The sequence is compatible with passing to open subschemes $U \subseteq X$, arbitrary localizations U of X and étale morphisms $U \to X$ in the following sense: If $S(X, \mathcal{B}) \times_X U$ denotes the pullback of $S(X, \mathcal{B})$ to U, then the associated contracted sequence $(S(X, \mathcal{B}) \times_X U)_{contr}$ coincides with $S(U, \mathcal{B}_U)$.

(b) There is also a canonical finite contracted sequence $S_0(X, \mathcal{B})$ of strict $\mathcal{B}$-permissible blowups with the same properties (except that now each $\mathcal{B}_{n+1}$ is the strict transform of $\mathcal{B}_n$).

If X is reduced, then every X_i is reduced, so that X_n is regular and $\mathcal{B}_n$ is normal crossing with X_n; moreover $X_{\mathcal{B}\text{sqreg}} = X_{\mathcal{B}\text{sreg}}$. In particular, Theorem 0.1 can be obtained as the case (b) for $\mathcal{B} = \emptyset$ (where $X_{\mathcal{B}\text{sreg}} = X_{\text{reg}}$ and $\mathcal{B}_n = \emptyset$ for all n), i.e., as the sequence $S_0(X, \emptyset)$. If we apply (a) for reduced X and $\mathcal{B} = \emptyset$, we have $X_{\mathcal{B}\text{sreg}} = X_{\text{reg}}$ as well, but then, for the sequence $S(X, \emptyset)$, $\mathcal{B}_n$ is not empty for $n > 0$, and we obtain the extra information that the collection of the strict transforms of all created exceptional divisors is a simple normal crossing divisor on X_n.

Definition 5.7 *Let $\mathcal{C}$ be a category of schemes which is closed under localization. Say that canonical, functorial resolution with boundaries holds for $\mathcal{C}$, if the statements in Theorem 5.6 (a) hold for all schemes in $\mathcal{C}$ and all boundaries on them. Say that canonical, functorial resolution holds for $\mathcal{C}$, if the statements of Theorem 0.1 (i.e., of Theorem 5.6 (b) with $\mathcal{B} = \emptyset$) hold for all schemes in $\mathcal{C}$.*

Now we will consider the case (E).

Definition 5.8 *(Case (E)) A sequence of complete (resp. strict) $\mathcal{B}$-permissible blowups over (X, Z) is a sequence of blowups:*

$$\mathcal{B} = \mathcal{B}_0 \qquad \mathcal{B}_1 \qquad \mathcal{B}_2 \qquad \qquad \mathcal{B}_{n-1} \qquad \mathcal{B}_n \quad \ldots$$

(5.2)
$$
\begin{array}{ccccccccccc}
Z = & Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \leftarrow & Z_{n-1} & \xleftarrow{\pi_n} & Z_n & \ldots \\
 & \cup & & \cup & & \cup & & \cup & & \cup & \\
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{n-1} & \xleftarrow{\pi_n} & X_n & \ldots
\end{array}
$$

where for any $i \ge 0$

$$
\begin{array}{ccccc}
Z_{i+1} & = & B\ell_{D_i}(Z_i) & \xrightarrow{\pi_{i+1}} & Z_i \\
 & & \cup & & \cup \\
X_{i+1} & = & B\ell_{D_i}(X_i) & \xrightarrow{\pi_{i+1}} & X_i
\end{array}
$$

64

are the blow-ups in a center $D_i \subset X_i$ which is permissible and n.c. with $\mathcal{B}_i$, and where $\mathcal{B}_{i+1} = \mathcal{B}'_i$ is the complete transform of $\mathcal{B}_i$ (resp. $\mathcal{B}_{i+1} = \widetilde{\mathcal{B}}_i$ is the strict transform of $\mathcal{B}_i$). Call a sequence as in (5.2) contracted if none of the morphisms π_n is an isomorphism. For a given sequence of $\mathcal{B}$-permissible blowups, define the associated contracted sequence by suppressing all isomorphisms in the sequence and renumbering as in (5.2) again. We abbreviate (5.2) as $(X, Z, \mathcal{B}) = (X_0, Z_0, \mathcal{B}_0) \xleftarrow{\pi_1} (X_1, Z_1, \mathcal{B}_1) \leftarrow \dots$, for short.

We will prove Theorem 0.3 in the following form.

Theorem 5.9 *Let $(X, Z, \mathcal{B})$ be as in (E), with X of dimension at most two.*
(a) There is a canonical finite contracted sequence $S(X, Z, \mathcal{B})$ of complete $\mathcal{B}$-permissible blow-ups over X

$$(X, Z, \mathcal{B}) = (X_0, Z_0, \mathcal{B}_0) \xleftarrow{\pi_1} (X_1, Z_1, \mathcal{B}_1) \xleftarrow{\pi_2} \dots \xleftarrow{\pi_n} (X_n, Z_n, \mathcal{B}_n)$$

such that π_{i+1} is an isomorphism over $(Z - X) \cup (X_i)_{\mathcal{B}_i \mathrm{sqreg}}$, $0 \le i < n$ and X_n is quasi-$\mathcal{B}_n$-regular. In particular, the morphism $Z_n \to Z$ is an isomorphism over $(Z - X) \cup X_{\mathcal{B}\mathrm{sqreg}}$. Moreover the following functoriality holds:

(F1) (equivariance) The action of the automorphism group of $(Z, X, \mathcal{B})$ (those automorphisms of Z which respect $\mathcal{B}$ and X) extends to the sequence in a unique way.

(F2) (localization) The sequence is compatible with passing to open subschemes $U \subseteq Z$, arbitrary localizations U of Z and étale morphisms $U \to Z$ in the following sense: If $S(X, Z, \mathcal{B}) \times_Z U$ denotes the pullback of $S(X, Z, \mathcal{B})$ to U, then the associated contracted sequence $(S(X, Z, \mathcal{B}) \times_Z U)_{contr}$ coincides with $S(X \times_Z U, U, \mathcal{B}_U)$.

(b) There is also a canonical finite contracted sequence $S_0(X, Z, \mathcal{B})$ of strict $\mathcal{B}$-permissible blowups over (X, Z) with the same properties (except that now each $\mathcal{B}_{n+1}$ is the strict transform of $\mathcal{B}_n$).

Again, for reduced X all X_i are reduced, $X_{\mathcal{B}\mathrm{sqreg}} = X_{\mathcal{B}\mathrm{sreg}}$, and X_n is regular and normal crossing with the simple normal crossings divisor $\mathcal{B}_n$.

Definition 5.10 *Let $\mathcal{C}$ be a category of schemes which is closed under localization. Say that canonical, functorial embedded resolution with boundaries holds for $\mathcal{C}$, if the statements in Theorem 5.9 (a) hold for all triples $(X, Z, \mathcal{B})$ where Z is a regular excellent scheme, $\mathcal{B}$ is a simple normal crossing divisor on Z and X is a closed subscheme of X which is in $\mathcal{C}$.*

Remark 5.11 *It follows from Lemma 4.21 that Theorem 5.6 implies Theorem 5.9, in the following way: If $S(X, \mathcal{B}_X)$ is constructed, one obtains $S(X, Z, \mathcal{B})$ by consecutively blowing up Z_i in the same center as X_i, and identifying X_{i+1} with the strict transform of X_i in Z_{i+1}. Conversely, the restriction of $S(X, Z, \mathcal{B})$ to X is $S(X, \mathcal{B}_X)$. More generally, by the same approach, canonical, functorial embedded resolution with boundaries holds for a category $\mathcal{C}$ of schemes as in Definition 5.10 if canonical, functorial resolution with boundaries holds for $\mathcal{C}$.*

We set up the strategy of proof for the above theorems in a more general setting. Let X be an excellent scheme.

Definition 5.12 *Call an excellent scheme Y equisingular, if H_Y is constant on Y. Call Y locally equisingular if all connected components are equisingular.*

As we will see below, our strategy will be to make X locally equisingular.

Remark 5.13 *(a) For $U \subseteq X$ open and $x \in U$ one has $H_U(x) = H_X(x)$. Hence U is equisingular if and only if $U \subseteq X(\nu)$ for some $\nu \in \Sigma_X$.*

(b) By Lemma 5.3 (a) there are two possibilities for a connected component $U \subseteq X$: Either U is equisingular (and irreducible), or U_{max} is nowhere dense in U. In the first case, it follows from Theorem 1.33 (1) and Theorem 2.10 (1) that, for any permissible blow-up $\pi : X' \to X$, $\pi^{-1}(U)$ is equisingular as well (viz., $\pi^{-1}(U) \subset X'(\nu)$ if $U \subset X(\nu)$), because by definition, permissible centers are nowhere dense in X.

(c) If X is reduced, then a connected component $U \subset X$ is equisingular if and only if U is regular (cf. Remark 1.32). Hence X is locally equisingular iff it is equisingular iff it is regular.

(d) By way of example, the following situation can occur for non-reduced schemes: X is the disjoint union of three irreducible components U_1, U_2 and U_2, where $\Sigma_{U_1} = \{\nu_1\}$, $\Sigma_{U_2} = \{\nu_1, \nu_2\}$ and $\Sigma_{U_3} = \{\nu_3\}$, such that $\nu_1 < \nu_2 < \nu_3$. By just blowing up in X_{max} we cannot make X locally equisingular.

Motivated by the remarks above, we define:

Definition 5.14 *Let X be connected and not equisingular. For $\nu \in \Sigma_X^{max}$, a ν-elimination for X is a morphism $\rho : X' \to X$ that is the composite of a sequence of morphisms:*

$$X = X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n = X'$$

such that for $0 \le i < n$, $\pi_i : X_{i+1} \to X_i$ is a blowup in a permissible center $D_i \subseteq X_i(\nu)$ and $X_n(\nu) = \emptyset$.

Let $\nu_1, \ldots, \nu_r$ be the elements of Σ_X^{max} and assume given a ν_i-elimination $\rho_i : X_i \to X$ of X for each $i \in \{1, \ldots, r\}$. Noting that ρ_i is an isomorphism over $X - X(\nu_i)$ and that $X(\nu_i) \cap X(\nu_j) = \emptyset$ if $1 \le i \ne j \le r$, we can glue the ρ_i over X – which is a composition of permissible blow-ups again – to get a morphism $\rho : X' \to X$ which is a Σ^{max}-elimination where we define:

Definition 5.15 *Let X be connected and not equisingular. A morphism $\rho : X' \longrightarrow X$ is called a Σ^{max}-elimination for X if the following conditions hold:*
(ME1) ρ is the composition of permissible blowups and an isomorphism over $X - X_{max}$.
(ME2) $\Sigma_{X'} \cap \Sigma_X^{max} = \emptyset$.

Note that, by Theorem 2.10 (1), (ME1) and (ME2) imply:

(ME3) For each $\mu \in \Sigma_{X'}$ there exists a $\nu \in \Sigma_X^{max}$ with $\mu < \nu$.

Definition 5.16 *For any excellent scheme X, a morphism $\rho : X' \to X$ is called a Σ^{max}-elimination, if it is a Σ^{max}-elimination after restriction to each connected component which is not equisingular, and an isomorphism on the other connected components.*

Theorem 5.17 *Let X be an excellent scheme, and let $X = X_0 \leftarrow X_1 \leftarrow \cdots$ be a sequence of morphisms such that $\pi_n : X_{n+1} \to X_n$ is a $\Sigma^{\max}$-elimination for each n. Then there is an $N \in \mathbb{N}$ such that X_N is locally equisingular. (So π_n is an isomorphism for $n \geq N$.)*

Proof (See also Theorem 15.3 for an alternative, more self-contained proof of Theorem 5.17.) Suppose there exists an infinite sequence $X = X_0 \leftarrow X_1 \leftarrow \ldots$ of $\Sigma^{\max}$-eliminations such that no X_n is locally equisingular. For each $n \geq 0$ let $X_n^0 \subseteq X_n$ be the union of those connected components of X_n which are not equisingular, and let $\Sigma_n = \Sigma_{X_n^0}$ and $\Sigma_n^{max} = \Sigma_{X_n^0}^{max}$. For each n, choose an element $\nu_n \in \Sigma_n^{max}$. It follows from the finiteness of Σ_{X_0} and Theorem 2.10 that there is an $m \in \mathbb{N}$ such that all Hilbert-Samuel functions occurring on the X_i are contained in the set HF_m of all Hilbert functions H (of standard graded algebras) with $H(1) \leq m$. Thus it follows from Theorem 1.15, i.e., the noetherianess of HF_m, that $\nu_i \leq \nu_j$ for some $i < j$ (see [AP] Proposition 1.3 for several equivalent conditions characterizing noetherian ordered sets). Choose $x_i \in X_i^0$ and $x_j \in X_j^0$ with $\nu_i = H_{X_i}(x_i)$ and $\nu_j = H_{X_j}(x_j)$, and for $i \leq k \leq j$, let y_k be the image of x_j in X_k. By Remark 5.13 (b) the morphism $X_{n+1} \to X_n$ maps X_{n+1}^0 to X_n^0, so we have $y_k \in X_k^0$, and by Theorem 2.10 and Remark 5.13 we have $\nu_j \leq \mu_k := H_{X_k}(y_k)$. Since $\nu_i \in \Sigma_i^{max} \subset \Sigma_i$, we conclude that the inequalities $\nu_i \leq \nu_j \leq \mu_k$ are all equalities. Therefore $\nu_i = \mu_{i+1} \in \Sigma_{i+1}$, contradicting the assumption that $X_{i+1} \to X_i$ is a Σ^{max}-elimination.

Corollary 5.18 *To prove (canonical, functorial) resolution of singularities for all excellent reduced schemes of dimension $\leq d$, it suffices to prove that for every connected non-regular excellent reduced scheme X of dimension $\leq d$ there exists a (canonical functorial) $\Sigma^{\max}$-elimination $X' \to X$. Equivalently, it suffices to show that for every such scheme and every $\nu \in \Sigma_X^{max}$, there is a (canonical functorial) ν-elimination for X. Here functoriality means that the analogues of the properties (F1) and (F2) in Theorem 5.6 hold for the $\Sigma^{\max}$- and ν-eliminations, respectively, where the analogue of property (F2) for a ν-elimination is the following: Either $U(\nu) = \emptyset$, or $\nu \in \Sigma_U^{max}$ and the pullback of the sequence to U is the canonical ν-elimination on U, after passing to the associated reduced sequence.*

In fact, under these assumptions one gets a (canonical, functorial) sequence $X \leftarrow X_1 \leftarrow \ldots$ of $\Sigma^{\max}$-eliminations, and by Theorem 5.17 some X_n is locally equisingular, which means that X_n is regular (Remark 5.13 (c)).

Now we consider the non-reduced case.

Corollary 5.19 *To prove (canonical, functorial) resolution of singularities for all excellent schemes of dimension $\leq d$, it suffices to prove that there exists a (canonical, functorial) $\Sigma^{\max}$-elimination $X' \to X$ for every connected excellent scheme X of dimension $\leq d$ which is not equi-singular. Equivalently, it suffices to show that for every such scheme and every $\nu \in \Sigma_X^{max}$, there is a (canonical, functorial) ν-elimination for X. Here functoriality is defined as in Corollary 5.18.*

In fact, here we first get a (canonical, functorial) sequence $X \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ of $\Sigma^{\max}$-eliminations such that X_m is locally equisingular. By Corollary 5.18 we get a similar sequence $(X_m)_{red} \leftarrow X'_{m+1} \leftarrow \ldots \leftarrow X'_n$ such that X'_n is regular. Blowing up in the same

centers we get a sequence of blow-ups $X_m \leftarrow X_{m+1} \leftarrow \ldots \leftarrow X_n$, where X'_i is identified with $(X_i)_{red}$, and X'_{i+1} with the strict transform of X'_i in X_{i+1} (since $D_i \subseteq (X_i)_{red}$, X'_{i+1} is reduced, and homeomorphic to X_{i+1}). For each $i \geq m$, X_i is again equisingular (see Remark 5.13), and by Theorem 2.3 the blow-up $X_{i+1} \to X_i$ is permissible. It follows that $(X_n)_{red}$ is regular and X_n is normally flat along $(X_n)_{red}$. Now assume that the first sequence and the sequence $(X_m)_{red} \leftarrow \ldots$ are functorial. Then it is immediate that the sequence $X \leftarrow \ldots \leftarrow X_n$ is functorial for localizations as well. As for automorphisms, it follows inductively via localization that the automorphisms of X_i ($i \geq m$) respect the center of the blow-up $X_{i+1} \to X_i$ and therefore extend to X_{i+1} in a unique way.

Remark 5.20 *We point out the choice of strategy here. It might be tempting to start with desingularizing X_{red}. But if $D \subset X_{red}$ is permissible in X_{red}, D will not in general be permissible in X. However, once X is made equi-singular (by the first series of blow-ups), the arguments above show that we get permissible blow-ups both for X_{red} and X, and can achieve that X_{red} becomes regular and X equi-singular at the same time, so that X is normally flat along X_{red}.*

We now consider a variant of the above for schemes with boundary. Let X be an excellent scheme, and let $\mathcal{B}$ be a boundary for X, i.e., a boundary on X (case (NE)) or on Z (case (E)). In the following we only consider complete transforms for the boundaries, i.e., sequences of complete $\mathcal{B}$-permissible blow-ups, and we will simply speak of sequences of $\mathcal{B}$-permissible blow-ups. It is easy to see that the analogous results also hold for the case of strict transforms, i.e., sequences of strict $\mathcal{B}$-permissible blow-ups.

Definition 5.21 *Call X O-equisingular if H_X^O is constant on X, and locally O-equisingular, if every connected component is O-equisingular.*

Remark 5.22 *(a) It follows from Lemma 5.3 (b) that a connected component $U \subseteq X$ is either O-equisingular, or U_{max}^O is nowhere dense in U. In the first case $U \subseteq X^O(\widetilde{\nu})$ for some $\widetilde{\nu} \in \Sigma_X^O$, and for every $\mathcal{B}$-permissible blow-up $\pi : X' \to X$ one has $\pi^{-1}(U) \subseteq X'(\widetilde{\nu})$.*

(b) If X is reduced, then X is locally O-equisingular if and only if X is O-regular.

(c) Even for a regular scheme X it can obviously happen that X is the union of three irreducible components U_1, U_2 and U_3 such that $\Sigma_{U_1}^O = \{(\nu_X^{reg}, 1)\}$, $\Sigma_{U_2}^O = \{(\nu_X^{reg}, 1), (\nu_X^{reg}, 2)\}$ and $\Sigma_{U_3}^O = \{(\nu_X^{reg}, 3)\}$, so that X cannot be made O-equisingular by blowing up in X_{max}^O.

Definition 5.23 *Let O be a history function for $\mathcal{B}$ such that $(\mathcal{B}, O)$ is admissible, and let*

$$(5.3) \qquad\qquad X = X_0 \xleftarrow{\pi_1} X_1 \leftarrow \ldots \leftarrow X_{n-1} \xleftarrow{\pi_n} X_n$$

be a sequence of $\mathcal{B}$-permissible blow-ups (where we have not written the boundaries $\mathcal{B}_i$, and neither the regular schemes Z_i in case (E)). For each $i = 0, \ldots, n-1$ let $(\mathcal{B}_{i+1}, O_{i+1})$ be the complete transform of $(\mathcal{B}_i, O_i)$ (where $(\mathcal{B}_0, O_0) = (\mathcal{B}, O)$). Let D_i be the center of the blow-up $X_i \xleftarrow{\pi_{i+1}} X_{i+1}$, and $\rho = \pi_n \circ \ldots \circ \pi_1 : X_n \to X$.

(1) If X is connected and not O-equisingular, and $\widetilde{\nu} \in \Sigma_X^{O,max}$, then (5.3) or ρ is called a $\widetilde{\nu}$-elimination, if $D_i \subseteq X_i(\widetilde{\nu})$ for $i = 1, \ldots, n-1$ and $X_n(\widetilde{\nu}) = \emptyset$.

(2) If X is connected and not O-equisingular, then (5.3) or ρ is called a $\Sigma^{O,max}$-elimination for $(X, \mathcal{B}, O)$, if $D_i \subseteq (X_i)_{max}^O$ for $i = 0, \ldots, n-1$ and $\Sigma_{X_n}^O \cap \Sigma_X^{O,max} = \emptyset$.

(3) If X is arbitrary, then (5.3) or ρ is called a $\Sigma^{O,max}$-elimination for $(X, \mathcal{B}, O)$, if it is a $\Sigma^{O,max}$-elimination after restriction to each connected component of X which is not O-equisingular, and an isomorphism after restriction to the connected components which are O-singular.

(4) Call the sequence (5.3) contracted, if none of the morphisms is an isomorphism, and in general define the associated contracted sequence by omitting the isomorphisms and renumbering (so the final index n may decrease).

Remark 5.24 *By glueing, one gets a (canonical, functorial) $\Sigma^{O,max}$-elimination for a connected, not O-equisingular X, if one has (canonical, functorial) $\widetilde{\nu}$-eliminations for all $\widetilde{\nu} \in \Sigma_X^{O,max}$, and a (canonical, functorial) $\Sigma^{O,max}$-elimination for a non-connected X, if one has this for all connected components. Here functoriality is defined as in Theorem 5.6.*

In a similar way as in Theorem 5.17 one proves:

Theorem 5.25 *For any infinite sequence $X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots$ of $\Sigma^{O,max}$-eliminations there is an n such that $(X_n, \mathcal{B}_n, O_n)$ is O-equisingular.*

The following result is now obtained both in the embedded and the non-embedded case.

Corollary 5.26 *Case (NE): To show (canonical, functorial) resolution of singularities with boundaries for all reduced excellent schemes of dimension $\leq d$ it suffices to show the existence of (canonical, functorial) $\Sigma^{O,max}$-eliminations for all connected reduced excellent schemes X of dimension $\leq d$ and all admissible boundaries with history $(\mathcal{B}, O)$ for X, for which X is not O-regular. (Here 'functorial' in the last statement means that the obvious analogues of the conditions (F1) and (F2) in Theorem 5.6 hold for the sequences considered here.)*
Case (E): The obvious analogous statement holds.

In fact, if $(X, \mathcal{B})$ is given, we start with the history function $O(x) = \mathcal{B}(x)$. Then X is O-regular if and only if X is strongly $\mathcal{B}$-regular at all $x \in X$. If this holds, we are done. If not, then by assumption there is a (canonical, functorial) $\Sigma^{O,max}$-elimination $X_1 \to X$, and we let $(\mathcal{B}_1, O_1)$ be the strict transform of $(\mathcal{B}, O)$ in X_1 (which is obtained by successive transforms for the sequence of $\mathcal{B}$-permissible blowups whose composition is $X_1 \to X$). Then $(\mathcal{B}_1, O_1)$ is admissible by Lemma 4.33. If X_1 is O_1-regular, we are done by Lemma 3.28. If not we repeat the process, this time with $(X_1, \mathcal{B}_1, O_1)$, and iterate if necessary. By Theorem 5.25, after finitely many steps this process obtains an X_n which is O_n-regular and hence achieves the resolution of X by Lemma 3.28 (case (E)) and Lemma 4.32 (case (NE)).

In the non-reduced case we obtain:

Corollary 5.27 *Case (NE): To show (canonical, functorial) resolution of singularities with boundaries for all excellent schemes of dimension $\leq d$ it suffices to show the existence of (canonical, functorial) $\Sigma^{O,max}$-eliminations for all connected excellent schemes X of dimension $\leq d$ and all admissible boundaries with history $(\mathcal{B}, O)$ for X, for which H_X^O is not constant. (Here 'functorial' in the last statement means that the obvious analogues of the conditions (F1) and (F2) in Theorem 5.6 hold for the sequences considered here.)*
Case (E): The obvious analogous statement holds.

This follows from Corollary 5.26 in a similar way as Corollary 5.19 follows from 5.18: First we get a (canonical, functorial) sequence of $\mathcal{B}$-permissible blow-ups $X \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ such that X_m is locally O-equisingular. Then we look at the (canonical, functorial) resolution sequence $(X_m)_{red} \leftarrow X'_{m+1} \leftarrow \ldots \leftarrow X'_n$ from Corollary 5.26 such that X'_n is $\mathcal{B}'_n$-regular, where $\mathcal{B}'_n$ comes from $\mathcal{B}$ via complete transforms. By blowing up in the same centers we obtain a sequence of $\mathcal{B}$-permissible blow-ups $X_m \leftarrow X_{m+1} \leftarrow \ldots \leftarrow X_n$ such that $(X_n)_{red}$ identifies with X'_n and thus is $\mathcal{B}'_n$-regular; moreover, X_n is normally flat along $(X_n)_{red}$, because $H^O_{X_n}$ is constant on all connected components.

We now prove Theorem 5.6. Then, by Remark 5.11, Theorem 5.9 follows as well.

By Corollary 5.27, it suffices to produce canonical, functorial $\Sigma^{O,max}$-eliminations for all connected excellent schemes X of dimension at most two and all admissible boundaries with history $(\mathcal{B}, O)$ on X such that H^O_X is not constant on X. By the remarks after Definition 5.23 it suffices to produce canonical functorial $\widetilde{\nu}$-eliminations for all $\widetilde{\nu} \in \Sigma^{O,max}_X$. We will slightly modify this procedure and deduce Theorem 5.6 from the following, partly weaker and partly more general result.

Theorem 5.28 *(Case (NE)) Let X be an excellent connected scheme, let $(\mathcal{B}, O)$ be an admissible boundary with history on X such that H^O_X is not constant on X, and let $\widetilde{\nu} \in \Sigma^{O,max}_X$. Assume the following:*

(1) $\mathrm{char}(k(x)) = 0$, or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$ for any $x \in X(\widetilde{\nu})$,

(2) $\dim(X(\widetilde{\nu})) \leq 1$,

and there is an integer e with $0 \leq e \leq 2$ such that for any closed point $x \in X(\widetilde{\nu})$,

(3e) $\overline{e}_x(X) \leq e$,

(4e) either $N(x) \pitchfork \mathrm{Dir}^O_x(X)$ or $e^O_x(X) \leq e - 1$.

Then there exists a canonical reduced $\widetilde{\nu}$-elimination $S(X, \widetilde{\nu})$

$$(X, \mathcal{B}) = (X_0, \mathcal{B}_0) \leftarrow (X_1, \mathcal{B}_1) \leftarrow \ldots \leftarrow (X_n, \mathcal{B}_n)$$

for $(X, \mathcal{B})$. It satisfies the analogues of properties (F1) and (F2) from Theorem 5.6, where the analogue of (F2) is the following: Either $U(\widetilde{\nu}) = \emptyset$, or else $\widetilde{\nu} \in \Sigma^{O,max}_U$ and the reduced sequence associated to the pullback of the sequence to U is the canonical $\widetilde{\nu}$-elimination for $(U, \mathcal{B}_U)$.

Theorem 5.6 follows from this: If the conditions of Theorem 5.28 hold for X, and $X' \to X$ is a blowup in a $\mathcal{B}$-permissible center $D \subseteq X(\widetilde{\nu})$, the conditions hold for X' as well, with the same e. In fact, (1) holds for X' since $\dim(X') = \dim(X)$, condition (3e) holds for X' by Theorem 2.10, and condition (4e) holds for X' by Corollary 4.28. Moreover, condition (2) holds for X' as we will see in the proof of Theorem 5.28. Now assume that X is of dimension $d \leq 2$. Then condition (1) holds, and condition (3e) holds with $e = d$. If moreover H^O_X is not constant on X, then condition (2) holds by Lemma 5.3, and (2) holds for X' as well. On the other hand, in the presence of condition (1) it suffices to consider admissible boundaries with history $(\mathcal{B}, O)$ which satisfy condition

(4e). In fact, in the procedure outlined in the proof of Corollary 5.27, property $(4e)$ is trivially fulfilled in the beginning where $O(x) = \mathcal{B}(x)$, i.e., $N(x) = \emptyset$, and as remarked a few lines above, it is also fulfilled in a sequence of $\mathcal{B}$-permissible blowups. Therefore we can keep all assumptions of Theorem 5.28 in a sequence of $\mathcal{B}$-permissible blowups, and the arguments for Corollaries 5.26 and 5.27 apply in this modified setting.

Remark 5.29 *(1) Before we start with the proof of Theorem 5.28, we outline the strategy which works under the conditions (1) to (4) of Theorem 5.28, but can be stated more generally. It would be interesting to see if it also works for higher-dimensional schemes.*

Let X be an excellent connected scheme, let $(\mathcal{B}, O)$ be an admissible boundary with history on X such that H_X^O is not constant on X, and let $\widetilde{\nu} \in \Sigma_X^{O,max}$. We construct a canonical contracted sequence $S(X, \widetilde{\nu})$

$$(5.4) \qquad\qquad X = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \ldots$$

of $\mathcal{B}$-permissible blowups over X as follows. We introduce some notations. If X_n has been constructed, then let $Y_n = X_n(\widetilde{\nu})$. Give labels (or 'years') to the irreducible components of Y_n in an inductive way as follows. The irreducible components of Y_0 all have label 0. If an irreducible component of Y_n dominates an irreducible component of Y_{n-1}, it inherits its label. Otherwise it gets the label n. Then we can write

$$(5.5) \qquad\quad X_n(\widetilde{\nu}) = Y_n = Y_n^{(0)} \cup Y_n^{(1)} \cup \ldots \cup Y_n^{(n-1)} \cup Y_n^{(n)} \,,$$

where $Y_n^{(i)}$ is the union of irreducible components of Y_n with label i.

Now we start the actual definition of (5.4). We have $\dim(Y_0) < \dim(X)$. By induction on dimension we have a canonical resolution sequence

$$Y_0 := Y_{0,0} \leftarrow Y_{0,1} \leftarrow Y_{0,2} \leftarrow \ldots \leftarrow Y_{0,m_0}$$

for $(Y_0, \mathcal{B}_{Y_0})$, i.e., a sequence of $\mathcal{B}$-permissible blowups, so that Y_{0,m_0} is $\mathcal{B}$-regular. By successively blowing up X in the same centers – which are also permissible for the $(X_i, \mathcal{B}_i)$ by Lemma 4.34 (3), we obtain a sequence of $\mathcal{B}$-permissible blowups

$$X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_{m_0}$$

in which $Y_{0,i}$ is the strict transform of $Y_{0,i-1}$ and, moreover, $Y_{0,i} = Y_i^{(0)}$ so that $Y_{m_0}^{(0)} = Y_{0,m_0}$ is $\mathcal{B}$-regular.

Then we blow up $Y_{m_0}^{(0)}$ to get X_{m_0+1}. We call the obtained sequence (from X_0 to X_{m_0+1}) of $\mathcal{B}$-permissible blow-ups the first (resolution) cycle. If $Y_{m_0+1}^{(0)}$ is non-empty and not $\mathcal{B}$-regular, we proceed as above and have a canonical resolution sequence

$$Y_{m_0+1}^{(0)} = Y_{0,m_0+1} \leftarrow Y_{0,m_0+2} \leftarrow \ldots \leftarrow Y_{0,m_1}$$

such that Y_{0,m_1} is $\mathcal{B}$-regular. Then we get a sequence

$$X_{m_0+1} \leftarrow X_{m_0+2} \leftarrow \ldots \leftarrow X_{m_1}$$

of $\mathcal{B}$-permissible blow-ups by blowing up in the same centers, for which $Y_{0,j} = Y_j^{(0)}$, and $Y_{m_1}^{(0)} = Y_{0,m_1}$ is $\mathcal{B}$-regular. Then we blow up $Y_{m_1}^{(0)}$ to get X_{m_1+1}. We call the sequence

from X_{m_0+1} to X_{m_1+1} the second cycle. Repeating this process finitely many times, i.e., producing further cycles, we get $X_{m_\ell} = X_{n_1}$ for which $Y_{n_1}^{(0)}$ is empty (in our situation, which is a non-trivial fact). Then we proceed with $Y_{n_1}^{(1)}$ as we did before with $Y_0^{(0)}$, in several cycles, until we reach X_{n_2} for which $Y_{n_2}^{(1)}$ is empty, and proceed with $Y_{n_2}^{(2)}$, etc. This procedure ends, i.e., there is an n_r such that Y_{n_r} is empty (in our situation, which is a non-trivial fact), so that we have eliminated the $\widetilde{\nu}$-locus.

(2) In the situation of Theorem 5.28, we will see that $Y_n^{(i)}$ is already $\mathcal{B}$-regular for all $n \geq m_0$ and $i \geq 0$.

(3) We will see below in Proposition 5.31 that the sequence constructed in (1) is functorial for automorphisms and morphisms with geometrically regular fibers.

(4) By (3), Remark 5.24 and Corollaries 5.26 and 5.27 we have: If the procedure in (1) is always finite, i.e., gives $\widetilde{\nu}$-eliminations for $\Sigma^{O,max}$-eliminations, then we have a canonical, functorial resolution with boundaries as formulated in 5.26 and 5.27.

Proof of Theorem 5.28 Let $X' \to X$ be a blowup in a $\mathcal{B}$-permissible center $D \subset X(\widetilde{\nu})$. As we have seen above, conditions (1), (3e) and (4e) hold for X' and the complete transform $(\mathcal{B}', O')$ of $(\mathcal{B}, O)$ as well. Now we will show the same for condition (2). We may assume that D is irreducible. By (2) D is a closed point or a regular irreducible curve.

Step 1 Let x be any closed point in $X(\widetilde{\nu})$ and consider $\pi : X' := B\ell_x(X) \to X$. Note that $x \hookrightarrow X$ is $\mathcal{B}$-permissible for trivial reasons. By Theorem 4.20 we have

$$(5.6) \qquad D' := X'(\widetilde{\nu}) \cap \pi^{-1}(x) \subset \mathbb{P}(\mathrm{Dir}_x^O(X)) \simeq \mathbb{P}_{k(x)}^t,$$

where $t = e_x^O(X) - 1 \leq e_x(X) - 1 \leq e - 1 \leq 1$ (by convention $\mathbb{P}_{k(x)}^t = \emptyset$ if $t < 0$). Hence condition (2) is also satisfied for X'. Moreover, if $\dim(D') \geq 1$, then $D' = \mathbb{P}(\mathrm{Dir}_x^O(X))$, so that D' is O'-permissible, and condition (3e) implies $e = 2$ and $N(x) \pitchfork \mathrm{Dir}_x^O(X)$, so that D' is n.c. with N' by Lemma 4.25. Hence D' is a union of closed points or a projective line over $k(x)$, and in both cases it is $\mathcal{B}'$-permissible.

Step 2 Now let $D \subset X(\widetilde{\nu})$ be regular irreducible of dimension 1 and n.c. with $\mathcal{B}$. By Theorem 4.13, $D \subset X$ is $\mathcal{B}$-permissible. Let η be the generic point of D. Consider $\pi : X' := B\ell_D(X) \to X$. Let $x \in D$ be a closed point. By Theorem 4.20, we have

$$X'(\widetilde{\nu}) \cap \pi^{-1}(x) \subset \mathbb{P}(\mathrm{Dir}_x^O(X)/T_x(D)) \simeq \mathbb{P}_{k(x)}^s,$$

where $s = e_x^O(X) - 2 \leq e_x(X) - 2 \leq e - 2 \leq 0$ by (3e) for X. Hence there is at most one point in $X'(\widetilde{\nu}) \cap \pi^{-1}(x)$ so that $\dim(X'(\widetilde{\nu}) \cap \pi^{-1}(D)) \leq 1$ and condition (2) is satisfied for X'. Similarly we have

$$X'(\widetilde{\nu}) \cap \pi^{-1}(\eta) \subset \mathbb{P}(\mathrm{Dir}_\eta^O(X)) \simeq \mathbb{P}_{k(\eta)}^r,$$

where $r = e_\eta^O(X) - 1 \leq e_\eta(X) - 1 \leq e_x(X) - 2 \leq 0$ by Theorem 2.6. Hence, if $X'(\nu) \cap \pi^{-1}(\eta)$ is not empty, then it consists of a unique point η', and one has $k(\eta) \simeq k(\eta')$. This implies that π induces an isomorphism $D' \xrightarrow{\sim} D$ where $D' = X'(\widetilde{\nu}) \cap \pi^{-1}(D)$. Thus D' is regular, and O'-permissible by Theorem 4.13. Moreover, $e_x(X) = 2$ in this case so that condition

(3e) for X implies $e = 2$ and $N(x) \pitchfork \mathrm{Dir}_x(X)$. By Lemma 4.26, D' is n.c. with N', hence with $\mathcal{B}'$. Hence D' is a collection of closed points or a regular irreducible curve, and in both cases it is $\mathcal{B}'$-permissible.

Step 3 Consider the special case where $\dim(X) = 1$. Here $\dim X(\widetilde{\nu}) = 0$, so every point $x \in X(\widetilde{\nu})$ is isolated in $X(\widetilde{\nu})$, and moreover we have $e_x(X) \leq \dim(X) = 1$. The canonical $\widetilde{\nu}$-elimination sequence as defined in Remark 5.29 consists of blowing up all points in $X(\widetilde{\nu})$ and repeating this process as long as $X(\widetilde{\nu}) \neq \emptyset$. By Theorem 5.35 below this process stops after finitely many steps. So Theorem 5.28 holds. As noticed after Theorem 5.28, this shows that Theorem 5.6 holds for $\dim(X) = 1$, i.e., there exists a canonical, functorial resolution sequence for $(X, \mathcal{B})$.

Step 4 Now we consider the general case and construct a canonical reduced sequence $S(X, \widetilde{\nu})$

$$(5.7) \qquad X = X_0 \xleftarrow{\pi_1} X_1 \leftarrow \ldots \leftarrow X_{n-1} \xleftarrow{\pi_n} X_n \ldots$$

of $\mathcal{B}$-permissible blowups over X as follows. Let $Y_0 = X_0(\widetilde{\nu})$. If X_n has been constructed, then let $Y_n = X_n(\widetilde{\nu})$. Give labels to the irreducible components of Y_n in an inductive way as follows. The irreducible components of Y_0 all have label 0. If an irreducible component of Y_n dominates an irreducible component of Y_{n-1}, it inherits its label. Otherwise it gets the label n. Then we can write

$$(5.8) \qquad Y_n = Y_n^{(0)} \cup Y_n^{(1)} \cup \ldots \cup Y_n^{(n-1)} \cup Y_n^{(n)},$$

where $Y_n^{(i)}$ is the union of irreducible components of Y_n with label i.

Step 5 By assumption, $\dim(Y_0) \leq 1$. Let $\mathcal{B}_{Y_o} = \mathcal{B} \times_X Y_0$ be the pull-back, and let

$$Y_0 = Y_{0,0} \leftarrow Y_{0,1} \leftarrow \ldots Y_{0,m-1} \leftarrow Y_{0,m}$$

be the canonical resolution sequence of Theorem 5.6 for $(Y_0, \mathcal{B}_{Y_0})$ (which exists and is finite by Step 3), so that $Y_{0,m}$ is regular and normal crossing with $\mathcal{B}_{0,m}$, where we write $\mathcal{B}_{0,i}$ for the boundary obtained on $Y_{0,i}$. Let

$$(5.9) \qquad X = X_0 \xleftarrow{\pi_1} X_1 \leftarrow \ldots \leftarrow X_{m-1} \xleftarrow{\pi_m} X_m$$

be the sequence of blow-ups obtained inductively by blowing up X_i in the center D_i of the blowup $Y_{0,i+1} \to Y_{0,i}$ and identifying $Y_{0,i+1}$ with the strict transform of $Y_{0,i}$ in X_{i+1}. D_i is a collection of closed points and hence $\mathcal{B}_i$-permissible, where we write $\mathcal{B}_i$ for the boundary obtained on X_i. Therefore (5.9) is a sequence of $\mathcal{B}$-permissible blow-ups. (We could also use Lemma 4.34 (3) as in Remark 5.29.) By Lemma 4.34 (4) we have $\mathcal{B}_{0,i} = (\mathcal{B}_i)_{Y_{0,i}}$. Since each D_i is a nowhere dense subscheme of $Y_{0,i}$, each $Y_{0,j}$ is contained in $X_j(\widetilde{\nu}) = Y_j$, and is in fact equal to the label 0 part $Y_j^{(0)}$ of Y_j as defined above. This is the first stage of (5.7).

Claim 1 For m as above, and all $i \geq 0$, the subschemes $Y_m^{(i)}$ are regular, of dimension at most 1, and $\mathcal{B}_m$-permissible.

In fact, for $Y_m^{(0)}$ this holds by construction. Moreover, from the statements in Step 1 we conclude that, for $0 < i \leq m$, all schemes $Y_i^{(i)}$ are disjoint unions of closed points and

projective lines and hence regular, moreover they are $\mathcal{B}_i$-permissible. Let $[Y_j]_0$ be the union of the 0-dimensional components of Y_j. Since $X_{j+1} \to X_j$ is a blow-up in closed points of $Y_j^{(0)}$, and $Y_j^{(0)} \cap [Y_j^{(i)}]_0 = \emptyset$ for $i > 0$ (no closed point can dominate the curve $Y_{j-1}^{(0)}$), the morphism $Y_j^{(i)} \to Y_i^{(i)}$ is an isomorphism for $j = i, \ldots, m$. Hence $Y_j^{(i)}$ is regular, and normal crossing with $\mathcal{B}_j$ (direct check, or application of Lemma 3.2). Since $Y_j^{(i)}$ is regular and contained in $X_j(\widetilde{\nu})$, we conclude it is $\mathcal{B}_j$-permissible.

Step 6 Next we blow up the subscheme $Y_m^{(0)}$, which is regular and $\mathcal{B}_m$-permissible, and obtain X_{m+1}.

Claim 2 For all $i \geq 0$, the subschemes $Y_{m+1}^{(i)}$ are regular, of dimension at most 1, and $\mathcal{B}_{m+1}$-permissible. Moreover, the intersection of $Y_{m+1}^{(m+1)}$ with $Y_{m+1}^{(i)}$ is empty for all $i \in \{0, \ldots, m\}$.

The first part follows by similar arguments as above. In fact, for $Y_{m+1}^{(0)}$ the arguments are exactly the same as above. For $Y_{m+1}^{(i)}$ with $i > 0$ we have to be careful, since $Y_{m+1}^{(i)}$ consists of irreducible components of the strict transform of $Y_m^{(i)}$, i.e., the blowup of $Y_m^{(i)}$ in $Y_m^{(i)} \times_{X_m} Y_m^{(0)}$, which is a zero-dimensional scheme with a possibly non-reduced structure. But since $Y_m^{(i)}$ is regular of dimension at most 1 and $Y_m^{(0)} \cap [Y_m^{(i)}]_0 = \emptyset$ for $i > 0$, $Y_{m+1}^{(i)} \to Y_m^{(i)}$ is an isomorphism . As for the second part, $Y_{m+1}^{(m+1)}$ consists of finitely many closed points which, by definition, are not contained in Y_{m+1}^{j} for $j \leq m$.

Step 7 Next we blow up X_{m+1} in $Y_{m+1}^{(0)}$ if this is non-empty, and in $Y_{m+1}^{(1)}$, otherwise, and obtain X_{m+2}. We proceed in this way for $n > m$, blowing up X_n in $Y_n^{(j)}$ where $j \geq 0$ is the smallest number with $Y_n^{(j)} \neq \emptyset$, to obtain X_{n+1}. This is well-defined, because we always have:

Claim 3 For all $n \geq m$ and $i \geq 0$, the subschemes $Y_n^{(i)}$ are regular, of dimension at most 1, and $\mathcal{B}_n$-permissible. For $i \geq m+1$, the intersection of $Y_n^{(i)}$ with $Y_n^{(j)}$ is empty for $j \neq i$.

The first part follows as for $n = m+1$. For $i = n$ the second part follows as in claim 2. For $m + 1 \leq i < n$, we may assume by induction that $Y_{n-1}^{(i)} \cap Y_{n-1}^{(j)} = \emptyset$ for all $j = 0, \ldots, n-1$ with $j \neq i$. By definition, for all $j = 0, \ldots, n-1$, $\pi(Y_n^{(j)}) \subset Y_{n-1}^{(j)}$ where $\pi : X_n \to X_{n-1}$. This implies $Y_n^{(i)} \cap Y_n^{(j)} = \emptyset$ for all $i \in \{m + 1, \ldots, n-1\}$ and all $j = 0, \ldots, n-1$ with $j \neq i$, which proves the desired assertion.

Step 8 Thus we have defined the wanted canonical sequence $S(X, \widetilde{\nu})$, which is reduced by construction. Now we show the finiteness of this sequence. We have

Lemma 5.30 *Let $X = X_0$ be a scheme satisfying the assumptions of Theorem 5.28. Let $C = C_0$ be an irreducible regular curve in $X(\widetilde{\nu})$. Let $\pi_1 : X_1 = B\ell_C(X_0) \to X_0$, and let $C_1 = X_1(\widetilde{\nu}) \cap \pi_0^{-1}(C_0)$. By Step 2, $\dim C_1 \leq 1$ and if $\dim C_1 = 1$, then C_1 is regular, $C_1 \subset X_1$ is $\mathcal{B}$-permissible and $C_1 \simeq C_0$. In this case we put $X_2 = B\ell_{C_1}(X_1)$. Repeat this procedure to get a sequence*

$$
(5.10) \quad
\begin{array}{ccccccccccc}
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{m-1} & \xleftarrow{\pi_m} & X_m & \ldots \\
& \cup & & \cup & & \cup & & \cup & & \cup & \\
C = & C_0 & \xleftarrow{\sim} & C_1 & \xleftarrow{\sim} & C_2 & \xleftarrow{\sim} \ldots \xleftarrow{\sim} & C_{m-1} & \xleftarrow{\sim} & C_m & \ldots
\end{array}
$$

such that $\pi_i : X_i = B\ell_{C_{i-1}}(X_{i-1}) \to X_{i-1}$ *is the blow-up morphism, and* $C_i = X_i(\widetilde{\nu}) \cap \pi_i^{-1}(C_{i-1})$.
Then the process stops after finitely many steps, i.e., there is an $m \geq 0$ *with* $C_0 \stackrel{\sim}{\leftarrow} C_1 \stackrel{\sim}{\leftarrow} \ldots \stackrel{\sim}{\leftarrow} C_m$ *and* $\dim(C_{m+1}) \leq 0$.

Proof Let η be the generic point of C. As remarked in Step 2, we have $e_\eta^O(X) \leq 1$. If $e_\eta^O(X) = 0$, then $C_1 = \emptyset$ so that $r = 1$. If $e_\eta^O(X) = 1$, we get a longer sequence, which however must be finite by Theorem 5.35/Corollary 5.37 below, applied to the localization $X_\eta = \mathrm{Spec}(\mathcal{O}_{X,\eta})$ of X at η, and the point η in it, for which $(X_\eta)_{max}^O = \{\eta\}$. Note that $e_\eta^O(X_\eta) = e_\eta^O(X)$ by definition.

By this result, there is an $N \geq 0$ such that $Y_n^{(i)} \cap Y_n^{(j)} = \emptyset$ for all $i,j \in \{0,\ldots,m\}$ with $i \neq j$, for all $n \geq N$, because $[Y_m^{(i)}]_0 \cap Y_m^{(j)} = \emptyset$ for all $i \neq j$, where $[Y_m^{(i)}]_0$ is the set of zero-dimensional components in $Y_n^{(i)}$. Note that all schemes X_n satisfy the conditions in Theorem 5.28. Therefore we have shown:

Claim 4 There is an $N > 0$ such that $Y_n = X_n(\widetilde{\nu})$ is regular for all $n \geq N$.

It is clear that the resolution sequence at each step X_n has the following property, because the centers of the blowups always lie in the subscheme Y_n: Let $Y_{n,1},\ldots,Y_{n,s}$ be the connected components of Y_n, and for each $i \in \{1,\ldots,s\}$, let $V_{n,i} \subset X_n$ be an open subscheme containing $Y_{n,i}$ but not meeting $Y_{n,j}$ for $j \neq i$. Then the resolution sequence for X is obtained by glueing the resolution sequences for the subsets $V_{n,i}$. To show finiteness of the resolution sequence we may thus assume that Y_n is regular and irreducible. Applying Lemma 5.30 again, we may assume that Y_n is a collection of finitely many closed points which are isolated in their O-Hilbert-Samuel stratum. Moreover, it is clear that in this case the remaining part of the resolution sequence $X_n \leftarrow X_{n+1} \leftarrow \ldots$ is just the canonical resolution sequence $S(X_n, \widetilde{\nu})$ for X_n.

Step 9 Thus we have reduced to the case of an isolated point $x \in X(\widetilde{\nu})$; in fact, we may assume that $X(\widetilde{\nu})$ just consists of x. The first step of the canonical sequence then is to form the blowup $X_1 = B\ell_x(X) \to X$. If $e_x^O(X) = 0$, then $X_1(\widetilde{\nu}) = \emptyset$ and we are done. If $e_x^O(X) = 1$, then $X_1(\widetilde{\nu})$ is empty or consists of a unique point x_1 lying above x. In the latter case we have $k(x_1) = k(x)$, and therefore $e_{x_1}(X_1) \leq e_x(X)$ by Theorem 2.10 (4). If $e_{x_1}(X_1) = 1$, then $e_{x_1}^O(X_1) \leq e_{x_1}(X_1) \leq 1$. Otherwise we must have $e_{x_1}(X_1) = e_x(X) = 2$ by assumption (3e). Then $e_{x_1}^O(X_1) \leq e_x^O(X) = 1$ by Theorem 4.27 (1). Thus we obtain a sequence of blow-ups $X = X_0 \leftarrow X_1 \leftarrow \ldots$ in points $x_i \in X_i(\widetilde{\nu})$ such that either $e_{x_n}(X_n) = 0$ for some n so that $X_{n+1}(\widetilde{\nu}) = \emptyset$ and the sequence stops, or we have a sequence where $e_{x_i}(X_i) = 1$ for all i. But this sequence must be finite by Theorem 5.35/Corollary 5.37 below. It remains the case where $e_x^O(X) = e_x(X) = \overline{e}_x(X) = 2$. This follows from Theorem 5.40 below.

Step 10 As a last step we show the functoriality, i.e., the properties (F1) and (F2) in Theorem 5.28. Property (F1) is of course an easier special case of (F2), and was only written for reference. We show more generally:

Proposition 5.31 *The canonical sequence defined in Remark 5.29 is functorial for arbitrary flat morphisms with geometrically regular fibers* $\alpha : X^* \to X$, *in the following sense: Let* $\widetilde{\nu} \in \Sigma_X^{O,max}$, *and let* $\mathcal{X} = (X_n, \mathcal{B}_n, O_n, Y_n, D_n)$ *be the sequence* $S(X, \widetilde{\nu})$ *as defined in Remark 5.29, so that*

$$Y_n = X_n(\widetilde{\nu}) \quad \text{and} \quad X_{n+1} = B\ell_{D_n}(X_n).$$

If $X^*(\widetilde{\nu}) \neq \emptyset$, then $\widetilde{\nu} \in \Sigma_{X^*}^{O^*,max}$, where O^* is the natural history function induced on X^*, see Lemma 4.36, and we let $\mathcal{X}^* = (X_n^*, \mathcal{B}_n^*, O_n^*, Y_n^*, D_n^*)$ be the analogous sequence $S(X^*, \widetilde{\nu})$ for X^* and $\widetilde{\nu}$. If $X^*(\widetilde{\nu}) = \emptyset$, let $\mathcal{X}^* = (X^*, \mathcal{B}^*, O^*, \emptyset, \emptyset)$ (sequence consisting of one term), where $\mathcal{B}^*$ and O^* are obtained by pull-back of O and $\mathcal{B}$ to X^* (notations as in Lemma 4.36). For any closed subscheme $Z \subseteq X_n$, let

$$Z_* = X^* \times_X Z$$

be its base change with $X^* \to X$. (Note that all X_n are X-schemes in a canonical way.) In particular let

$$\mathcal{X}_* = (X_{n,*}, \mathcal{B}_{n,*}, O_{n,*}, Y_{n,*}, D_{n,*})$$

be the base change of $\mathcal{X}$ with $X^* \to X$, so that $X_{n,*} = X^* \times_X X_n, Y_{n,*} = X^* \times_X Y_n$ etc.. Then $\mathcal{X}^*$ is canonically isomorphic to the associated contracted sequence $(\mathcal{X}_*)_{contr}$ (obtained by omitting the isomorphisms). More precisely we claim the following.

(i) There is a canonical isomorphism $X^*(\widetilde{\nu}) \xrightarrow{\sim} X(\widetilde{\nu})_*$. Define the function $\varphi : \mathbb{N} \to \mathbb{N}$ inductively by $\varphi(0) = 0$ and the following property for all $n \geq 1$:

(5.11)
$$\varphi(n) = \begin{cases} \varphi(n-1) & \text{if } D_{n,*} = \emptyset \\ \varphi(n-1) + 1 & \text{otherwise.} \end{cases}$$

Then the following holds:

(a) There are canonical isomorphisms $X_{n+1,*} \cong B\ell_{D_{n,*}}(X_{n,*})$.

(b) Let $\beta_0 : X_0^* = X^* \to X^* \times_X X = X_{0,*}$ be the canonical isomorphism induced by $\alpha : X^* \to X$. For all $n \geq 1$ there are unique morphisms $\beta_n : (X^*)_{\varphi(n)} \to X_{n,*}$ such that all diagrams

(5.12)
$$
\begin{array}{ccc}
X_{\varphi(n)}^* & \xrightarrow{\beta_n} & X_{n,*} \\
\downarrow & & \downarrow \\
X_{\varphi(n-1)}^* & \xrightarrow{\beta_{n-1}} & X_{n-1,*}
\end{array}
$$

are commutative. Moreover all β_n are isomorphisms. Here the left hand morphism is the identity for $\varphi(n) = \varphi(n-1)$, and the morphism occurring in $\mathcal{X}^*$ for $\varphi(n) = \varphi(n-1) + 1$. The right hand morphism is the base change of $X_n \to X_{n-1}$ with $X^* \to X$.

(c) The morphism β_n induces isomorphisms

(5.13)
$$Y_{\varphi(n)}^* \xrightarrow{\sim} Y_{n,*}$$

and

(5.14)
$$(Y_{\varphi(n)}^*)^{(\varphi(j))} \xrightarrow{\sim} X^* \times_X Y_n^{(j)} = (Y_n^{(j)})_*$$

if $(Y_n^{(j)})_* \neq \emptyset$ and

(5.15)
$$D_{\varphi(n)}^* \xrightarrow{\sim} D_{n,*}$$

if $D_{n,*} \neq \emptyset$.

(ii) The analogous statement holds, if the $\widetilde{\nu}$-elimination sequence of Remark 5.29 is replaced by the $\Sigma^{O,max}$-elimination sequence deduced from these $\widetilde{\nu}$-eliminations by gluing, and for the canonical resolution sequences outlined in the proof of Corollary 5.26 or Corollary 5.27 using the mentioned max-eliminations.

Proof We prove both claims by induction on dimension. It is quite clear how to deduce (ii) from (i) once the dimension is fixed, and both claims are trivial for dimension zero. Therefore it suffices to prove (i) when (ii) is assumed for smaller dimension. Since $X^* \to X = X_0$ is flat with regular fibers, the morphism

$$(5.16) \qquad X^*(\widetilde{\nu}) \to X^* \times_X X(\widetilde{\nu}) = Y_{0,*}$$

is an isomorphism by Lemma 4.36 (2), and property (a) follows since $X_{n+1} = B\ell_{D_n}(X_n)$, and $X^* \to X$ is flat (compare the proof of Lemma 4.36 (4)). For the other claims we use induction on n.

(b) is empty for $n = 0$, and we show (c) for $n = 0$. The isomorphism (5.13) follows immediately from (5.16), since $Y_0^* = X^*(\widetilde{\nu})$ by definition. Moreover, $Y_0 = Y_0^{(0)}$ and $Y_0^* = (Y_0^*)^{(0)}$ so that (5.14) follows trivially for $n = 0$. Now we show (5.15) for $n = 0$. By the strategy of Remark 5.29, the following holds. If Y_0 is $\mathcal{B}$-regular, then the first resolution cycle is already completed, and we have to blow up in Y_0, so that $D_0 = Y_0$. Since $Y_0^* \to Y_0$ is flat with geometrically regular fibers, we conclude that Y_0^* is regular as well, and $D_0^* = Y_0^*$, so that (5.15) follows from (5.13). If Y_0 is not $\mathcal{B}$-regular, then D_0 is determined as the first center of the canonical resolution sequence for $(Y_0, \mathcal{B}_{Y_0})$, so that $D_0 = Y_{max}^O$. Similarly, $D_0^* = (Y^*)_{max}^O$. Now we note again that $Y^* \to Y$ is flat with regular fibers and that $D_{0,*} = X^* \times_X D_0 = Y^* \times_Y D_0$. Hence, if $D_{0,*} \neq \emptyset$, then $(Y^*)_{max}^O = Y^* \times_Y Y_{max}^O$ by Lemma 4.36 (2), which shows (5.15) for $n = 0$.

For $n \geq 1$ suppose we have already constructed the morphisms β_m for $m \leq n - 1$ and proved (b) and (c) for them. For $m \leq n - 1$ define the morphisms

$$\alpha_m : X_{\varphi(m)}^* \xrightarrow{\beta_m} X_{m,*} = X^* \times_X X_m \xrightarrow{pr_2} X_m$$

as the composition of the isomorphism β_m with the canonical projection. Then α_m is flat with regular fibers, because this holds for pr_2.

If $D_{n-1,*} = \emptyset$, then $X_{n,*} \xrightarrow{\sim} X_{n-1,*}$ is an isomorphism by (a), we have $\varphi(n) = \varphi(n-1)$, and there is a unique morphism β_n, necessarily an isomorphism, making the diagram in (c) commutative, and trivially cartesian. Define α_n as the composition of β_n with the projection to X_n.

If $D_{n-1,*} \neq \emptyset$, then $\varphi(n) = \varphi(n-1) + 1$, and β_{n-1} induces an isomorphism

$$D_{\varphi(n-1)}^* \xrightarrow{\sim} D_{n-1,*} = X^* \times_X D_{n-1} \cong X_{\varphi(n-1)}^* \times_{X_{n-1}} D_{n-1} ,$$

by induction. As α_{n-1} is flat, there is a unique morphism α_n making the diagram

$$(5.17)$$

$$\begin{array}{ccc}
B\ell_{D_{n-1}^*}(X_{\varphi(n-1)}^*) = X_{\varphi(n)}^* & \xrightarrow{\alpha_n} & X_n = B\ell_{D_{n-1}}(X_{n-1}) \\
\downarrow & & \downarrow \\
X_{\varphi(n-1)}^* & \xrightarrow{\alpha_{n-1}} & X_{n-1}
\end{array}$$

commutative, and the diagram is cartesian (see Lemma 4.36 (4)). By base change of the right column with X^* over X we obtain a diagram as wanted in (b).

By flatness of α_n and regularity of its fibers, and by Lemma 4.36 (2), we have the isomorphism

$$(5.18) \qquad Y^*_{\varphi(n)} = X^*(\widetilde{\nu}) \xrightarrow{\sim} X^*_{n-1} \times_{X_{n-1}} Y_n \cong X^* \times_X Y_n = Y_{n,*}$$

in (5.13) in both cases ($D_{n-1,*}$ empty or non-empty). We note that we have a cartesian diagram

$$(5.19) \qquad \begin{array}{ccc} Y^*_{\varphi(n)} & \xrightarrow{\alpha_n} & Y_n \\ \pi^* \downarrow & & \downarrow \pi \\ Y^*_{\varphi(n-1)} & \xrightarrow{\alpha_{n-1}} & Y_{n-1} \, . \end{array}$$

In fact, we have

$$Y^*_{\varphi(n-1)} \times_{Y_{n-1}} Y_n \cong (X^* \times_X Y_{n-1}) \times_{Y_{n-1}} Y_n \cong X^* \times_X Y_n \cong Y^*_{\varphi(n)} \, .$$

Now we show (5.14) for n, i.e., that the isomorphism (5.18) induces isomorphisms

$$(5.20) \qquad (Y^*_{\varphi(n)})^{(\varphi(j))} \xrightarrow{\sim} X^*_{\varphi(n-1)} \times_{X_{n-1}} Y_n^{(j)} \cong X^* \times_X Y_n^{(j)}$$

for all $j \leq n$. By the factorization

$$\alpha_{n-1} : Y^*_{\varphi(n-1)} \xrightarrow{\sim} X^* \times_X Y_{n-1} \xrightarrow{pr_2} Y_{n-1} \, ,$$

condition (5.14) for $n-1$ means that $(\alpha_{n-1})^{-1}(Y_{n-1}^{(j)}) = (Y^*_{\varphi(n-1)})^{(\varphi(j))}$ for all $j \leq n-1$, or, equivalently, that all generic points of $(Y^*_{\varphi(n-1)})^{(\varphi(j))}$ map to generic points of $Y_{n-1}^{(j)}$, and we have to show the corresponding property for n in place of $n-1$, where we can assume that $D_{n-1,*} \neq \emptyset$. Since α_{n-1} and α_n are flat, the generic points of $Y^*_{\varphi(n-1)}$ are those points which lie over generic points η of Y_{n-1} and are generic points in the fiber over η, and the analogous statement holds for n in place of $n-1$.

Let $\varphi(j) \leq \varphi(n) = \varphi(n-1) + 1$, and let η_n^* be a generic point of $(Y^*_{\varphi(n)})^{(\varphi(j))}$. If $\varphi(j) \leq \varphi(n-1)$, then η_n^* maps to a generic point η_{n-1}^* of $(Y^*_{\varphi(n-1)})^{(\varphi(j))}$ which in turn maps to a generic point η_{n-1} of $Y_{n-1}^{(j)}$. Let η_n be the image of η_n^* in Y_n. Then η_n is a generic point of Y_n and lies in $Y_n^{(j)}$ since it maps to η_{n-1}. Note that in (5.18) the fibers of π^* are obtained from those of π by base change with a field extension. If $\varphi(j) = \varphi(n-1) + 1 (= \varphi(n))$, then η_n is still a generic point of Y_n, but η_{n-1}^* is not a generic point of Y_{n-1}^*. By (5.19), the fiber F_n of α_n over η_n is flat over the fiber F_{n-1} of α_{n-1} over η_{n-1}, because it is obtained by base change with the residue field extension $k(\eta_n)/k(\eta_{n-1})$. Therefore η_n^* is a generic point of the fiber over η_n. This implies that η_{n-1} is not a generic point of Y_{n-1}, and hence that η_n lies in $Y_n^{(n)}$.

Finally we show that α_n induces an isomorphism

$$D^*_{\varphi(n)} \xrightarrow{\sim} D_{n,*} = X^* \times_X D_n \cong X^*_{\varphi(n)} \times_{X_n} D_n \, ,$$

provided that $D_{n,*} \neq \emptyset$. Suppose that

$$(5.21) \qquad D_n \subset Y_n^{(j)} \, .$$

By the prescription in Remark 5.29 this means that either $j = 0$ or that $j > 0$ and $Y_n^{(j-1)} = \emptyset$. In the first case we have $\varphi(0) = 0$, and in the second case we have $(Y_{\varphi(n)}^*)^{(\varphi(j-1))} = \emptyset$ by (5.20). As $\varphi(j) - 1 \leq \varphi(j-1)$, we then also have $(Y_{\varphi(n)}^*)^{(\varphi(j)-1)} = \emptyset$. By (5.21) we get $Y_n^{(j)} \neq \emptyset$ and hence also $(Y_{\varphi(n)}^*)^{(\varphi(j))} \neq \emptyset$ by (5.20). Therefore in both cases we have $D_n^* \subset (Y_n^*)^{(\varphi(j))}$ by our strategy.

If $Y_n^{(j)}$ is regular, then $D_n = Y_n^{(j)}$ by our strategy, and $(Y_n^*)^{(\varphi(j))}$ is regular, too, because $(Y_n^*)^{(\varphi(j))} \to Y_n^{(j)}$ is flat with regular fibers by (5.14). Then also $D_n^* = (Y_n^*)^{(\varphi(j))}$, and the claim follows from (5.20).

If $Y_n^{(j)}$ is not regular, then D_n lies in the singular locus of $Y_n^{(j)}$ by our strategy, and we are in one of the resolution cycles for $Y_?^{(j)}$. Suppose this cycle has started with $Y_r^{(j)}$ $(r \leq n)$. Then we obtain the centers from the canonical resolution sequence for $Y_r^{(j)}$.

First assume that $r = 0$. Then $j = 0$, and we start a resolution cycle with $Y_0^* = (Y_0^*)^{(0)}$ as well. By induction on dimension we can apply (ii) to $Y = Y_0$, and we have a canonical commutative diagram with cartesian squares

$$
\begin{array}{ccccccccc}
Y_n & \to & Y_{n-1} & \to & \ldots & \to & Y_1 & \to & Y_0 & = Y \\
\uparrow & & \uparrow & & \ldots & & \uparrow & & \uparrow & \\
Y_{\psi(n)}^* & \to & Y_{\psi(n-1)}^* & \to & \ldots & \to & Y_{\psi(1)}^* & \to & Y_0^* & = Y^*
\end{array}
$$

where the upper line is part of the canonical resolution sequence for Y, the Y_m^* in the lower line come from the canonical resolution sequence of Y_0^*, and the function ψ is defined in an analogous way as φ in (5.11), but using $Y^* \to Y$ and the $D_{i-1} \subseteq Y_{i-1}$ for all $i \leq n$ instead of $X^* \to X$ and the $D_{i-1} \subseteq X_{i-1}$: Note that, by definition of the resolution cycle, the center of the blow-up $Y_i \to Y_{i-1}$ is the same center D_{i-1} as for the blow-up $X_i \to X_{i-1}$. Again by induction the vertical morphisms induce isomorphisms $D_{\psi(i)} \xrightarrow{\sim} Y^* \times_Y D_i$ for all $i \leq n$ if the last scheme is non-empty. We claim that $\psi(i) = \varphi(i)$, and that $D_{\varphi(i)}^* \to D_i$ coincides with the morphism induced by $\alpha_i : X_{\varphi(i)}^* \to X_i$, for all $i \leq n$, which then proves the claim, since $Y^* \times_Y D_i = X^* \times_X D_i$.

The claim is true for $i = 0$, and we assume it is true up to some $i \leq n - 1$. But then we have $Y^* \times_Y D_i = X^* \times_X D_i = D_{i,*}$, and hence the first scheme is empty if and only if the last one is, which shows $\psi(i+1) = \varphi(i+1)$ since $\psi(i) = \varphi(i)$. Moreover the existence and uniqueness of the morphisms $Y_{\varphi(i+1)}^* \to Y_{*,i+1}$ induced by the above diagram and by α_i implies the remaining claim.

If $r > 0$, then $Y_{r-1}^{(j)}$ is regular, $D_{r-1} = Y_{r-1}^{(j)}$ and $X_r = Bl_{D_{r-1}}(X_{r-1})$, and by induction and (5.14), $(Y_{\varphi(r-1)}^*)^{(\varphi(j))}$ is regular as well (note that $r \leq n$ and that $(Y_{\varphi(r-1)}^*)^{(\varphi(j))} \to Y_{r-1}^{(j)}$ is flat with regular fibers). Hence, by our strategy, with $(Y_{\varphi(r)}^*)^{(\varphi(j))}$ we start a resolution cycle for X^* as well. Now the proof is the same as for the cycle starting with Y_0, up to some renumbering. $\square$

Corollary 5.32 *If in the situation of Proposition 5.31 the $\widetilde{\nu}$-elimination sequence (resp. the $\Sigma^{O,max}$-elimination sequence, resp. the resolution sequence) is finite for X, then it is also finite for X^*.*

We now turn to the two key theorems used in the proof above. Let X be an excellent scheme, and let $(\mathcal{B}, O)$ be a boundary with history on X. Let $x \in X$ and assume that

$(F1)$ $\mathrm{char}(k(x)) = 0$ or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$.

$(F2)$ $N(x) \pitchfork \mathrm{Dir}_x^O(X)$.

$(F3)$ $e_x^O(X) \leq 1$ or $e_x(X) = \bar{e}_x(X)$.

Consider

$$\pi_1 : X_1 = B\ell_x(X) \to X,$$

$$C_1 := \mathbb{P}(\mathrm{Dir}_x^O(X)) \subset \mathbb{P}(\mathrm{Dir}_x(X)) \subset X_1.$$

Let η_1 be the generic point of C_1. We note $C_1 \simeq \mathbb{P}_k^{t-1}$, where $t = e_x^O(X)$. By Theorem 4.20, any point of X_1 which is O-near to x, lies in C_1.

Lemma 5.33 (1) If η_1 is O-near to x, then so is any point of C_1.

(2) If η_1 is very O-near to x, then so is any point of C_1.

Proof Take any point $y \in C_1$. By Theorems 4.11 and 4.18, we have

$$H_{X'}^O(\eta_1) \leq H_{X'}^O(y) \leq H_X^O(x).$$

(1) follows from this. By Theorems 2.6 and 2.10, we have

$$e_{\eta_1}(X') \leq e_y(X') - \dim(\mathcal{O}_{C_1,y}) \leq e_x(X) - \delta_{y/x} - \dim(\mathcal{O}_{C_1,y}) = e_x(X) - \delta_{\eta_1/x},$$

where we used $(F3)$ for the second inequality. In fact, if $e_x^O(X) \leq 1$ then $k(y) = k(x)$, so in both cases of $(F3)$ we can apply 2.10 (4). Hence the assumption of (2) implies that y is very near to x. Then, by Theorems 4.23 and 4.18, we get

$$e_{\eta_1}^O(X') \leq e_y^O(X') - \dim(\mathcal{O}_{C_1,y}) \leq e_x^O(X) - \delta_{y/x} - \dim(\mathcal{O}_{C_1,y}) = e_x^O(X) - \delta_{\eta_1/x},$$

which implies the conclusion of (2). $\square$

We now assume that $t \geq 1$ and η_1 is very O-near to x. This implies

$$(5.22) \qquad\qquad e_{\eta_1}^O(X_1) = e_x^O(X) - \dim(C_1) = 1.$$

By $(F2)$, Lemma 4.25 and Remark 4.15, C_1 is $\mathcal{B}$-permissible with respect to the complete transform $(\mathcal{B}_1, O_1)$ of $(\mathcal{B}, O)$ for X_1. Consider the blow-up

$$\pi_2 : X_2 = B\ell_{C_1}(X_1) \to X_1$$

and the complete transform $(\mathcal{B}_2, O_2)$ of $(\mathcal{B}_1, O_1)$ for X_2. By (5.22) and Theorem 4.20, there is at most one point $\eta_2 \in X_2$ which is O-near to η_1. If it exists, let C_2 be the closure of η_2 in X_2. Then $k(\eta_1) = k(\eta_2)$, π_2 induces an isomorphism $C_2 \cong C_1$, and C_2 is O_2-permissible. By Lemma 5.33 and Theorem 4.27, $(F2)$ implies $N(y) \pitchfork \mathrm{Dir}_y(X_1)$ for any point $y \in C_1$. By Lemma 4.26, C_2 is n.c. with $N\mathcal{B}_2(y') = \mathcal{B}_2(y') - O_2(y')$ at the unique point $y' \in C_2$ above y, so that C_2 is $\mathcal{B}_2$-permissible. Consider

$$\pi_3 : X_3 = B\ell_{C_2}(X_2) \to X_2$$

and proceed in the same way as before. This construction (which occurred in the proof of Theorem 5.28 for $t = 1, 2$) leads us to the following:

Definition 5.34 *Assume $e_x^O(X) \geq 1$ and let m be a non-negative integer or ∞. The fundamental sequence of $\mathcal{B}$-permissible blowups over x of length m is the canonical (possibly infinite) sequence of permissible blowups:*

$$\mathcal{B} = \quad \mathcal{B}_0 \qquad\quad \mathcal{B}_1 \qquad\quad \mathcal{B}_2 \qquad\qquad\quad \mathcal{B}_{n-1} \qquad\quad \mathcal{B}_n$$

(5.23)
$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \leftarrow \ldots \leftarrow X_{n-1} \xleftarrow{\pi_n} X_n \leftarrow \ldots$$
$$\uparrow \qquad \cup \qquad \cup \qquad\qquad \cup \qquad\quad \cup$$
$$x \quad\leftarrow\quad C_1 \quad\tilde{\leftarrow}\quad C_2 \quad\tilde{\leftarrow}\ldots\tilde{\leftarrow}\quad C_{n-1} \quad\tilde{\leftarrow}\quad C_n \leftarrow \ldots$$

which satisfies the following conditions:

(i) $X_1 = Bl_x(X)$ *and*

$$C_1 = \mathbb{P}(\mathrm{Dir}_x^O(X)) \cong \mathbb{P}_{k(x)}^{t-1} \quad (t = e_x^O(X)).$$

(ii) *For $1 \leq q < m$,*

$$C_q = \{\xi \in \phi_q^{-1}(x) \mid H_{X_q}^O(\xi) = H_X^O(x)\} \quad \text{with } \phi_q : X_q \to X.$$

Let η_q be the generic point of C_q.

(iii) *For $2 \leq q < m$, $X_q = Bl_{C_{q-1}}(X_{q-1})$ and $\pi_q : C_q \xrightarrow{\sim} C_{q-1}$ is an isomorphism.*

(iv) *If $m = 1$, then the generic point of C_1 is not O-near to x. If $1 < m < \infty$, then $X_m = Bl_{C_{m-1}}(X_{m-1})$ and there is no point in X_m which is near to η_{m-1}. If $m = \infty$, then the sequence is infinite.*

Here $(\mathcal{B}_q, O_q)$ is the complete transform of $(\mathcal{B}_{q-1}, O_{q-1})$.

The proof of the following first key theorem will be given in §9.

Theorem 5.35 *Assume that there is no regular closed subscheme*

$$D \subseteq \{\xi \in \mathrm{Spec}(\mathcal{O}_{X,x}) \mid H_X^O(\xi) \geq H_X^O(x)\}$$

of dimension $e_x^O(X)$. Then, for the sequence (5.23), we have $m < \infty$, i.e., it stops in finitely many steps.

Remark 5.36 *We note that the assumption of the theorem holds in particular, if*

(5.24)
$$\dim\left(\{\xi \in \mathrm{Spec}(\mathcal{O}_{X,x}) \mid H_X^O(\xi) \geq H_X^O(x)\}\right) < e_x^O(X).$$

Thus a special case of Theorem 5.35 is the following.

Corollary 5.37 *If x is isolated in the O-Hilbert-Samuel locus of X and $e_x^O(X) = 1$, then the fundamental sequence (5.23) consists of a sequence of blowups in closed points and is finite.*

In particular, if $\dim(X) = 2$, we obtain the canonical sequences constructed in steps 8 and 9 of the proof of Theorem 5.28. Hence we obtain their finiteness as needed in that proof.

Now we consider the fundamental sequence of $\mathcal{B}$-permissible blowups over x as in Definition 5.34 for the second case needed in the proof of Theorem 5.28, namely where x is isolated in $X(\widetilde{\nu})$ and

$$e_x^O(X) = e_x(X) = \bar{e}_x(X) = 2.$$

Here we have $C_q \cong \mathbb{P}^1_{k(x)}$ for $1 \le q < m$. Again by Theorem 5.35 (and Remark 5.36 (a)) we deduce that the fundamental sequence (5.23) is finite, i.e., there exists an $m < \infty$ such that there is no point in X_m which is O-near to η_{m-1}. Let $\widetilde{\nu} = H_X^O(x)$. If $X_m(\widetilde{\nu})$ is non-empty, all its points lie above $X_{m-1}(\widetilde{\nu})$ (by maximality of $\widetilde{\nu}$), and the image in $X_{m-1}(\widetilde{\nu})$ consists of finitely many closed points (since it does not contain the generic point of in $X_{m-1}(\widetilde{\nu})$). By the argument used in Step 2 of the proof of Theorem 5.28, each of these points has at most one point in $X_m(\widetilde{\nu})$ above it, so that the latter set consist of finitely many closed points as well. Pick one such point y. Since we already treated the case where y is isolated with $e_y^O(X_m) = 1$, and we always have $e_y^O(X_m) \le 2$, we are led to the following definition.

Definition 5.38 *A sequence of $\mathcal{B}$-permissible blowups $(\mathcal{X}, \mathcal{B})$:*

$$
\begin{array}{ccccccccc}
\mathcal{B} = & \mathcal{B}_0 & & \mathcal{B}_1 & & \mathcal{B}_2 & & \mathcal{B}_{m-1} & & \mathcal{B}_m \\
\end{array}
$$

$$
\begin{array}{ccccccccccc}
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{m-1} & \xleftarrow{\pi_m} & X_m \\
& \uparrow & & \cup & & \cup & & \cup & & \uparrow \\
& x & \leftarrow & C_1 & \xleftarrow{\sim} & C_2 & \xleftarrow{\sim} \ldots \xleftarrow{\sim} & C_{m-1} & \leftarrow & x_m
\end{array}
$$

is called a fundamental unit of $\mathcal{B}$-permissible blowups of length m if the following conditions are satisfied:

(i) *x is a closed point of X such that $e_x^O(X) = e_x(X) = \bar{e}_x(X) = 2$.*

(ii) *$X_1 = B\ell_x(X)$ and*
$$C_1 = \mathbb{P}(\mathrm{Dir}_x^O(X)) \cong \mathbb{P}^1_{k(x)}.$$

(iii) *For $2 \le q \le m$, $C_q = \phi_q^{-1}(x) \cap X_q(\widetilde{\nu})$ and $X_q = B\ell_{C_{q-1}}(X_{q-1})$, where $\widetilde{\nu} = H_X^O(x)$ and $\phi_q : X_q \to X$ is the natural morphism.*

(iv) *For $2 \le q < m$, π_q induces an isomorphism $C_q \cong C_{q-1}$.*

(v) *$\pi_m : C_m \to C_{m-1}$ is not surjective.*

(vi) *x_m is a closed point of X_m above x such that*
$$H_{X_m}^O(x_m) = H_X^O(x) \quad and \quad e_{x_m}^O(X_m) = e_{x_m}(X_m) = \bar{e}_{x_m}(X_m) = 2.$$

Here $H_{X_q}^O$ is considered for the successive complete transform $(\mathcal{B}_q, O_q)$ of $(\mathcal{B}, O)$ for X_q.

By convention, a fundamental unit of $\mathcal{B}$-permissible blowups of length 1 is a sequence of $\mathcal{B}$-permissible blowups such as

$$\mathcal{B} \;=\; \mathcal{B}_0 \qquad \mathcal{B}_1$$

$$
\begin{array}{ccccccc}
X & = & X_0 & \overset{\pi_1}{\leftarrow} & X_1 & = & B\ell_x(X) \\
\uparrow & & \uparrow & & \uparrow & & \\
x & = & x_0 & \leftarrow & x_1 & &
\end{array}
$$

where $x \in X$ is as in (i) and x_1 is as in (vi) with $m = 1$.
We call $(x, X, \mathcal{B})$ (resp. $(x_m, X_m, \mathcal{B}_m)$) the initial (resp. terminal) part of $(\mathcal{X}, \mathcal{B})$.

We remark that, in this definition, we have not assumed that x (resp. x_m) is isolated in X^O_{max} (resp. $(X^O_m)_{\max}$).

Definition 5.39 *A chain of fundamental units of $\mathcal{B}$-permissible blowups is a sequence of $\mathcal{B}$-permissible blowups:*

$$\mathcal{X}_1 \leftarrow \mathcal{X}_2 \leftarrow \mathcal{X}_3 \leftarrow \ldots$$

where $\mathcal{X}_i = (\mathcal{X}_i, \mathcal{B}_i)$ is a fundamental unit of $\mathcal{B}$-permissible blowups such that the terminal part of $\mathcal{X}_i$ coincides with initial part of $\mathcal{X}_{i+1}$ for $\forall\, i \geq 1$.

The finiteness of the canonical $\widetilde{\nu}$-elimination $S(X, \widetilde{\nu})$ for the case where $X(\widetilde{\nu}) = \{x\}$ and $e^O_x(X) = e_x(X) = \bar{e}_x(X) = 2$, as needed in the proof of Theorem 5.28, is now a consequence of the following second key theorem whose proof will be given in §12 and §13.

Theorem 5.40 *Let $\mathcal{X}_1 \leftarrow \mathcal{X}_2 \leftarrow \mathcal{X}_3 \leftarrow \ldots$ be a chain of fundamental units of $\mathcal{B}$-permissible blowups. Let $(x^{(i)}, X^{(i)}, B^{(i)})$ be the initial part of $(\mathcal{X}_i)$ for $i \geq 0$. Assume that, for each i, there is no regular closed subscheme $C \subseteq (X^{(i)})^O_{\max}$ of dimension 1 with $x^{(i)} \in C$ (which holds if $x^{(i)}$ is isolated in $(X^{(i)})^O_{\max}$). Then the chain must stop after finitely many steps.*

In fact, to show the finiteness of $S(X, \widetilde{\nu})$ in the considered case, we have to show that there is no infinite sequence of closed points $x_n \in X_n(\widetilde{\nu})$ such that $x_0 = x$ and x_{n+1} lies above x_n. This can only happen if $e^O_{x_n}(X_n) = 2$ ($= e_{x_n}(X_n) = \bar{e}_{x_n}(X_n)$) for all n, and by construction, the canonical sequence $S(X, \widetilde{\nu})$ would then give rise to an infinite chain of fundamental units.

We remark that the claims on the fundamental sequences, fundamental units and chains of fundamental units depend only on the localization $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$ of X at x. Moreover, by the results in Lemma 1.27 and Lemma 1.37 we may assume that $X = \mathrm{Spec}(\mathcal{O})$ for a complete local ring. Thus we assume that there is an embedding $X \hookrightarrow Z$ into a regular excellent scheme Z, and moreover, by Lemma 4.21, that there is a simple normal crossings boundary $\mathcal{B}_Z$ on Z whose pull-back to X is $\mathcal{B}$. Thus we may consider an embedded version of the constructions above, where each blowup $X_{n+1} = B\ell_{C_n}(X_n) \to X_n$ (where $C_0 = \{x\}$) can be embedded into a diagram

$$
\begin{array}{ccccc}
Z_{m+1} & = & B\ell_{C_m}(Z_m) & \overset{\pi_{m+1}}{\longrightarrow} & Z_m \\
 & & \cup & & \cup \\
X_{m+1} & = & B\ell_{C_m}(X_m) & \overset{\pi_{m+1}}{\longrightarrow} & X_m\,.
\end{array}
$$

In the proofs of Theorems 5.35 and 5.40, this situation will be assumed.

6 (u)-standard bases

Let R be a regular noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let $J \subseteq \mathfrak{m}$ be an ideal. It turns out that the directrix $\mathrm{Dir}(R/J)$ is an important invariant of the singularity of $X = \mathrm{Spec}(R/J)$, and that it is useful to consider a system $(y_1, \ldots, y_r, u_1, \ldots, u_e)$ of regular parameters for R such that:

(6.1) $\qquad I\mathrm{Dir}(R/J) = \langle Y_1, \ldots, Y_r \rangle \subset \mathrm{gr}_{\mathfrak{m}}(R)$, where $Y_i := in_{\mathfrak{m}}(y_i) \in \mathrm{gr}_{\mathfrak{m}}^1(R)$.

Then $(U_1, \ldots, U_e)$ with $U_j := in_{\mathfrak{m}}(u_j)$ form coordinates of the affine space $\mathrm{Dir}(R/J) \cong \mathbb{A}_k^e$. Consequently, it will be useful to distinguish the Y- and U-coordinates in $\mathrm{gr}_{\mathfrak{m}}(R) = k[Y, U]$. This observation leads us to the following:

Definition 6.1 *(1) A system $(y, u) = (y_1, \ldots, y_r, u_1, \ldots, u_e)$ of regular parameters for R is called strictly admissible for J if it satisfies the above condition* (6.1)*.*

(2) A sequence $u = (u_1, \ldots, u_e)$ of elements in $\mathfrak{m} \subseteq R$ is called admissible (for J), if it can be extended to a strictly admissible system (y, u) for J.

(3) Let (y, u) be strictly admissible for J, and let $f = (f_1, \ldots, f_m)$ be a system of elements in J. Then (f, y, u) is called admissible if $in_{\mathfrak{m}}(f_i) \in k[Y]$ for all $i = 1, \ldots, m$.

Let $T_1, \ldots, T_e$ be a tuple of new variables over k. Note that $(u) = (u_1, \ldots, u_e)$ is admissible if and only if we have an isomorphism of k-algebras

$$k[T_1, \ldots, T_e] \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}(R)/I\mathrm{Dir}(R/J) \ ; \ T_i \to in_{\mathfrak{m}}(u_i) \mod I\mathrm{Dir}(R/J).$$

The map induces the following isomorphism which we will use later.

(6.2) $$\psi_{(u)} : \mathbb{P}(\mathrm{Dir}(R/J)) \xrightarrow{\sim} \mathrm{Proj}(k[T_1, \ldots, T_e]) = \mathbb{P}_k^{e-1}$$

The admissibility will play an essential role in the next section. For the moment we shall work in the following general setup:

Setup A: Let $J \subset \mathfrak{m} \subset R$ be as above. Let $(u) = (u_1, \ldots, u_e)$ be a system of elements in $\mathfrak{m}$ such that (u) can be extended to a system of regular parameters (y, u) for some $y = (y_1, \ldots, y_r)$. In what follows we fix u and work with various choices of y as above. Such a choice induces an identification

$$\mathrm{gr}_{\mathfrak{m}}(R) = k[Y, U] = k[Y_1 \ldots, Y_r, U_1, \ldots, U_e]. \quad (Y_i = in_{\mathfrak{m}}(y_i), \ U_j = in_{\mathfrak{m}}(u_j)).$$

Let $\tilde{R} = R/\langle u \rangle$ and $\tilde{\mathfrak{m}} = \mathfrak{m}/\langle u \rangle$ where $\langle u \rangle = \langle u_1, \ldots, u_e \rangle \subset R$. For $f \in R - \{0\}$ put

$$n_{(u)}(f) = v_{\tilde{\mathfrak{m}}}(\tilde{f}) \quad \text{with } \tilde{f} = f \mod \langle u \rangle \ \in \tilde{R}.$$

Note $n_{(u)}(f) \geq v_{\mathfrak{m}}(f)$ and $n_{(u)}(f) = \infty$ if and only if $f \in \langle u \rangle$. Let $f \in R - \{0\}$ and write an expansion in the $\mathfrak{m}$-adic completion $\hat{R}$ of R:

(6.3) $$f = \sum_{(A,B)} C_{A,B} \, y^B u^A \quad \text{with} \quad C_{A,B} \in R^\times \cup \{0\}$$

84

where for $A = (a_1, \ldots, a_e) \in \mathbb{Z}_{\geq 0}^e$ and $B = (b_1, \ldots, b_r) \in \mathbb{Z}_{\geq 0}^r$,

$$y^B = y_1^{b_1} \ldots y_r^{b_r} \quad \text{and} \quad u^A = u_1^{a_1} \ldots u_e^{a_e}.$$

If $n_{(u)}(f) < \infty$, then we define the 0-initial form of f by:

$$(6.4) \qquad in_0(f) = in_0(f)_{(y,u)} = \sum_{\substack{B \\ |B|=n_{(u)}(f)}} \overline{C}_{0,B} \, Y^B \in k[Y],$$

where $\overline{C_{A,B}} = C_{A,B} \mod \mathfrak{m} \in k = R/\mathfrak{m}$. If $n_{(u)}(f) = \infty$, we define $in_0(f)_{(y,u)} = 0$. It is easy to see that $in_0(f)$ depends only on (y, u), not on the presentation (6.3).

We will need to make the expansion (6.3) uniquely determined by f. By [EGA IV] Ch. 0 Th. (19.8.8), we can choose a ring S of coefficients of $\hat{R}$: S is a subring of $\hat{R}$ which is a complete local ring with the maximal ideal pS where $p = \text{char}(k)$ such that $\mathfrak{m} \cap S = pS$ and $S/pS = R/\mathfrak{m}$. We choose a set $\Gamma \subset S$ of representatives of k. Note that $S \simeq k$ and $\Gamma = k$ if $\text{char}(k) = \text{char}(K)$ where K is the quotient field of R. Back in the general situation each $f \in R$ is expanded in $\hat{R}$ in a unique way as:

$$(6.5) \qquad f = \sum_{(A,B)} C_{A,B} \, y^B u^A \quad \text{with} \quad C_{A,B} \in \Gamma.$$

We will use the following map of sets

$$(6.6) \qquad \omega = \omega_{(y,u,\Gamma)} : k[[Y,U]] \to \hat{R} \,; \quad \sum_{(A,B)} c_{A,B} \, Y^B U^A \to \sum_{(A,B)} C_{A,B} \, y^B u^A,$$

where $C_{A,B} \in \Gamma$ is the representative of $c_{A,B} \in k$. For $F, G \in k[[Y,U]]$ we have

$$(6.7) \qquad \omega(F+G) - \omega(F) - \omega(G) \in p\hat{R} \quad \text{and} \quad \omega(F \cdot G) - \omega(F) \cdot \omega(G) \in p\hat{R}.$$

We now introduce the notion of a (u)-standard base (see Definition 6.7), which generalizes that of a standard base (cf. Definition 1.17). The following facts are crucial: Under a permissible blowup a standard base is not necessarily transformed into a standard base but into a (u)-standard base (see Theorem 8.1), on the other hand there is a standard procedure to transform a (u)-standard base into a standard base (see Theorem 7.26).

A linear form $L : \mathbb{R}^e \longrightarrow \mathbb{R}$ given by

$$L(A) = \sum_{i=1}^{e} c_i a_i \text{ with } c_i \in \mathbb{R} \quad (A = (a_i) \in \mathbb{R}^e)$$

is called positive (resp. semi-positive) if $c_i > 0$ (resp. $c_i \geq 0$) for all $1 \leq i \leq e$.

Definition 6.2 *Let (y, u) be as in Setup A and let L be a non-zero semi-positive linear form L on $\mathbb{R}^e$.*

(1) For $f \in \hat{R} - \{0\}$ define the L-valuation of f with respect to (y, u) as:

$$v_L(f) = v_L(f)_{(y,u)} := \min\{|B| + L(A) \mid C_{A,B} \neq 0\},$$

where the $C_{A,B}$ come from a presentation (6.5) and $|B| = b_1 + \cdots b_r$ for $B = (b_1, \ldots, b_r)$. We set $v_L(f) = \infty$ if $f = 0$.

(2) Fix a representative Γ of k in $\hat{R}$ as in (6.5). The initial form of $f \in \hat{R} - \{0\}$ with respect to L, (y, u) and Γ is defined as:

$$in_L(f) = in_L(f)_{(y,u,\Gamma)} := \sum_{A,B} \overline{C_{A,B}}\, Y^B U^A$$

where A, B range over $\mathbb{Z}_{\geq 0}^e \times \mathbb{Z}_{\geq 0}^r$ satisfying $|B| + L(A) = v_L(f)$. We set $in_L(f) = 0$ if $f = 0$.

(3) For an ideal $J \subset R$, we define

$$\widehat{In}_L(J) = \widehat{In}_L(J)_{(y,u,\Gamma)} = \langle in_L(f) \mid f \in J \rangle \subset k[[U]][Y].$$

In case L is positive we define

$$In_L(J) = In_L(J)_{(y,u,\Gamma)} = \langle in_L(f) \mid f \in J \rangle \subset k[U, Y] = \mathrm{gr}_{\mathfrak{m}}(R).$$

It is easy to see that this is well-defined, i.e., one has the following:

(i) $in_L(f)$ is an element of $k[[U]][Y]$, the polynomial ring of Y with coefficients in the formal power series ring $k[[U]]$,

(ii) If L is positive, $in_L(f) \in k[U, Y] = \mathrm{gr}_{\mathfrak{m}}(R)$, and is independent of the choice of Γ.

Remark 6.3 Note that $v_{\mathfrak{m}}(f) = v_{L_0}(f)$ and $in_{\mathfrak{m}}(f) = in_{L_0}(f)$, where

$$L_0(A) = |A| = a_1 + \cdots + a_e \quad \text{for } A = (a_1, \ldots, a_e).$$

The proofs of the following lemmas 6.4 and 6.6 are easy and left to the readers.

Lemma 6.4 Let the assumptions be as in Definition 6.2.

(1) $v_L(f)$ is independent of the choice of Γ. We have

$$v_L(fg) = v_L(f) + v_L(g) \quad \text{and} \quad v_L(f + g) \geq \min\{v_L(f), v_L(g)\}.$$

(2) Assume $v_L(\mathrm{char}(k)) > 0$ (which is automatic if L is positive). If $f = \sum_{i=1}^{m} f_i$ and $v_L(f_i) \geq v_L(f)$ for all $i = 1, \ldots, m$, then $in_L(f) = \sum_{1 \leq i \leq m} in_L(f_i)$, where the sum ranges over such i that $v_L(f_i) = v_L(f)$.

(3) Let $z = (z_1, \ldots, z_r) \subset R$ be another system of parameters such that (z, u) is a system of regular parameters of R. Assume $v_L(z_i - y_i)_{(z,u)} \geq 1$ for all $i = 1, \ldots, r$. Then, for any $f \in R$, we have $v_L(f)_{(z,u)} \geq v_L(f)_{(y,u)}$.

Definition 6.5 *Let the assumption be as Definition 6.2. Let $f = (f_1, \ldots, f_m)$ be a system of elements in $R - \{0\}$. A non-zero semi-positive linear form L on $\mathbb{R}^e$ is called effective for (f, y, u) if $in_L(f_i) \in k[Y]$ for all $i = 1, \ldots, m$.*

Lemma 6.6 *Let the assumption be as in Definition 6.5.*

(1) The following conditions are equivalent:

    (i) L is effective for (f, y, u).

    (ii) $v_L(f_i) = n_{(u)}(f_i) < \infty$ and $in_L(f_i) = in_0(f_i)$ for all $i = 1, \ldots, m$.

    (iii) For $1 \leq i \leq m$ write as (6.3)

$$f_i = \sum_{(A,B)} C_{i,A,B} \, y^B u^A \quad \text{with} \quad C_{i,A,B} \in R^\times \cup \{0\} \,.$$

    Then, for $1 \leq i \leq m$ and $A \in \mathbb{Z}_{\geq 0}^e$ and $B \in \mathbb{Z}_{\geq 0}^r$, we have

$$|B| + L(A) > n_{(u)}(f_i) \text{ if } |B| < n_{(u)}(f_i) \text{ and } C_{i,A,B} \neq 0.$$

(2) There exist a positive linear form L on $\mathbb{R}^e$ effective for (f, y, u) if and only if f_i is not contained in $\langle u \rangle \subset R$ for any $1 \leq i \leq m$.

(3) If L is effective for (f, y, u) and Λ is a linear form such that $\Lambda \geq L$, then Λ is effective for (f, y, u). More precisely one has the following for $1 \leq i \leq m$:

$$v_\Lambda(f_i) = v_L(f_i) = n_{(u)}(f_i) \quad \text{and} \quad in_\Lambda(f_i) = in_L(f_i) = in_0(f_i)$$

Definition 6.7 *Let u be as in Setup A. Let $f = (f_1, \ldots, f_m) \subset J$ be a system of elements in $R - \{0\}$.*

(1) f is called a (u)-effective base of J, if there is a tuple $y = (y_1, \ldots, y_r)$ as in Setup A and a positive form L on $\mathbb{R}^e$ such that L is effective for (f, y, u) and

$$In_L(J) = \langle in_0(f_1), \ldots, in_0(f_m) \rangle \subset \mathrm{gr}_{\mathfrak{m}}(R).$$

(2) f is called a (u)-standard base, if in addition $(in_0(f_1), \ldots, in_0(f_m))$ is a standard base of $In_L(J)$.

In both cases (1) and (2), (y, L) is called a reference datum for the (u)-effective (or (u)-standard) base $(f_1, \ldots, f_m)$.

Lemma 6.8 *Let u be as in Setup A.*

(1) Let $f = (f_1, \ldots, f_m)$ be a standard base of J such that $in_{\mathfrak{m}}(f_i) \in k[Y]$ for $i = 1, \ldots, m$. Then f is a (u)-standard base of J with reference datum (y, L_0), where L_0 is as in Remark 6.3.

(2) Assume that (u) be admissible for J (cf. Definition 6.1). A standard base $f = (f_1, \ldots, f_m)$ of J is a (u)-standard base of J.

Proof (1) is an immediate consequence of Definition 6.1(3) and Remark 6.3. We show (2). First we note that

$$\nu^*(J) = (n_1, \ldots, n_m, \infty, \ldots) \quad \text{with } n_i = v_{\mathfrak{m}}(f_i).$$

and that $(in_{\mathfrak{m}}(f_1), \ldots, in_{\mathfrak{m}}(f_m))$ is a standard base of $In_{\mathfrak{m}}(J)$. Choose $y = (y_1, \ldots, y_r)$ such that (y, u) is strictly admissible for $J \subset R$ and identify $\mathrm{gr}_{\mathfrak{m}}(R) = k[Y, U]$. Then there exist $\psi_1, \ldots, \psi_m \in k[Y]$ which form a standard base of $In_{\mathfrak{m}}(J)$. Note that ψ_i is homogeneous of degree n_i for $i = 1, \ldots, m$. We have

$$\langle in_{\mathfrak{m}}(f_1), \ldots, in_{\mathfrak{m}}(f_m) \rangle = In_{\mathfrak{m}}(J) = \langle \psi_1, \ldots, \psi_m \rangle.$$

Writing

$$in_{\mathfrak{m}}(f_i) = \phi_i + \sum_{A \in \mathbb{Z}^e_{\geq 0} - \{0\}} U^A P_{i,A}, \quad \text{with } \phi_i, \ P_{i,A} \in k[Y] \ (i = 1, \ldots, m),$$

this implies that $(\phi_1, \ldots, \phi_m)$ is a standard base of $In_{\mathfrak{m}}(J)$. Hence it suffices to show that there exists a positive linear form $L : \mathbb{R}^e \to \mathbb{R}$ such that $in_L(f_i) = \phi_i$ for all $i = 1, \ldots, m$. We may write

$$P_{i,A} = \sum_{B \in \mathbb{Z}^r_{\geq 0}} c_{i,A,B} Y^B, \quad (c_{i,A,B} \in k),$$

where the sum ranges over $B \in \mathbb{Z}^r_{\geq 0}$ such that $|B| + |A| = n_i := v_{\mathfrak{m}}(f_i)$. It is easy to see that there exists a positive linear form L satisfying the following for all $A \in \mathbb{Z}^e_{\geq 0} - \{0\}$:

$$L(A) > |A| \quad \text{and} \quad L(\frac{A}{n_i - |B|}) > 1 \text{ if } c_{i,A,B} \neq 0.$$

Then $in_L(f_i) = \phi_i$ and the proof of Lemma 6.8 is complete. $\square$

A crucial fact on (u)-standard bases is the following:

Theorem 6.9 Let $f = (f_1, \ldots, f_m)$ be a (u)-effective (resp. standard) base of J. Then, for any $y = (y_1 \ldots, y_r)$ as in Setup A and for any positive linear form L on $\mathbb{R}^e$ effective for (f, y, u), (y, L) is a reference datum for f.

Before going to the proof of Theorem 6.9, we deduce the following:

Corollary 6.10 Let $f = (f_1, \ldots, f_m)$ be a (u)-effective (resp. standard) base of J. Let $g = (g_1, \ldots, g_m) \subset J$ be such that $in_0(g_i) = in_0(f_i)$ for all $i = 1, \ldots, m$. Then g is a (u)-effective (resp. standard) base of J.

Proof The assumption implies that no f_i or g_i is contained in $\langle u \rangle \subset R$. By Lemma 6.6 (2) and (3) there exists a positive linear form L on $\mathbb{R}^e$ effective for both (f, y, u) and (g, y, u). By the assumption on f, Theorem 6.9 implies that $in_L(f_1), \ldots, in_L(f_m)$ generate (resp. form a standard base of) $in_L(J)$. By Lemma 6.6 $(1)(ii)$, we get

$$in_L(g_i) = in_0(g_i) = in_0(f_i) = in_L(f_i) \quad \text{for } i = 1, \ldots, m,$$

which implies Corollary 6.10. $\square$

Proof of Theorem 6.9 Let (z, Λ) be a reference datum for f which exists by the assumption. By definition (z, u) is a system of regular parameters of R and Λ is a positive linear form on $\mathbb{R}^e$ such that $in_\Lambda(f_1)_{(z,u)}, \ldots, in_\Lambda(f_m)_{(z,u)}$ generate (resp. form a standard base of) $in_\Lambda(J)_{(z,u)}$. First we assume $y = z$. Then the theorem follows from Proposition 6.11 below in view of Lemma 6.6 (Note that condition (4) of Proposition 6.11 is always satisfied if L and Λ are both positive). We consider the general case. Since (y, u) and (z, u) are both systems of regular parameters of R, there exists $M = (\alpha_{ij}) \in GL_r(R)$ such that

$$y_i = l_i(z) + d_i, \quad \text{where } l_i(z) = \sum_{j=1}^{r} \alpha_{ij} z_j \text{ and } d_i \in \langle u \rangle.$$

Take any positive linear form L' on $\mathbb{R}^e$ such that $L'(A) > |A|$ for $\forall A \in \mathbb{R}^e_{\geq 0} - \{0\}$. An easy computation shows that for $A \in \mathbb{Z}^e_{\geq 0}$ and $B = (b_1, \ldots, b_r) \in \mathbb{Z}^r_{\geq 0}$ we have

$$y^B u^A = u^A \cdot l_1(z)^{b_1} \cdots l_1(z)^{b_1} + w \quad \text{with} \quad v_{L'}(w) > v_{L'}(y^B u^A) = |B| + L'(A).$$

This implies $in_{L'}(g)_{(z,u)} = \phi(in_{L'}(g)_{(y,u)})$ for $g \in R - \{0\}$, where

$$\phi : k[Y] \simeq k[Z] \; ; \; Y_i \to \sum_{j=1}^{r} \overline{\alpha}_{ij} Z_j. \quad (Z_j = in_{\mathfrak{m}}(z_j), \; \overline{\alpha}_{ij} = \alpha_{ij} \mod \mathfrak{m} \in k)$$

In view of Lemma 6.6, the proof of Theorem 6.9 is now reduced to the case $y = z$. $\square$

Proposition 6.11 *Let (y, u) be as in Setup A, and let $f = (f_1, \ldots, f_m) \subset J$. Let Λ and L be semi-positive linear forms on $\mathbb{R}^e$. Assume $v_\Lambda(\text{char}(k)) > 0$ (cf. Lemma 6.4) Assume further the following conditions:*

(1) $v_\Lambda(f_i) = v_L(f_i) = n_{(u)}(f_i) < \infty$ for $i = 1, \ldots, m$.

(2) $in_\Lambda(f_i) = in_0(f_i) := F_i(Y) \in k[Y]$ for $i = 1, \ldots, m$.

(3) $\widehat{In}_\Lambda(J) = \langle F_1(Y), \ldots, F_m(Y) \rangle \subset k[[U]][Y]$.

Then for any $g \in J$ and any $M \geq 0$, there exist $\lambda_1, \ldots, \lambda_m \in \hat{R}$ such that

$$v_L(\lambda_i f_i) \geq v_L(g), \quad v_\Lambda(\lambda_i f_i) \geq v_\Lambda(g), \quad v_\Lambda \left(g - \sum_{i=1}^{m} \lambda_i f_i \right) > M.$$

If Λ is positive, one can take $\lambda_i \in R$ for $i = 1, \ldots, m$. Assume further:

(4) there exist $c > 0$ such that $L \geq c\Lambda$.

Then we have
$$\widehat{In}_L(J) = \langle in_L(f_1), \ldots, in_L(f_m) \rangle \subset k[[U]][Y].$$

If L is positive, then

$$In_L(J) = \langle in_L(f_1), \ldots, in_L(f_m) \rangle \subset k[U, Y].$$

Proof Let $g \in J$ and expand as in (6.5):

$$g = \sum_{A,B} C_{A,B} y^B u^A \quad \text{in } \hat{R}, \quad C_{A,B} \in \Gamma.$$

Then we have

$$in_\Lambda(g) = \sum_{\substack{A,B \\ |B|+\Lambda(A)=v_\Lambda(g)}} \overline{C_{A,B}} \ Y^B U^A.$$

Put

$$B_{\max} = B_{\max}(g, \Lambda) = \max\{B \mid \overline{C_{A,B}} \neq 0, \ |B| + \Lambda(A) = v_\Lambda(g) \text{ for some } A \in \mathbb{Z}_{\geq 0}^e\}$$

where the maximum is taken with respect to the lexicographic order.

Lemma 6.12 *Under the assumption (1) and (2) of Proposition 6.11, there exist $\lambda_1, \ldots, \lambda_m \in \hat{R}$ such that:*

(1) $v_\Lambda(\lambda_i f_i) = v_\Lambda(g)$ if $\lambda_i \neq 0$, and $v_L(\lambda_i f_i) \geq v_L(g)$.

(2) $v_L(g_1) \geq v_L(g)$ where $g_1 = g - \sum_{i=1}^m \lambda_i f_i$.

(3) $v_\Lambda(g_1) \geq v_\Lambda(g)$ and $B_{\max}(g_1, \Lambda) < B_{\max}(g, \Lambda)$ if $v_\Lambda(g_1) = v_\Lambda(g)$.

If Λ is positive, one can take $\lambda_i \in R$ for $i = 1, \ldots, m$.

Proof By the assumptions of Proposition 6.11, we can write

$$in_\Lambda(g) = \sum_{\substack{A,B \\ |B|+\Lambda(A)=v_\Lambda(g)}} \overline{C_{A,B}} \ Y^B U^A = \sum_{1 \leq i \leq m} H_i F_i(Y)$$

for some $H_1, \ldots, H_m \in k[[U]][Y]$, where $F_i(Y) \in k[Y]$ is homogeneous of degree $n_i := v_\Lambda(f_i) = v_L(f_i)$ for $i = 1, \ldots, m$. Writing $H_i = \sum_{A \in \mathbb{Z}_{\geq 0}^e} h_{i,A}(Y) U^A$ with $h_{i,A}(Y) \in k[Y]$, this implies

$$(6.8) \qquad \sum_{\substack{B \\ |B|+\Lambda(A)=v_\Lambda(g)}} \overline{C_{A,B}} \ Y^B = \sum_{1 \leq i \leq m} h_{i,A}(Y) F_i(Y) \quad \text{for each } A \in \mathbb{Z}_{\geq 0}^e.$$

Take any A_0 with $|B_{\max}| + \Lambda(A_0) = v_\Lambda(g)$ and $\overline{C_{A_0, B_{\max}}} \neq 0$. Looking at the homogeneous part of degree $|B_{\max}|$ in (6.8) with $A = A_0$, we get

$$\sum_{\substack{B \\ |B|=|B_{\max}|}} \overline{C_{A_0,B}} \ Y^B = \sum_{1 \leq i \leq m} S_i(Y) F_i(Y)$$

where $S_i(Y) \in k[Y]$ is the homogenous part of degree $|B_{\max}| - n_i$ of $h_{i,A_0}(Y)$. Therefore

$$(6.9) \qquad Y^{B_{\max}} - \sum_{i=1}^m P_i(Y) F_i(Y) = \sum_{\substack{|B|=|B_{\max}| \\ B \neq B_{\max}}} a_B Y^B,$$

90

where $P_i(Y) = \left(\overline{C_{A_0,B_{\max}}}\right)^{-1} S_i(Y) \in k[Y]$ and $a_B = -\left(\overline{C_{A_0,B_{\max}}}\right)^{-1}\overline{C_{A_0B}} \in k$. Now put

$$g_1 = g - \sum_{i=1}^{m} \lambda_i f_i \quad \text{with } \lambda_i = \begin{cases} \tilde{P}_i(y)\tilde{Q}(u) & \text{if } P_i(Y) \neq 0, \\ 0 & \text{if } P_i(Y) = 0, \end{cases}$$

where $\tilde{P}_i(Y) \in R[Y]$ is a lift of $P_i(Y) \in k[Y]$ and

$$\tilde{Q}(u) = \sum_{\substack{A \\ |B_{\max}|+\Lambda(A)=v_\Lambda(g)}} C_{A,B_{\max}}u^A \in \hat{R}.$$

Note that if Λ is positive, then the sum is finite and $\tilde{Q}(u) \in R$ and $\lambda_i \in R$. For $1 \leq i \leq m$ with $P_i(Y) \neq 0$, we have

$$v_\Lambda(\lambda_i) = (|B_{\max}| - n_i) + (v_\Lambda(g) - |B_{\max}|) = v_\Lambda(g) - n_i,$$

$$\begin{aligned} v_L(\lambda_i) =& (|B_{\max}| - n_i) + v_L(\tilde{Q}(u)) \\ \geq& (|B_{\max}| - n_i) + v_L(g) - |B_{\max}| = v_L(g) - n_i, \end{aligned}$$

which shows Lemma 6.12 (1) in view of Proposition 6.11 (1). Here the last inequality holds because

$$v_L(g) = \min\{|B| + L(A) \mid \overline{C_{A,B}} \neq 0\},$$

$$v_L(\tilde{Q}(u)) + |B_{\max}| = \min\{|B| + L(A) \mid \overline{C_{A,B}} \neq 0,\ B = B_{\max},\ |B_{\max}| + \Lambda(A) = v_\Lambda(g)\}.$$

Therefore, by Lemma 6.4 (1) we get

$$v_\Lambda(g_1) \geq \min\{v_\Lambda(g),\ v_\Lambda(\lambda_i f_i)\ (1 \leq i \leq m)\} \geq v_\Lambda(g).$$

If $v_\Lambda(g_1) = v_\Lambda(g)$, then Lemma 6.4 (2) implies

$$\begin{aligned} in_\Lambda(g_1) &= in_\Lambda(g) + in_\Lambda\left(\sum_{i=1}^{m} \lambda_i f_i\right) \\ &= in_\Lambda(g) + \sum_{\substack{1 \leq i \leq m \\ \lambda_i \neq 0}} in_\Lambda(\lambda_i)in_\Lambda(f_i) \\ &= in_\Lambda(g) - Q(U)\sum_{i=1}^{m} P_i(Y)F_i(Y), \end{aligned}$$

where

$$Q(U) = \sum_{\substack{A \\ |B_{\max}|+\Lambda(A)=v_\Lambda(g)}} \overline{C_{A,B_{\max}}}U^A \in k[[U]].$$

Hence, by (6.9), we get $B_{\max}(g_1, \Lambda) < B_{\max}(g, \Lambda)$, which proves Lemma 6.12 (3). Finally,

$$v_L(g_1) \geq \min\{v_L(g), v_L(\lambda_i f_i)\ (1 \leq i \leq m)\} \geq v_L(g),$$

because $v_L(\lambda_i f_i) \geq v_L(g)$. This proves Lemma 6.12 (2) and the proof of the lemma is complete. $\square$

We now proceed with the proof of Proposition 6.11. From $g = g_0$ we construct g_1 as in Lemma 6.12, and by applying Lemma 6.12 repeatedly, we get a sequence $g_0, g_1, g_2, \ldots, g_\ell, \ldots$ in J such that for all $\ell \geq 1$ we have

$$g_\ell = g_{\ell-1} - \sum_{i=1}^{m} \lambda_{\ell,i} f_i \quad \text{with} \quad \lambda_{\ell,i} \in R,$$

$$v_\Lambda(\lambda_{\ell,i} f_i) \geq v_\Lambda(g_{\ell-1}) \geq v_\Lambda(g) \text{ and } v_L(\lambda_{\ell,i} f_i) \geq v_L(g_{\ell-1}) \geq v_L(g),$$

$$v_L(g_\ell) \geq v_L(g_{\ell-1}) \text{ and } v_\Lambda(g_\ell) \geq v_\Lambda(g_{\ell-1}),$$

$$B_{\max}(g_\ell, \Lambda) < B_{\max}(g_{\ell-1}, \Lambda) \text{ if } v_\Lambda(g_\ell) = v_\Lambda(g_{\ell-1}).$$

Then we have

$$g_\ell = g - \sum_{i=1}^{m} \mu_{\ell,i} f_i \quad \text{with } \mu_{\ell,i} = \sum_{q=1}^{\ell} \lambda_{q,i},$$

(6.10)
$$v_\Lambda(\mu_{\ell,i} f_i) \geq \min_{1 \leq q \leq \ell} \{v_\Lambda(\lambda_{qi} f_i)\} \geq v_\Lambda(g),$$
$$v_L(\mu_{\ell,i} f_i) \geq \min_{1 \leq q \leq \ell} \{v_L(\lambda_{qi} f_i)\} \geq v_L(g).$$

Note that $B_{\max}(g_\ell, \Lambda)$ cannot drop forever in the lexicographic order so that we must have $v_\Lambda(g_\ell) \neq v_\Lambda(g_{\ell-1})$ for infinitely many ℓ. Noting $v_\Lambda(R)$ is a discrete subset of $\mathbb{R}$, this implies that for given $M \geq 0$, taking ℓ sufficiently large,

$$v_\Lambda(g_\ell) = v_\Lambda(g - \sum_{i=1}^{m} \mu_{\ell,i} f_i) > M.$$

This shows the first assertion of Proposition 6.11 in view of (6.10). It implies by (4) that for any $g \in J$, there exist $\lambda_1, \ldots, \lambda_m \in \hat{R}$ such that

$$v_L(\lambda_i f_i) \geq v_L(g), \quad v_L(g - \sum_{i=1}^{m} \lambda_i f_i) > v_L(g),$$

which implies, by Lemma 6.4 (2), that $in_L(g) = \sum\limits_{i=1}^{m} in_L(\lambda_i) in_L(f_i)$, where the sum ranges over all i for which $v_L(\lambda_i f_i) = v_L(g)$. This shows $\widehat{In}_L(J) = \langle in_L(f_1), \ldots, in_L(f_m) \rangle$, which also implies the last assertion of Proposition 6.11 by the faithful flatness of $k[[U]][Y]$ over $k[U, Y]$. This completes the proof of Proposition 6.11 $\square$

Lemma 6.13 *Let u be as in Setup A and assume that (u) is admissible for J (cf. Definition 6.1). Choose $y = (y_1, \ldots, y_r)$ such that (y, u) is strictly admissible for J.*

(1) If $f = (f_1, \ldots, f_m)$ is a (u)-effective base of J, then

$$In_{\mathfrak{m}}(J) = \langle in_0(f_1), \ldots, in_0(f_m) \rangle \quad and \quad J = (f_1, \ldots, f_m).$$

Here the equality makes sense via the isomorphism $gr_{\mathfrak{m}}(R) \simeq k[Y, U]$ induced by the above choice of (y, u).

(2) If $f = (f_1, \ldots, f_m)$ is a (u)-standard base, then $(in_0(f_1), \ldots, in_0(f_m))$ is a standard base of $In_{\mathfrak{m}}(J)$ and $\nu^*(J) = (n_{(u)}(f_1), \ldots, n_{(u)}(f_m), \infty, \infty, \ldots)$.

Proof (2) follows at once from (1). The second assertion of (1) follows from the first in view of [H3], (2.21.d). We now show the first assertion of (1). Take a positive linear form Λ which is effective for (f, y, u). By Theorem 6.9, (y, Λ) is a reference datum for f. By Definition 6.7 and Lemma 6.6 (1) this implies $v_\Lambda(f_i) = n_{(u)}(f_i) < \infty$ for $i = 1, \ldots, m$ and

$$In_\Lambda(J) = \langle F_1(Y), \ldots, F_m(Y) \rangle \quad \text{with } F_i(Y) = in_\Lambda(f_i) = in_0(f_i) \in k[Y].$$

It suffices to show $In_\Lambda(J) = In_{\mathfrak{m}}(J)$. By the strict admissibility of (y, u), Lemma 1.22 implies that there exists a standard base $g = (g_1, \ldots, g_s)$ which is admissible for (y, u) (cf. Definition 6.1). By Lemma 6.8 (2) this implies $v_{\mathfrak{m}}(g_i) = v_{L_0}(g_i) = n_{(u)}(g_i) < \infty$ for $i = 1, \ldots, s$ and

$$In_{\mathfrak{m}}(J) = In_{L_0}(J) = \langle G_1(Y), \ldots, G_s(Y) \rangle \quad \text{with } G_i(Y) = in_{L_0}(g_i) \in k[Y].$$

Take a positive linear form L such that $L \geq \Lambda$ and $L \geq L_0$. Then Proposition 6.11 and Lemma 6.6 (3) imply $In_\Lambda(J) = In_L(J) = In_{L_0}(J) = In_{\mathfrak{m}}(J)$. This completes the proof.
$\square$

7 Characteristic polyhedra of $J \subset R$

In this section we are always in Setup A (beginning of §6). We introduce a polyhedron $\Delta(J, u)$ which plays a crucial role in this paper. It will provide us with useful invariants of singularities of $\mathrm{Spec}(R/J)$ (see §10). It also give us a natural way to transform a (u)-standard base of J into a standard base of J (see Corollary 7.26).

Definition 7.1 *(1) An F-subset $\Delta \subseteq \mathbb{R}^e_{\geq 0}$ is a closed convex subset of $\mathbb{R}^e_{\geq 0}$ such that $v \in \Delta$ implies $v + \mathbb{R}^e_{\geq 0} \subseteq \Delta$. The essential boundary $\partial \Delta$ of an F-subset Δ is the subset of Δ consisting of those $v \in \Delta$ such that $v \notin v' + \mathbb{R}^e_{\geq 0}$ with $v' \in \Delta$ unless $v = v'$. We write $\Delta^+ = \Delta - \partial \Delta$.*

(2) For a semi-positive positive linear form $L : \mathbb{R}^e \to \mathbb{R}$, put

$$\delta_L(\Delta) = \min\{L(v) \mid v \in \Delta\}.$$

Then $E_L = \Delta \cap \{v \in \mathbb{R}^e \mid L(v) = \delta_L(\Delta)\}$ is called a face of Δ with slope L. One easily sees that E_L is bounded if and only if L is positive. If E_L consists of a unique point v, we call v a vertex of Δ.

(3) When $L = L_0$ as in Remark 6.3, we call $\delta(\Delta) = \delta_{L_0}(\Delta) = \min\{a_1 + \ldots + a_e \mid (a_1, \ldots, a_e) \in \Delta\}$ the δ-invariant of Δ and E_{L_0} the δ-face of Δ.

Definition 7.2 *Let (y, u) be as in Setup A in §6. Let $f \in \mathfrak{m}$ be not contained in $\langle u \rangle \subset R$. Write as in (6.3):*

$$f = \sum_{(A,B)} C_{A,B} \, y^B u^A \quad \text{with} \quad C_{A,B} \in R^\times \cup \{0\}.$$

(1) The polyhedron

$$\Delta(f, y, u) \subseteq \mathbb{R}^e_{\geq 0}$$

is defined as the smallest F-subset containing all points of

$$\left\{ v = \frac{A}{n_{(u)}(f) - |B|} \mid C_{A,B} \neq 0, |B| < n_{(u)}(f) \right\}.$$

This is in fact a polyhedron in $\mathbb{R}^e_{\geq 0}$, which depends only on f, y, u, and does not depend on the presentation (6.3).

(2) For $v \in \mathbb{R}^e - \Delta(f, y, u)^+$, the v-initial of f is defined as

$$in_v(f) = in_v(f)_{(y,u)} = in_0(f) + in_v(f)^+ \in k[Y, U],$$

where writing as (6.3),

$$in_v(f)^+ = in_v(f)^+_{(y,u)} = \sum_{(A,B)} \overline{C}_{A,B} \, Y^B U^A \in k[Y, U]$$

where the sum ranges over such (A, B) that $|B| < n_{(u)}(f)$ and $\dfrac{A}{n_{(u)}(f) - |B|} = v$.

(3) *For a semi-positive linear form $L : \mathbb{R}^e \to \mathbb{R}$, we write $\delta_L(f, y, u) = \delta_L(\Delta(f, y, u))$. By definition*

$$\delta_L(f, y, u) = \min \left\{ \frac{L(A)}{n_{(u)}(f) - |B|} \mid C_{A,B} \neq 0, \ |B| < n_{(u)}(f) \right\}.$$

and $E_L = \Delta(f, y, u) \cap \{A \in \mathbb{R}^e \mid L(A) = \delta_L(f, y, u)\}$ is a face of $\Delta(f, y, u)$ of slope L. When E_L is the δ-face of $\Delta(f, y, u)$ (namely $L = L_0$ as in Definition 7.1 (3)), we write simply $\delta(f, y, u) = \delta_L(f, y, u)$.

(4) *Let E_L be as in (3). We define the E_L-initial of f by*

$$in_{E_L}(f) = in_{E_L}(f)_{(y,u)} = in_0(f) + \sum_{(A,B)} \overline{C}_{A,B} \, Y^B U^A \in k[[U]][Y]$$

where the sum ranges over such (A, B) that

$$|B| < n_{(u)}(f) \quad and \quad L(A) = \delta_L(f, y, u)(n_{(u)}(f) - |B|).$$

We note that $in_{E_L}(f)$ is different from $in_L(f)$ in Definition 6.2 (2). When E_L is the δ-face of $\Delta(f, y, u)$, we write $in_\delta(f)$ for $in_{E_L}(f)$.

One easily sees the following:

Lemma 7.3 *Let the notation be as in Definition 7.2.*

(1) *$in_v(f)_{(y,u)}$ is independent of the presentation (6.3).*

(2) *If E_L is bounded, $in_{E_L}(f) \in k[U, Y]$ and it is independent of the presentation (6.3). Otherwise it may depend on (6.3) (so there is an abuse of notation).*

(3) *If E_L is bounded,*
$$in_{E_L}(f) = in_0(f) + \sum_{v \in E_L} in_v(f)^+.$$

(4) *$in_v(f) = in_0(f)$ if $v \notin \Delta(f, y, u)$ and $in_v(f) \neq in_0(f)$ if v is a vertex of $\Delta(f, y, u)$.*

Lemma 7.4 (1) *$\delta(f, y, u) \geq 1$ if and only if $n_{(u)}(f) = v_{\mathfrak{m}}(f)$.*

(2) *$\delta(f, y, u) = 1$ if and only if $in_{\mathfrak{m}}(f) = in_\delta(f)$.*

(3) *$\delta(f, y, u) > 1$ if and only if $in_{\mathfrak{m}}(f) = in_0(f) \in k[Y]$.*

Proof By definition $\delta(f, y, u) \geq 1$ is equivalent to the condition:

$$C_{A,B} \neq 0 \text{ and } |B| < n_{(u)}(f) \Rightarrow |A| + |B| \geq n_{(u)}(f),$$

which is equivalent to

$$|A| + |B| < n_{(u)}(f) \Rightarrow C_{A,B} = 0 \text{ or } |B| = n_{(u)}(f).$$

Lemma 7.4 (1) follows easily from this. (2) and (3) follow by a similar argument and the details are omitted. $\square$

Definition 7.5 *Let* $f = (f_1, \ldots, f_m) \subset \mathfrak{m}$ *be a system of elements such that* $f_i \notin \langle u \rangle$.

(1) Define the polyhedron

$$\Delta((f_1, \ldots, f_m), y, u) = \Delta(f, y, u) \subseteq \mathbb{R}^e_{\geq 0}$$

as the smallest F-subset containing $\underset{1 \leq i \leq m}{\cup} \Delta(f_i, y, u)$.

(2) For $v \in \mathbb{R}^e - \Delta(f, y, u)^+$, *the v-initial of f is defined as*

$$in_v(f) = (in_v(f_1), \ldots, in_v(f_m))$$

by noting

$$\Delta(f, y, u)^+ \supset \underset{1 \leq i \leq m}{\cup} \Delta(f_i, y, u)^+,$$

The E_L-initial $in_{E_L}(f)$ of f for a face E_L of $\Delta(f, y, u)$ is defined similarly.

(3) For a semi-positive linear form $L : \mathbb{R}^e \to \mathbb{R}$, we put

$$\delta_L(f, y, u) = \min\{\delta_L(f_i, y, u) | \ 1 \leq i \leq m\}.$$

(4) We let $V(f, y, u)$ denote the set of vertices of $\Delta(f, y, u)$. We put

$$\widetilde{V}(f, y, u) = \{v \in \mathbb{R}^e - \Delta(f, y, u)^+ | \ in_v(f_i) \neq in_0(f_i) \text{ for some } 1 \leq i \leq m\}.$$

We call it the set of the essential points of $\Delta(f, y, u)$. By definition $\Delta(f, y, u)$ is the smallest F-subset of $\mathbb{R}^e$ which contains $\widetilde{V}(f, y, u)$. By Lemma 7.3 we have

$$V(f, y, u) \subset \widetilde{V}(f, y, u) \subset \partial \Delta(f, y, u).$$

The following fact is easily seen:

Lemma 7.6 *We have*

$$\widetilde{V}(f, y, u) \subset \frac{1}{d!}\mathbb{Z}^e_{\geq 0} \subseteq \mathbb{R}^e \quad \text{with } d = \max\{n_{(u)}(f_i) | \ 1 \leq i \leq m\}.$$

In particular $\widetilde{V}(f, y, u)$ is a finite set.

Theorem 7.7 *Let the assumption be as in Definition 7.5. The following conditions are equivalent:*

(1) f is a (u)-standard base of J and $\delta(f, y, u) > 1$.

(2) f is a standard base of J and $in_{\mathfrak{m}}(f_i) \in k[Y]$ for $\forall i$.

If (u) is admissible for J, the conditions imply that (y, u) is strictly admissible for J.

Proof The implication $(2) \Rightarrow (1)$ follows from Lemma 7.4 in view of Remark 6.3. We show $(1) \Rightarrow (2)$. When $\delta(f, y, u) > 1$, $L = L_0$ is effective for (f, y, u) by Lemma 7.4, where $L_0(A) = |A|$ (cf. Remark 6.3). Thus the desired assertion follows from Theorem 6.9. Assume that (u) is admissible for J. By Lemma 6.13, the conditions imply that $In_{\mathfrak{m}}(J)$ is generated by polynomials in $k[Y]$. Thus we must have $IDir(R/J) = \langle Y_1, \ldots, Y_r \rangle$ by the assumption that (u) is admissible for J. This proves the last assertion. $\square$

Definition 7.8 *Let the assumption be as in Setup A in §6. The polyhedron $\Delta(J, u)$ is the intersection of all $\Delta(f, y, u)$ where $f = (f_1, \ldots, f_m)$ is a (u)-standard basis with reference datum (y, L) for some y and L.*

Remark 7.9 *(1) This is not the original definition given in [H3], (1.12) (which is formulated more intrinsically), but it follows from [H3], (4.8) that the definitions are equivalent.*

(2) As a polyhedron, $\Delta(f, y, u)$ and $\Delta(J, u)$ are defined by equations

$$L_1(A) \geq d_1, \ldots, L_t(A) \geq d_t$$

for different non-zero semi-positive linear forms L_i on $\mathbb{R}^e$.

Another important result of Hironaka provides a certain condition under which we have $\Delta(J, u) = \Delta(f, y, u)$ (see Theorem 7.16). First we introduce the notion of normalizedness.

Definition 7.10 *Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k and $I \subset S$ be a homogeneous ideal. We define:*

$$E(I) = \{LE(\varphi) \in \mathbb{Z}_{\geq 0}^n \mid \varphi \in I \text{ homogeneous}\},$$

where for a homogeneous polynomial $\varphi \in S$, $LE(\varphi)$ is its leading exponent, i.e., the biggest exponent (in the lexicographic order on $\mathbb{Z}_{\geq 0}^n$) occurring in φ: For $\varphi = \sum c_A X^A$ we have $LE(\varphi) = \max\{A \mid c_A \neq 0\}$. If I is generated by homogeneous elements $\varphi_1, \ldots, \varphi_m$, we also write $E(I) = E(\varphi_1, \ldots, \varphi_m)$. We note $E(I) + \mathbb{Z}_{\geq 0}^n \subset E(I)$.

Definition 7.11 *Assume given $G_1, \ldots, G_m \in k[[U]][Y] = k[[U_1, \ldots, U_e]][Y]$:*

$$G_i = F_i(Y) + \sum_{|B| < n_i} Y^B P_{i,B}(U), \quad (P_{i,B}(U) \in k[U])$$

where $F_i(Y) \in k[Y]$ is homogeneous of degree n_i and $P_{i,B}(U) \notin k - \{0\}$.

(1) $(F_1, \ldots, F_m)$ is normalized if writing

$$F_i(Y) = \sum_B C_{i,B} Y^B \quad \text{with } C_{i,B} \in k,$$

$C_{i,B} = 0$ *if $B \in E(F_1, \ldots, F_{i-1})$ for $i = 1, \ldots, m$.*

(2) $(G_1, \ldots, G_m)$ is normalized if $(F_1, \ldots, F_m)$ is normalized and $P_{i,B}(U) \equiv 0$ if $B \in E(F_1, \ldots, F_{i-1})$ for $i = 1, \ldots, m$.

It is easy to see that if $(F_1, \ldots, F_m)$ is normalized, then it is weakly normalized in the sense of Definition 1.3. There is a way to transform a weakly normalized standard base of a homogeneous ideal $I \subset k[Y]$ into a normalized standard base of I (cf. [H3], Lemma 3.14 and Theorem 7.19 below).

Definition 7.12 *Let the assumption be as in Definition 7.5.*

(1) (f, y, u) is weakly normalized if $(in_0(f_1), \ldots, in_0(f_m))$ is weakly normalized.

(2) (f, y, u) is 0-normalized if $(in_0(f_1), \ldots, in_0(f_m))$ is normalized in the sense of Definition 7.11 (1).

(3) (f, y, u) is normalized at $v \in \mathbb{R}^e - \Delta(f, y, u)^+$ if so is $(in_v(f_1), \ldots, in_v(f_m))$ in the sense of Definition 7.11 (2).

(4) (f, y, u) is normalized along a face E_L of $\Delta(f, y, u)$ if so is $(in_{E_L}(f_1), \ldots, in_{E_L}(f_m))$ in the sense of Definition 7.11 (2).

Now we introduce the notions of (non-) solvability and preparedness.

Definition 7.13 *Let the assumption be as Definition 7.5. For $v \in V(f, y, u)$, (f, y, u) is called solvable at v if there are $\lambda_1, \ldots, \lambda_r \in k[U]$ such that*

$$ in_v(f_i)_{(y,u)} = F_i(Y + \lambda) \quad \text{with } F_i(Y) = in_0(f_i)_{(y,u)} \quad (i = 1, \ldots, m) , $$

where $Y + \lambda = (Y_1 + \lambda_1, \ldots, Y_r + \lambda_r)$. In this case the tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ is called a solution for (f, y, u) at v.

Remark 7.14 *For $v \in V(f, y, u)$, it is not possible that $in_v(f_i) \in k[Y]$ for all $i = 1, \ldots, m$ (cf. Definition 7.5 (4)); hence $\lambda \neq 0$ if v is solvable.*

Definition 7.15 *Let the assumption be as Definition 7.5.*

(1) *Call (f, y, u) prepared at $v \in V(f, y, u)$ if (f, y, u) is normalized at v and not solvable at v.*

(2) *Call (f, y, u) prepared along a face E_L of $\Delta(f, y, u)$ if (f, y, u) is normalized along E_L and not solvable at any $v \in V(f, y, u) \cap E_L$.*

(3) *Call (f, y, u) δ-prepared if it is prepared along the δ-face of $\Delta(f, y, u)$.*

(4) *Call (f, y, u) well prepared if it is prepared at any $v \in V(f, y, u)$.*

(5) *Call (f, y, u) totally prepared if it is well prepared and normalized along all bounded faces of $\Delta(f, y, u)$.*

We can now state Hironaka's crucial result (cf. [H3], (4.8)).

Theorem 7.16 *Let the assumption be as in Definition 7.5. Assume that f is a (u)-standard base of J and the following condition holds, where $\tilde{R} = R/\langle u \rangle$, $\tilde{\mathfrak{m}} = \mathfrak{m}/\langle u \rangle$, $\tilde{J} = J\tilde{R}$:*

(*) *There is no proper k-subspace $T \subsetneq gr_{\tilde{\mathfrak{m}}}^1(\tilde{R})$ such that*

$$ (In_{\tilde{\mathfrak{m}}}(\tilde{J}) \cap k[T]) \cdot gr_{\tilde{\mathfrak{m}}}(\tilde{R}) = In_{\tilde{\mathfrak{m}}}(\tilde{J}). $$

Let v be a vertex of $\Delta(f, y, u)$ such that (f, y, u) is prepared at v. Then v is a vertex of $\Delta(J, u)$. In particular, if (f, y, u) is well-prepared, then $\Delta(J, u) = \Delta(f, y, u)$.

We note that condition $(*)$ is satisfied if (u) is admissible (Definition 6.1).

Corollary 7.17 *Let the assumption be as Theorem 7.16. Assume further that (u) is admissible for J. Then the following conditions are equivalent:*

(1) (f, y, u) is prepared at any $v \in V(f, y, u)$ lying in $\{A \in \mathbb{R}^e | \ |A| \leq 1\}$.

(2) $\delta(f, y, u) > 1$.

(3) (y, u) is strictly admissible for J and f is a standard base of J admissible for (y, u).

The above conditions hold if (f, y, u) is δ-prepared.

Proof Clearly (2) implies (1). The equivalence of (2) and (3) follows from Theorem 7.7. It remains to show that (1) implies (2). By Lemma 1.22 we can find a strictly admissible (z, u) and a standard base g of J admissible for (z, u). By Lemma 7.4 we have $\delta(g, z, u) > 1$ and hence

$$\Delta(J, u) \subset \Delta(g, z, u) \subset \{A \in \mathbb{R}^e | \ |A| > 1\}.$$

By Theorem 7.16 this implies $\delta(f, y, u) > 1$ since (f, y, u) is well-prepared at any vertex v with $|v| \leq 1$. Finally, if (f, y, u) is δ-prepared, Theorem 7.16 implies $\delta(f, y, u) = \delta(\Delta(J, u))$ and the same argument as above shows $\delta(f, y, u) > 1$. This completes the proof. $\square$

We have the following refinement of Theorem 2.2 (2)(iv).

Theorem 7.18 *Let the assumption be as in Definition 7.5. Assume that (u) is admissible for J and that (f) is a (u)-standard base of J. Let $X = \mathrm{Spec}(R/J)$ and $D = \mathrm{Spec}(R/\mathfrak{p})$ for $\mathfrak{p} = (y, u_1, \ldots, u_s) \subset \mathfrak{m} = (y, u_1, \ldots, u_e)$. Assume that $D \subset X$ is permissible and that there exists a vertex v on the face E_L such that (f, y, u) is prepared at v, where*

$$L : \mathbb{R}^e \to \mathbb{R}; \ (a_1, \ldots, a_e) \to \sum_{1 \leq i \leq s} a_i,$$

Then $v_{\mathfrak{p}}(f_i) = v_{\mathfrak{m}}(f_i) = n_{(u)}(f_i)$ for $i = 1, \ldots, m$. In particular we have $\delta(f, y, u) \geq 1$.

Proof of Theorem 7.18 By Theorem 7.16, the last assumption implies

$$(7.1) \qquad\qquad \delta_L(f, y, u) = \delta_L(\Delta(J, u)).$$

Let $n_i = n_{(u)}(f_i) = \nu^{(i)}(J)$ for $i = 1, \ldots, m$. By Lemma 6.13 (2) and (1.3) we have

$$v_{\mathfrak{p}}(f_i) \leq v_{\mathfrak{m}}(f_i) \leq n_{(u)}(f_i) = n_i \quad \text{for } i = 1, \ldots, m.$$

Thus it suffices to show that (7.1) implies $v_{\mathfrak{p}}(f_i) \geq n_i$. Let $g = (g_1, \ldots, g_m)$ be a (u)-standard base of J. As usual write

$$g_i = \sum_{i, A, B} C_{i, A, B} \, y^B u^A \quad \text{with} \quad C_{i, A, B} \in R^\times \cup \{0\}, \ A \in \mathbb{Z}_{\geq 0}^e, \ B \in \mathbb{Z}_{\geq 0}^r.$$

Note that $n_{(u)}(g_i) = n_{(u)}(f_i) = n_i$ by Lemma 6.13 (2). We have

$$v_{\mathfrak{p}}(g_i) \geq n_i \Leftrightarrow |B| + L(A) \geq n_i \text{ if } C_{i, A, B} \neq 0 \text{ and } |B| < n_i$$
$$(7.2) \qquad\qquad \Leftrightarrow L(\frac{A}{n_i - |B|}) \geq 1 \text{ if } C_{i, A, B} \neq 0 \text{ and } |B| < n_i.$$

Hence we get the following equivalences for a (u)-standard base g of J:

$$(7.3) \qquad \delta_L(g, y, u) \geq 1 \Leftrightarrow v_{\mathfrak{p}}(g_i) \geq \nu^{(i)}(J) \quad \text{for } i = 1, \ldots, m$$
$$\Leftrightarrow v_{\mathfrak{p}}(g_i) = v_{\mathfrak{m}}(g_i) = n_{(u)}(g_i) = \nu^{(i)}(J) \quad \text{for } i = 1, \ldots, m$$

Noting that any standard base is a (u)-standard base by Lemma 6.8 (2) (here we used the assumption (1)), Theorem 2.2 (2)(iv) implies that there exists a (u)-standard base g which satisfies the conditions of (7.3). Since $\Delta(g, y, u) \supset \Delta(J, u)$, this implies $\delta_L(f, y, u) \geq \delta_L(g, y, u) \geq 1$, which implies the desired assertion by (7.3). Finally the last assertion follows from Lemma 7.4. $\square$

We can not expect to get the desirable situation of Theorem 7.16 right away. So we need procedures to attain this situation. This is given by the following results (cf. [H3], (3.10), (3.14) and (3.15)). First we discuss **normalizations**.

Theorem 7.19 *Let the assumption be as in Definition 7.5. Assume that (f, y, u) is weakly normalized. For $v \in V(f, y, u)$, there exist $x_{ij} \in \langle u \rangle \subset R$ $(1 \leq j < i \leq m)$ such that the following hold for*

$$h = (h_1, \ldots, h_m), \qquad \text{where } h_i = f_i - \sum_{j=1}^{i-1} x_{ij} f_j.$$

(i) $\Delta(h, y, u) \subseteq \Delta(f, y, u)$.

(ii) *If $v \in \Delta(h, y, u)$, then $v \in V(h, y, u)$ and (h, u, y) is normalized at v.*

(iii) $V(f, y, u) - \{v\} \subset V(h, y, u)$.

(iv) *For $v' \in V(f, y, u) - \{v\}$, we have $in_{v'}(f)_{(y,u)} = in_{v'}(h)_{(y,u)}$.*

Remark 7.20 *In the above situation, the passage from (f, y, u) to (h, y, u) is called a normalization at v. It is easy to see that $in_0(h_i) = in_0(f_i)$ for all i, $1 \leq i \leq m$.*

We will need the following slight generalization of Theorem 7.19:

Theorem 7.21 *Let the assumption be as in Definition 7.5. Let E be a bounded face of $\Delta(f, y, u)$. Assume that (f, y, u) is normalized at any $v \in E \cap V(f, y, u)$. Then there exist $x_{ij} \in (u)R$ $(1 \leq j < i \leq m)$ such that putting*

$$h_i = f_i - \sum_{j=1}^{i-1} x_{ij} f_j \quad \text{for } 1 \leq i \leq m,$$

$\Delta(h, y, u) = \Delta(f, y, u)$ *and (h, y, u) is normalized along E.*

Proof Write $E = \Delta(f, y, u) \cap \{A \in \mathbb{R}^e | L(A) = 1\}$ for a positive linear form $L : \mathbb{R}^e \to \mathbb{R}$. For $i = 1, \ldots, m$, we can write

$$in_E(f_i) = F_i(Y) + \sum_{|B| < n_i} Y^B P_{i,B}(U) \in k[Y, U], \quad P_{i,B}(U) = \sum_{|B| + L(A) = n_i} c_{i,A,B} U^A \in k[U],$$

100

where $n_i = n_{(u)}(f_i) = v_L(f_i)$ and $F_i(Y) = in_0(f_i) \in k[Y]$ which is homogeneous of degree n_i. For each $i \geq 1$ put

$$\Sigma(f_1, \ldots, f_i) = \{B | \; P_{i,B}(U) \not\equiv 0, \; B \in E(F_1, \ldots, F_{i-1})\}.$$

If $\Sigma(f_1, \ldots, f_i) = \emptyset$ for all $i \geq 1$, there is nothing to be done. Assume the contrary and let $j = \min\{i | \; \Sigma(f_1, \ldots, f_i) \neq \emptyset\}$ and $B_{\max}$ be the maximal element of $\Sigma(f_1, \ldots, f_j)$ with respect to the lexicographic order. Note that

(7.4)
$$\{\frac{A}{n_j - |B_{\max}|} \; | \; c_{j,A,B_{\max}} \neq 0\} \subset E_L \backslash V(f, y, u)$$

by the assumption that (f, y, u) is normalized at any $v \in E_L \cap V(f, y, u)$. By the construction there exist $G_i(Y) \in k[Y]$, homogeneous of degree $|B_{\max}| - n_i$, for $1 \leq i \leq j - 1$ such that

$$H(Y) := Y^{B_{\max}} - \sum_{1 \leq i \leq j-1} G_i(Y) F_i(Y)$$

has exponents smaller than $B_{\max}$. Note that $H(Y)$ is homogeneous of degree $|B_{\max}|$. For $i = 1, \ldots, j - 1$, take $g_i \in R$ such that $in_{\mathfrak{m}}(g_i) = G_i(Y)$ and take

$$\tilde{P}_{j,B_{\max}} = \sum_{|B_{\max}|+L(A)=n_i} \tilde{c}_{j,A,B_{\max}} u^A \in R \quad \text{with } c_{j,A,B_{\max}} = \tilde{c}_{j,A,B_{\max}} \quad \text{mod } \mathfrak{m}.$$

Put

$$h_j = f_j - \tilde{P}_{j,B_{\max}} \sum_{1 \leq i \leq j-1} g_i f_i \quad \text{and} \quad h_i = f_i \quad \text{for } 1 \leq i \neq j \leq m.$$

By (7.4), we have

$$in_E(h_j) = in_E(f_j) - P_{j,B_{\max}}(U)(Y^{B_{\max}} - H(Y)),$$
$$in_v(h_j) = in_v(f_j) \quad \text{for } \forall v \in V(f, y, u).$$

Hence $\Sigma(h_1, \ldots, h_i) = \Sigma(f_1, \ldots, f_i)$ for all $i = 1, \ldots, j-1$ and all elements of $\Sigma(h_1, \ldots, h_j)$ are smaller than $B_{\max}$ in the lexicographical order. This proves Theorem 7.21 by induction. $\square$

Now we discuss **dissolutions**.

Theorem 7.22 *Let the assumption be as in Definition 7.5 and let $v \in V(f, y, u)$.*

(a) *Any solution for (f, y, u) at v is of the form λ with $\lambda_i = c_i U^v$, where $c_i \in k^\times$. In particular, if it exists, a solution is always non-trivial and $v \in \mathbb{Z}^e$.*

(b) *Let $d = (d_1, \ldots, d_r) \subset R$ with $d_i \in \langle u^v \rangle$ be such that the image of d_i in $\mathrm{gr}_{\mathfrak{m}}^{|v|}(R)$ is λ_i. Let $z = y - d = (y_1 - d_1, \ldots, y_r - d_r)$. Then*

  (i) $\Delta(f, z, u) \subseteq \Delta(f, y, u)$.

  (ii) $v \notin \Delta(f, z, u)$ *and* $V(f, y, u) - \{v\} \subset V(f, z, u)$.

  (iii) *For $v' \in V(f, y, u) - \{v\}$, we have*

$$in_{v'}(f)_{(z,u)} = in_{v'}(f)_{(y,u)|Y=Z} \in k[Z, U]. \quad (Z = in_{\mathfrak{m}}(z) \in \mathrm{gr}_{\mathfrak{m}}^1(R))$$

Remark 7.23 *In the above situation, the passage from (f, y, u) to (f, z, u) is called the dissolution at v. It is easy to see $in_0(f_i)_{(z,u)} = in_0(f_i)_{(y,u)|Y=Z}$ for $1 \leq \forall i \leq m$.*

We now come to the **preparation** of (f, y, u): Let the assumption be as in Definition 7.5. We apply to (f, y, u) alternately and repeatedly normalizations and dissolutions at vertices of polyhedra. To be precise we endow $\mathbb{R}^e_{\geq 0}$ with an order defined by

$$v > w \Leftrightarrow |v| > |w| \text{ or } |v| = |w| \text{ and } v > w \text{ in the lexicographical order.}$$

Let $v \in V(f, y, u)$ be the smallest point and apply the normalization at v from Theorem 7.19 and then the dissolution at v from Theorem 7.22 if v is solvable, to get (g, z, u). Then (g, z, u) is prepared at v. Repeating the process, we arrive at the following conclusion (cf. [H3], (3.17)).

Theorem 7.24 *Let the assumption be as in Definition 7.5. Assume that (f, y, u) is weakly normalized. For any integer $M > 0$, there exist*

$$x_{ij} \in \langle u \rangle \quad (1 \leq j < i \leq m), \quad d_\nu \in \langle u \rangle \quad (\nu = 1, \dots, r),$$

such that putting

$$z = (z_1, \dots, z_r) \text{ with } z_\nu = y_\nu - d_\nu,$$

$$g = (g_1, \dots, g_m) \text{ with } g_i = f_i - \sum_{j=1}^{i-1} x_{ij} f_j,$$

we have $\Delta(g, z, u) \subseteq \Delta(f, y, u)$, and (g, z, u) is prepared along all bounded faces contained in $\{A \in \mathbb{R}^e | \, |A| \leq M\}$. If R is complete, we can obtain the stronger conclusion that (g, z, u) is well-prepared.

Remark 7.25 *By Theorem 7.21 we can make (g, z, u) in Theorem 7.24 satisfy the additional condition that it is normalized along all bounded faces of the polyhedron contained in $\{A \in \mathbb{R}^e | \, |A| \leq N\}$. If R is complete, we can make (g, z, u) totally prepared.*

Corollary 7.26 *Let the assumption and notation be as in Theorem 7.24.*

(1) If f is a (u)-standard base of J, then so is g.

(2) If f is a (u)-standard base of J and (u) is admissible (cf. 6.1), then $\delta(g, z, u) > 1$ and (z, u) is strictly admissible and g is a standard base admissible for (z, u).

Proof   (1) follows from Corollary 6.10. (2) follows from (1) and Theorem 7.17. $\square$

At the end of this section we prepare a key result which relates certain localizations of our ring to certain projections for the polyhedra. Let (f, y, u) be as in Definition 7.5. For $s = 1, \dots, e$, we let

$$\mathfrak{p}_s = \langle y, u_{\leq s} \rangle = \langle y_1, \dots, y_r, u_1, \dots, u_s \rangle.$$

Let R_s be the localization of R at $\mathfrak{p}_s$, let $J_s = JR_s$, and $\mathfrak{m}_s = \mathfrak{p}_s R_s$ (the maximal ideal of R_s). We want to relate $\Delta(f, y, u) \subset \mathbb{R}^e$ to $\Delta(f, y, u_{\leq s}) \subset \mathbb{R}^s$, the characteristic polyhedron for $J_s \subset R_s$. Assume given a presentation as in (6.3):

$$(7.5) \qquad f_i = \sum_{(A,B)} P_{i,A,B}\, y^B u^A \quad \text{with} \quad P_{i,A,B} \in R^\times \cup \{0\} \quad (i = 1, \ldots, m).$$

(7.5) can be rewritten as:

$$(7.6) \qquad f_i = \sum_{(C,B)} P^{\leq s}_{i,C,B}\, y^B u^C_{\leq s},$$

where for $C = (a_1, \ldots, a_s) \in \mathbb{Z}^s_{\geq 0}$, $u^C_{\leq s} = u_1^{a_1} \cdots u_s^{a_s}$ and

$$(7.7) \qquad P^{\leq s}_{i,C,B} = \sum_{A=(a_1,\ldots,a_s,a_{s+1},\ldots,a_e)} P_{i,A,B} u_{s+1}^{a_{s+1}} \cdots u_e^{a_e} \in \hat{R}.$$

We now introduce some conditions which are naturally verified in the case where $\operatorname{Spec}(R/\langle u_1 \rangle)$ is the exceptional divisor of a blowup at a closed point (see Lemma 9.4 below). Assume

(P0) $J \subset \mathfrak{p}_1$ and there is a subfield $k_0 \subset R/\mathfrak{p}_1$ such that $P_{i,A,B} \mod \mathfrak{p}_1 \in k_0$.

By (P0) we get the following for $C = (a_1, \ldots, a_s) \in \mathbb{Z}^s_{\geq 0}$:

$$\overline{P}^{\leq s}_{i,C,B} := P^{\leq s}_{i,C,B} \mod \mathfrak{p}_s = \sum_{A=(a_1,\ldots,a_s,a_{s+1},\ldots,a_e)} \overline{P}_{i,A,B} \overline{u}_{s+1}^{a_{s+1}} \cdots \overline{u}_e^{a_e} \in k_0[[\overline{u}_{s+1}, \ldots, \overline{u}_e]] \subset \hat{R}/\mathfrak{p}_s,$$

$$\overline{P}_{i,C,B} := P_{i,A,B} \mod \mathfrak{p}_s \in k_0 \hookrightarrow R/\mathfrak{p}_s, \quad \overline{u}_j = u_j \mod \mathfrak{p}_s \in R/\mathfrak{p}_s.$$

Hence we have the following equivalences for $C \in \mathbb{Z}^s_{\geq 0}$

$$(P1) \qquad \begin{aligned} P^{\leq s}_{i,C,B} \in \mathfrak{p}_s &\Leftrightarrow P_{i,A,B} = 0 \quad \text{for all } A \in \mathbb{Z}^e_{\geq 0} \text{ such that } \pi_s(A) = C \\ &\Leftrightarrow P^{\leq s}_{i,C,B} = 0 \end{aligned}$$

where $\pi_s : \mathbb{R}^e \to \mathbb{R}^s$; $(a_1, \ldots, a_e) \to (a_1, \ldots, a_s)$. We further assume

(P2) For fixed B and $a \in \mathbb{Z}_{\geq 0}$, there are only finitely many A such that $P_{i,A,B} \neq 0$ and $\pi_1(A) = a$.

This condition implies $P^{\leq s}_{i,C,B} \in R$.

Theorem 7.27 *Let $L : \mathbb{R}^s \to \mathbb{R}$ be a semi-positive linear form and $L_s = L \circ \pi_s$.*

  (1) If (P0) holds, then

$$\Delta(f, y, u_{\leq s}) = \pi_s(\Delta(f, y, u)) \quad \text{and} \quad \delta_L(f, y, u_{\leq s}) = \delta_{L_s}(f, y, u).$$

  (2) If (P0) and (P2) hold and $L(1, 0, \ldots, 0) \neq 0$, then the initial form along the face E_L of $\Delta(f, y, u_{\leq s})$ (with respect to the presentation (7.6), cf. Definition 7.2) lies in the polynomial ring $R/\mathfrak{p}_s[Y, U_{\leq s}]$ and we have

$$in_{E_L}(f)_{(y,u_{\leq s})} = in_{E_{L_s}}(f)_{(y,u)|U_i = \overline{u}_i \ (s+1 \leq i \leq e)},$$

considered as an equation in $k_0[\overline{u}_{s+1}, \ldots, \overline{u}_e][Y, U_{\leq s}] \subset R/\mathfrak{p}_s[Y, U_{\leq s}]$ (cf. (P2)).

(3) *Assume* (P0) *and* (P2) *and* $L(1, 0, \ldots, 0) \neq 0$. *Assume further*:

    (i) *There is no proper k-subspace $T \subset \bigoplus\limits_{1 \leq i \leq r} k \cdot Y_i$ ($k = R/\mathfrak{m}$) such that*

$$F_i(Y) := in_0(f_i) \in k[T] \subset k[Y] \quad \text{for all } j = 1, \ldots, m.$$

    (ii) (f, y, u) *is prepared along the face E_{L_s} of $\Delta(f, y, u)$.*

    Then $(f, y, u_{\leq s})$ is prepared along the face E_L of $\Delta(f, y, u_{\leq s})$.

Proof By (P1) we get

$$\delta_L(f, y, u_{\leq s}) = \min\{\frac{L(C)}{n_i - |B|} \mid |B| < n_i, \ P_{i,C,B}^{\leq s} \neq 0\}$$

$$= \min\{\frac{L_s(A)}{n_i - |B|} \mid |B| < n_i, \ P_{i,A,B} \neq 0\} = \delta_{L_s}(f, y, u)$$

and (1) follows from this. To show (2), we use (7.6) to compute

$$in_{E_L}(f)_{(y, u_{\leq s})} = F_i(Y) + \sum_{B,C} \overline{P}_{i,C,B}^{\leq s} Y^B U_{\leq s}^C$$

(7.8)

$$= F_i(Y) + \sum_{B,A} \overline{P}_{i,A,B} \overline{u}_{s+1}^{a_{s+1}} \cdots \overline{u}_e^{a_e} \cdot Y^B U_{\leq s}^{\pi_s(A)} \ ,$$

where the first (resp. second) sum ranges over those B, C (resp. B, A) for which $|B| < n_i$ and $L(C) = \delta_L(f, y, u_{\leq s})(n_i - |B|)$ (resp. $|B| < n_i$ and $L_s(A) = \delta_{L_s}(f, y, u)(n_i - |B|)$). (2) follows easily from this.

We now show (3). By (2) the assumption (ii) implies that $in_{E_L}(f)_{(y, u_{\leq s})}$ is normalized. It suffices to show the following:

Claim 7.28 *Let v be a vertex on E_L. Then $(f, y, u_{\leq s})$ is not solvable at v.*

We prove the claim by descending induction on s (the case $s = e$ is obvious). From (1) we easily see that there exists a vertex $w \in \Delta(f, y, u)$ such that $\pi_s(w) = v$ and $v_t := \pi_t(w)$ is a vertex of $\Delta(f, y, , u_{\leq t})$ for all $s \leq t \leq e$. By induction hypothesis, we may assume

    (∗) $(f, y, u_{\leq s+1})$ is not solvable at v_{s+1}.

Assume $(f, y, u_{\leq s})$ solvable at v. Then $v \in \mathbb{Z}_{\geq 0}^s$ and there exist elements $\lambda_1, \ldots, \lambda_r$ of the fraction field K of $R/\mathfrak{p}_s$ such that

(7.9) $\quad in_v(f)_{(y, u_{\leq s})} = F_i(Y_1 + \lambda_1 U_{\leq s}^v, \ldots, Y_r + \lambda_r U_{\leq s}^v) \quad$ for all $i = 1, \ldots, m.$

Claim 7.29 λ_j *lies in the localization S of $R/\mathfrak{p}_s$ at $\mathfrak{p}_{s+1}/\mathfrak{p}_s$ for all $j = 1, \ldots, r$.*

Admit the claim for the moment. By the claim we can lift $\lambda_j \in S$ to $\widetilde{\lambda}_j \in R_{s+1}$, the localization of R at $\mathfrak{p}_{s+1}$. Set $z_j = y_j + \widetilde{\lambda}_j u_{\leq s}^v \in R_{s+1} \subset R_s$. Take a positive linear form $L_v : \mathbb{R}^s \to \mathbb{R}$ such that $E_{L_v} = \{v\}$ and hence $L_v(v) = \delta_{L_v}(f, y, u_{\leq s})$. By Theorem 7.22, $\Delta(f, z, u_{\leq s}) \subset \Delta(f, y, u_{\leq s}) - \{v\}$ so that

(7.10) $$\delta_{L_v}(f, z, u_{\leq s}) > \delta_{L_v}(f, y, u_{\leq s}).$$

Now we apply (1) to $J_{s+1} \subset R_{s+1}$ and $(f, z, u_{\leq s+1})$ instead of $J \subset R$ and (f, y, u). Note that in the proof of (1) we have used only $(P0)$ which carries over to the replacement. We get

$$\Delta(f, z, u_{\leq s}) = \pi\big(\Delta(f, z, u_{\leq s+1})\big) \quad \text{and} \quad \delta_{L_v \circ \pi}(f, z, u_{\leq s+1}) = \delta_{L_v}(f, z, u_{\leq s}),$$

where $\pi : \mathbb{R}^{s+1} \to \mathbb{R}^s$; $(a_1, \ldots, a_{s+1}) \to (a_1, \ldots, a_s)$. By the assumption, $v = \pi(v_{s+1})$ and

$$L_v \circ \pi(v_{s+1}) = L_v(v) = \delta_{L_v}(f, y, u_{\leq s}).$$

By Theorem 7.16, $(*)$ implies

(7.11) $$v_{s+1} \in \Delta(J_{s+1}, u_{\leq s+1}) \subset \Delta(f, z, u_{\leq s+1})).$$

Thus we get

$$\delta_{L_v}(f, z, u_{\leq s}) = \delta_{L_v \circ \pi}(f, z, u_{\leq s+1}) \leq \delta_{L_v \circ \pi}(\Delta(J_{s+1}, u_{\leq s+1})) \leq L_v \circ \pi(v_{s+1}) = \delta_{L_v}(f, y, u_{\leq s}),$$

where the inequalities follow from (7.11). This contradicts (7.10) and the proof of (3) is complete.

Now we show Claim 7.29. Note that S is a discrete valuation ring with a prime element $\pi := u_{s+1} \mod \mathfrak{p}_s$. Thus it suffices to show $v_\pi(\lambda_j) \geq 0$ for $j = 1, \ldots, r$. Assume the contrary. We may assume

$$v_\pi(\lambda_1) = -\epsilon < 0, \quad v_\pi(\lambda_1) \leq v_\pi(\lambda_j) \quad \text{for } j = 1, \ldots, r.$$

Set

$$Z_j = Y_j + \mu_j V \quad \text{where } V = U_{\leq s}^v \text{ and } \mu_j = \lambda_j \pi^\epsilon \in S,$$

and recall that $F_i(Y) = in_0(f_i) \in k_0[Y]$ and $k_0 \subset S$ by $(P0)$. Consider

$$F_i(\overline{Z}) = F_i(Y_1 + \overline{\mu}_1 V, \ldots, Y_r + \overline{\mu}_r V) \in \kappa[Y, V], \quad \kappa := S/\langle \pi \rangle = \kappa(\mathfrak{p}_{s+1}),$$

where $\overline{\mu}_i = \mu_j \mod \pi \in \kappa$. We claim that there is some i for which $F_i(\overline{Z}) \notin \kappa[Y]$. Indeed, by the structure theorem of complete local rings, $(P0)$ implies $\hat{R}/\mathfrak{p}_{s+1} \simeq k[[u_{s+2}, \ldots, u_e]]$ so that κ is contained in $k((u_{s+2}, \ldots, u_e))$ which is a separable extension of k. By $(3)(i)$ and Lemma 1.20 (2),

$$T = \bigoplus_{1 \leq j \leq r} \kappa \cdot (Y_1 + \overline{\mu}_j V) \subset \bigoplus_{1 \leq j \leq r} \kappa \cdot Y_j \oplus \kappa \cdot V$$

is the smallest κ-subspace such that $F_i(\overline{Z}) \in \kappa[T]$ for all $i = 1, \ldots, m$. Thus the claim follows from the fact that $\overline{\mu}_1 \neq 0$. For the above i, we expand

$$F_i(Y_1 + \lambda_1 V, \ldots, Y_r + \lambda_r V) = \sum_B \gamma_B Y^B V^{n_i - |B|} \quad (\gamma_B \in K),$$

and we get

$$F_i(Z) = F_i(Y_1 + \mu_i V, \ldots, Y_r + \mu_i V) = \sum_B \gamma_B \pi^{\epsilon(n_i - |B|)} \cdot Y^B V^{n_i - |B|} \in S[Y, V].$$

Since $F_i(\overline{Z}) \notin \kappa[Y]$, there is some B such that $|B| < n_i$ and $\gamma_B \pi^{\epsilon(n_i - |B|)}$ is a unit of S. Noting $\epsilon > 0$, this implies $\gamma_B \notin S$. On the other hand, (7.8) and (7.9) imply $\gamma_B \in S$, which is absurd. This completes the proof of Claim 7.29. $\square$

8   Transformation of standard bases under blow-ups

In this section we will study the transformation of a standard base under permissible blow-ups, in particular with respect to near points in the blow-up. We begin by setting up a local description of the situation in Theorem 2.14 in §2.

Setup B

Let Z be an excellent regular scheme and $X \subset Z$ be a closed subscheme and take a closed point $x \in X$. Put $R = \mathcal{O}_{Z,x}$ with the maximal ideal $\mathfrak{m}$ and put $k = R/\mathfrak{m} = k(x)$. Write $X \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/J)$ for an ideal $J \subset \mathfrak{m}$. Define the integers $n_1 \leq n_2 \leq \cdots \leq n_m$ by

$$\nu_x^*(X, Z) = \nu^*(J) = (n_1, \ldots, n_m, \infty, \infty, \ldots).$$

Let $D \subset X$ be a closed subscheme permissible at $x \in D$ and let $J \subset \mathfrak{p}$ be the prime ideal defining $D \subset Z$. By Theorem 2.2 (2) we have $T_x(D) \subset \mathrm{Dir}_x(X)$ so that we can find a system of regular parameters for R,

$$(y, u, v) = (y_1, \ldots, y_r, u_1, \ldots, u_s, v_1, \ldots, v_t)$$

such that $\mathfrak{p} = (y, u)$ and $(y, (u, v))$ is strictly admissible for J (cf. Definition 6.1). This gives us an identification

$$\mathrm{gr}_\mathfrak{m}(R) = k[Y, U, V] = k[Y_1, \ldots, Y_r, U_1, \ldots, U_s, V_1, \ldots, V_t] \quad (k = R/\mathfrak{m})$$

where $Y_i = in_\mathfrak{m}(y_i)$, $U_i = in_\mathfrak{m}(u_i)$, $V_i = in_\mathfrak{m}(v_i) \in \mathrm{gr}_\mathfrak{m}(R)$. Consider the diagram:

$$Bl_D(X) = \begin{matrix} X' & \subset & Z' & = & Bl_D(Z) \hookleftarrow \pi_Z^{-1}(x) = E_x \\ \downarrow \pi_X & & \downarrow \pi_Z & & \\ X & \subset & Z & & \end{matrix}$$

and note that

$$E_x := \pi_Z^{-1}(x) = \mathbb{P}(T_x(Z)/T_x(D)) = \mathrm{Proj}(k[Y, U]) \cong \mathbb{P}_k^{r+s-1}.$$

We fix a point

$$x' \in \mathrm{Proj}(k[U]) \subset \mathrm{Proj}(k[Y, U]) = E_x.$$

(By Theorem 2.14, if $\mathrm{char}(k(x)) = 0$ or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$, any point of X' near to x lies in $\mathrm{Proj}(k[U])$. Moreover, if x' is near to x, Theorem 2.10 implies

$$\nu_{x'}^*(X', Z') = \nu_x^*(X, Z).$$

Without loss of generality we assume further that x' lies in the chart $\{U_1 \neq 0\} \subset E_x$. Let $R' = \mathcal{O}_{Z',x'}$ with the maximal ideal $\mathfrak{m}'$ and let $J' \subset \mathfrak{m}'$ be the ideal defining $X' \subset Z'$ at x'. Put $k' = k(x') = R'/\mathfrak{m}'$. Then $\mathfrak{m}R' = (u_1, v) = (u_1, v_1, \ldots, v_t)$, and

$$(y', u_1, v), \quad \text{with } y' = (y'_1, \ldots, y'_r),\ y'_i = y_i/u_1,\ v = (v_1, \ldots, v_t)$$

is a part of a system of regular parameters for R'. Choose any $\phi_2, \ldots, \phi_{s'} \in \mathfrak{m}'$ such that (y', u_1, ϕ, v), with $\phi = (\phi_2, \ldots, \phi_{s'})$, is a system of regular parameters for R' (note $s - s' = \mathrm{trdeg}_k(k(x'))$). Then

$$\mathrm{gr}_{\mathfrak{m}'}(R') = k'[Y', U_1, \Phi, V] = k'[Y'_1, \ldots, Y'_r, U_1, \Phi_1, \ldots, \Phi_{s'}, V_1, \ldots, V_t],$$

where $Y_i' = in_{\mathfrak{m}'}(y_i')$, $\Phi_i = in_{\mathfrak{m}'}(\phi_i)$, $V_i = in_{\mathfrak{m}'}(v_i) \in \mathrm{gr}_{\mathfrak{m}'}(R')$. Assume now given

a standard base $f = (f_1, \ldots, f_m)$ of J which is admissible for $(y, (u, v))$.

By definition

$$(8.1) \quad F_i(Y) := in_{\mathfrak{m}}(f_i) \in k[Y] \ (i = 1, \ldots, m) \quad \text{and} \quad In_{\mathfrak{m}}(J) = \langle F_1(Y), \ldots, F_m(Y) \rangle.$$

By Lemma 6.13 (2) we have

$$(8.2) \qquad\qquad v_{\mathfrak{m}}(f_i) = n_{(u,v)}(f_i) = n_i \quad \text{for } i = 1, \ldots, m.$$

Finally we assume

$$(8.3) \qquad\qquad v_{\mathfrak{m}}(f_i) = v_{\mathfrak{p}}(f_i) \quad \text{for } i = 1, \ldots, m.$$

This assumption is satisfied if $D = \{x\}$ (for trivial reasons) or under the conditions of Theorem 7.18 (for example if $(f, y, (u, v))$ is well-prepared). The assumptions imply

$$(8.4) \quad f_i = \sum_{A,B} C_{i,A,B}\, y^B u^{A_u} v^{A_v}, \quad C_{A,B} \in \Gamma, \quad A = (A_u, A_v),\ A_u \in \mathbb{Z}_{\geq 0}^s,\ A_v \in \mathbb{Z}_{\geq 0}^t,$$

where the sum ranges over A_u, A_v, B such that

$$(8.5) \qquad\qquad\qquad\qquad |B| + |A_u| \geq n_i.$$

By [H1] Ch. III.2 p. 216 Lemma 6 we have $J' = \langle f_1', \ldots, f_m' \rangle$ with

$$(8.6) \qquad\qquad f_i' = f_i/u_1^{n_i} = \sum_{A,B} \left(C_{A,B}\, u'^{A_u} \right) y'^B u_1^{|A_u|+|B|-n_i} v^{A_v},$$

$$u'^{A_u} = u_2'^{a_2} \ldots u_s'^{a_s} \quad \text{for } A_u = (a_1, \ldots, a_s) \quad (u_i' = u_i/u_1).$$

This implies

$$(8.7) \qquad f_i' = \tilde{F}_i' \mod \langle u_1, v \rangle = \langle u_1, v_1, \ldots, v_t \rangle \quad \text{for } i = 1, \ldots, m, \text{ where}$$

$$\tilde{F}_i' = \sum_{|B|=n_i} C_{i,0,B}\, y'^B \quad \text{and} \quad n_i = v_{\mathfrak{m}'}(\tilde{F}_i') = n_{(u_1,\phi,v)}(f_i')$$

so that

$$(8.8) \qquad\qquad in_0(f_i')_{(y',(u_1,\phi,v))} = F_i(Y') \quad \text{for } i = 1, \ldots, m$$

For later use, we choose S (resp. S'), a ring of coefficients of $\hat{R}$ (resp. $\hat{R}'$) (cf. (6.5)). We also choose a set $\Gamma \subset S$ of representatives of k (resp. a set $\Gamma' \subset S'$ of representatives of k'). We note that the choices for R and R' are independent: We do not demand $S \subset S'$ nor $\Gamma \subset \Gamma'$.

(end of Setup B).

We want to compare the properties of $(f, y, (u, v))$ (downstairs) and $(f', y', (u_1, \phi, v))$ (upstairs), especially some properties of the polyhedra and initial forms.

Let $e = s + t$ and $e' = s' + t$. For a semi-positive linear form L on $\mathbb{R}^e$ (downstairs) (resp. on $\mathbb{R}^{e'}$ (upstairs)), v_L and $in_L(*)$ denote the L-valuation of R (resp. R') and the corresponding initial form of $* \in R$ (resp. $* \in R'$) with respect to $(y, (u, v), \Gamma)$ (resp. $(y', (u_1, \phi, v), \Gamma')$).

Theorem 8.1 *In Setup B, $f' = (f'_1, \ldots, f'_m)$ is a (u_1, ϕ, v)-effective basis of J'. If $(f_1, \ldots, f_m)$ is a standard base of J, then $(f'_1, \ldots, f'_m)$ is a (u_1, ϕ, v)-standard basis of J'. More precisely there exists a positive linear form L' on $\mathbb{R}^e$ (upstairs) such that:*

$$in_{L'}(f'_i) = F_i(Y') \quad (1 \le i \le m) \quad \text{and} \quad in_{L'}(J') = \langle F_1(Y'), \ldots, F_m(Y') \rangle.$$

First we need to show the following:

Lemma 8.2 *Let the assumption be as in Setup B. Choose $d > 1$ and consider the linear forms on $\mathbb{R}^e$ and $\mathbb{R}^{e'}$:*

$$L(A) = \frac{1}{d}\Big(\sum_{1 \le i \le s} a_i + \sum_{1 \le j \le t} a'_j \Big) \qquad \text{(downstairs)}$$

$$\Lambda'(A) = \frac{1}{d-1}\Big(a_1 + \sum_{1 \le j \le t} a'_j \Big) \qquad \text{(upstairs)}$$

respectively, where $A = (a_1, \ldots, a_*, a'_1, \ldots, a'_t)$ with $* = s$ (downstairs) and $* = s'$ (upstairs). Then the following holds for $g \in R$.

(1) $v_{\Lambda'}(g) = \frac{d}{d-1} v_L(g)$.

(2) Assuming that $g' := g/u_1^n \in R'$ with $v_L(g) = n$, we have $v_{\Lambda'}(g') = n$. Assuming further $in_L(g) = G(Y) \in k[Y]$, we have $in_{\Lambda'}(g') = G(Y')$.

Proof Let $g \in \hat{R}$ and write

$$g = \sum_{A,B} C_{A,B} \, y^B u^{A_u} v^{A_v}$$

as in (8.4). Then, in $\widehat{R'}$ we have

(8.9)
$$g = \sum_{A,B} \big(C_{A,B} \, u'^{A_u} \big) y'^B u_1^{|A_u| + |B|} v^{A_v},$$

where the notation is as in (8.6). Note that

$$C_{A,B} \, u'^{A_u} = 0 \quad \text{in} \quad \widehat{R'}/(y', u_1, v) \iff C_{A,B} = 0$$

108

because $C_{A,B} \neq 0$ implies $C_{A,B} \in R^\times$ so that $C_{A,B} \in (R')^\times$. Hence

$$v_{\Lambda'}(g) = \min\left\{|B| + \frac{|B|+|A|}{d-1} \mid C_{A,B} \neq 0\right\}$$

$$= \frac{d}{d-1} \min\left\{|B| + \frac{|A|}{d} \mid C_{A,B} \neq 0\right\} = \frac{d}{d-1} v_L(g),$$

which proves (1) of Lemma 8.2. Next assume $v_L(g) = n$ and $g' = g/u_1^n \in R'$. Then

$$v_{\Lambda'}(g') = v_{\Lambda'}(g) - v_{\Lambda'}(u_1^n) = \frac{d}{d-1} v_L(g) - \frac{n}{d-1} = n.$$

Note that

$$g' = \sum_{A,B} \left(C_{A,B}\, u'^{A_u}\right) y'^B u_1^{|A_u|+|B|-n} v^{A_v},$$

$$|B| + \frac{|A|}{d} = n \iff |B| + \frac{|A|+|B|-n}{d-1} = n.$$

Therefore, with (8.9) we see

$$in_L(g) \in k[Y] \iff C_{A,B} = 0 \quad \text{if} \quad |B| + \frac{|A|}{d} = n \quad \text{and} \quad |B| < n$$

$$\iff u'^{A_u}\, C_{A,B} = 0 \quad \text{if} \quad |B| + \frac{|A|+|B|-n}{d-1} = n \quad \text{and} \quad |B| < n$$

$$\implies in_{\Lambda'}(g') \in k'[Y'].$$

(Note that the last implication is independent of the choice of a representative Γ' of k'.) Moreover, if these conditions hold, we have

$$in_L(g) = \sum_{\substack{B \\ |B|=v_L(g)}} \overline{C_{0,B}}\, Y^B \quad \text{and} \quad in_{\Lambda'}(g') = \sum_{\substack{B \\ |B|=v_L(g)}} \overline{C_{0,B}}\, Y'^B.$$

This completes the proof of Lemma 8.2. $\square$

Proof of Theorem 8.1 By (8.1) and Lemma 7.4 we have $\delta := \delta(f, y, (u, v)) > 1$. Choose d with $1 < d < \delta$, and consider the linear forms L on $\mathbb{R}^e$ (downstairs) and Λ' on $\mathbb{R}^{e'}$ (upstairs) on $\mathbb{R}$ as in Lemma 8.2. As for the desired positive linear form in Theorem 8.1, we take

$$L'(A) = \frac{\sum a_i + \sum a'_j}{d-1} \quad \text{(upstairs)} \quad (A = ((a_i)_{1 \leq i \leq s'}, (a'_j)_{1 \leq i \leq t})).$$

By Lemma 6.6(3) and Proposition 6.11, we have $v_L(f_i) = v_{\mathfrak{m}}(f_i) = n_i$ and

(8.10) $\quad in_L(f_i) = in_{\mathfrak{m}}(f_i) = F_i(Y), \quad In_L(J) = \langle F_1(Y), \ldots, F_m(Y)\rangle.$

Clearly $L' \geq \Lambda'$, so that by Proposition 6.11 it suffices to show

(8.11) $\quad in_{\Lambda'}(f'_i) = F_i(Y') \quad \text{and} \quad \widehat{In}_{\Lambda'}(J) = \langle F_1(Y'), \ldots, F_m(Y')\rangle.$

(Λ' satisfies the condition $v_{\Lambda'}(\mathrm{char}(k')) > 0$ in the proposition since $\mathrm{char}(k') = \mathrm{char}(k) \in \mathfrak{m}R' = \langle u_1, v \rangle$). The first part follows from (8.10) and Lemma 8.2 (2) To show the second part, choose any $g' \in J'$. Take an integer $N > 0$ such that $g := u_1^N g' \in J$. By (8.10), Proposition 6.11 implies that there exist $\lambda_1, \ldots, \lambda_m \in R$ such that,

$$(8.12) \qquad v_{\mathfrak{p}}(\lambda_i f_i) \geq v_{\mathfrak{p}}(g)$$

$$(8.13) \qquad v_L(\lambda_i f_i) \geq v_L(g),$$

$$(8.14) \qquad v_L(g - \sum_i \lambda_i f_i) \gg N.$$

where $\mathfrak{p} = (y, u) = (y_1, \ldots, y_r, u_1, \ldots, u_s)$. Here we used the fact that $v_{\mathfrak{p}} = v_{L_{\mathfrak{p}}}$ with

$$L_{\mathfrak{p}}(A) = \sum_{1 \leq i \leq s} a_i \quad \text{(downstairs)} \quad (A = ((a_i)_{1 \leq i \leq s}, (a'_j)_{1 \leq i \leq t}).$$

Let v_{u_1} be the discrete valuation of R' with respect to the ideal $\langle u_1 \rangle \subset R'$. Because $\mathfrak{p}R' = \langle u_1 \rangle$, (8.12) implies

$$v_{u_1}(\lambda_i) = v_{\mathfrak{p}}(\lambda_i) \geq v_{\mathfrak{p}}(g) - n_i = v_{u_1}(g) - n_i \geq N - n_i,$$

where $n_i = v_{\mathfrak{p}}(f_i) = v_{\mathfrak{m}}(f_i)$ (cf. (8.3)). Therefore

$$\lambda'_i := \lambda_i / u_1^{N-n_i} \in R'.$$

We calculate

$$
(8.15) \qquad
\begin{aligned}
v_{\Lambda'}(\lambda'_i f'_i) &= v_{\Lambda'}(\lambda_i f_i / u_1^N) \\
&= \frac{d}{d-1} v_L(\lambda_i f_i) - \frac{N}{d-1} \quad \text{(by Lemma 8.2(1))} \\
&\geq \frac{d}{d-1} v_L(g) - \frac{N}{d-1} \quad \text{(by (8.13))} \\
&= v_{\Lambda'}(g/u_1^N) = v_{\Lambda'}(g') \quad \text{(by Lemma 8.2(1)).}
\end{aligned}
$$

(8.14) implies

$$
\begin{aligned}
v_{\Lambda'}\left(g' - \sum_{1 \leq i \leq m} \lambda'_i f'_i \right) &= v_{\Lambda'}\left(\left(g - \sum_{1 \leq i \leq m} \lambda_i f_i \right) / u_1^N \right) \\
&= \frac{d}{d-1} v_L\left(g - \sum_{1 \leq i \leq m} \lambda_i f_i \right) - \frac{N}{d-1} \gg 0 \,.
\end{aligned}
$$

Therefore we may assume

$$(8.16) \qquad v_{\Lambda'}\left(g' - \sum_i \lambda'_i f'_i \right) > v_{\Lambda'}(g').$$

By Lemma 6.4 (2), (8.15) and (8.16) imply

$$in_{\Lambda'}(g') = in_{\Lambda'}\left(\sum_{1 \leq i \leq m} \lambda_i' f_i'\right) = \sum_i in_{\Lambda'}(\lambda_i') in_{\Lambda'}(f_i'),$$

where the last sum ranges over all i such that $v_{\Lambda'}(\lambda_i' f_i') = v_{\Lambda'}(g')$. This proves the second part of (8.11) and the proof of Theorem 8.1 is complete.

We keep the assumptions and notations of Setup B and assume that $(f_1, \ldots, f_m)$ is a standard base of J. So by Theorem 8.1 $f' = (f_1', \ldots, f_m')$ is a (u_1, ϕ, v)-standard base of $J' = JR'$. Then Corollary 7.26 assures that we can form a standard base of J' from f' by preparation if (u_1, ϕ, v) is admissible for J'. Hence the following result is important.

Theorem 8.3 *Let $k' = k(x')$. If x' is very near to x (cf. Definition 2.13), there exist linear forms $L_1(U_1, V), \ldots, L_r(U_1, V) \in k'[U_1, V]$ such that*

$$I\mathrm{Dir}(R'/J') = \langle Y_1' + L_1(U_1, V), \ldots, Y_r' + L_r(U_1, V)\rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R') = k'[Y, U_1, \Phi, V].$$

In particular (u_1, ϕ, v) is admissible for J'.

Let K/k' be a field extension. Consider the following map

$$\psi : K[Y] \xrightarrow{\sim} K[Y'] \hookrightarrow K[Y', \Phi] = \mathrm{gr}_{\mathfrak{m}'}(R')_K / \langle U_1, V\rangle,$$

where the first isomorphism maps Y_i to Y_i' for $i = 1, \ldots, r$. Recall that

$$I\mathrm{Dir}(R/J) = \langle Y_1, \ldots, Y_r\rangle \subset \mathrm{gr}_{\mathfrak{m}}(R) = k[Y, U, V],$$

so that

$$I\mathrm{Dir}(R/J)_{/K}^{(1)} := I\mathrm{Dir}(R/J)_{/K} \cap \mathrm{gr}_{\mathfrak{m}}^1(R)_K \subset \bigoplus_{1 \leq i \leq r} K \cdot Y_i \subset K[Y].$$

Theorem 8.3 is an immediate consequence of the following more general result.

Theorem 8.4 *Assume that x' is near to x. Then we have*

$$\psi(I\mathrm{Dir}(R/J)_{/K}^{(1)}) \subset I\mathrm{Dir}(R'/J')_{/K} \mod \langle U_1, V\rangle \text{ in } \mathrm{gr}_{\mathfrak{m}'}(R')_K / \langle U_1, V\rangle.$$

If $e(R/J) = e(R/J)_K$, we have

$$I\mathrm{Dir}(R'/J')_{/K} \supset \langle Y_1' + L_1(U_1, V), \ldots, Y_r' + L_r(U_1, V)\rangle$$

for some linear forms $L_1(U_1, V), \ldots, L_r(U_1, V) \in K[U_1, V]$.

For the proof, we need the following.

Proposition 8.5 *If x' is near to x, there exist $h_{ij} \in R'$ for $1 \leq j < i \leq m$ such that setting*

$$g_i = f_i' - \sum_{j=1}^{i-1} h_{ij} f_j',$$

we have the following for $i = 1, \ldots, m$:

(8.17) $v_{\mathfrak{m}'}(g_i) = n_i,$ *and* $in_{\mathfrak{m}'}(g_i) \equiv F_i(Y') \mod \langle U_1, V\rangle \text{ in } \mathrm{gr}_{\mathfrak{m}'}(R').$

In particular, $(g_1, \ldots, g_m)$ is a standard base of J'.

111

Proof First we note that since x' is near to x, we have

(8.18) $\qquad \nu^*(J') = \nu^*(J) = (n_1, n_2, \ldots, n_m, \infty, \infty, \ldots) \quad (n_1 \le n_2 \le \ldots \le n_m)$

The last assertion of the proposition follows from (8.17) and (8.18) in view of [H1], Ch. III Lemma 3. Put

$$\tilde{F}'_{i,\Gamma'} := \omega_{(y',u_1,\phi,v,\Gamma')}(F_i(Y')), \quad \text{where} \quad \omega_{(y',u_1,\phi,v,\Gamma')} : k'[[Y', U_1, \Phi, V]] \to \hat{R}'$$

is the map (6.6) for (y', u_1, ϕ, v) and Γ' (notations as in Setup B). (8.7) implies

(8.19) $\qquad f'_i \equiv \tilde{F}'_{i,\Gamma'} + \lambda_i \mod (\mathfrak{m}')^{n_i+1} \quad \text{with } \lambda_i \in \langle u_1, v \rangle \quad \text{for } i = 1, \ldots, m.$

To prove (8.17), it suffices to show that there are $h_{ij} \in \hat{R}'$ for $1 \le j < i, \le m$ such that letting g_i be as in the proposition, we have

(8.20) $\qquad v_{\mathfrak{m}'}(g_i) = n_i, \quad \text{and} \quad g_i - \tilde{F}'_{i,\Gamma'} \in \langle u_1, v \rangle + (\mathfrak{m}')^{n_i+1} \subset \hat{R}'.$

Indeed (8.17) follows from (8.20) by replacing the h_{ij} with elements of R' sufficiently close to them. (8.19) implies

$$v_{\mathfrak{m}'}(f'_i) \le n_i = v_{\mathfrak{m}'}(\tilde{F}'_{i,\Gamma'}) \quad \text{for } i = 1, \ldots, m.$$

(8.18) and Corollary 1.4 imply $v_{\mathfrak{m}'}(f'_1) = n_1$ so that one can take $g_1 = f'_1$ for (8.20). Let ℓ be the maximal $t \in \{1, \ldots, m\}$ for which the following holds.

$(*_t)$ There exist $h_{ij} \in \hat{R}'$ for $1 \le j < i \le t$, such that (8.20) holds for

$$g_i = f'_i - \sum_{j=1}^{i-1} h_{ij} f'_j \text{ with } i = 1, \ldots, t.$$

We want to show $\ell = m$. Suppose $\ell < m$. Then $v_{\mathfrak{m}'}(f'_{\ell+1}) < n_{\ell+1}$ since otherwise (8.20) holds for $g_{\ell+1} = f'_{\ell+1}$, which contradicts the maximality of ℓ. This implies

(8.21) $\qquad in_{\mathfrak{m}'}(f'_{\ell+1}) = in_{\mathfrak{m}'}(\lambda_{\ell+1}) \in \langle U_1, V \rangle \subset k'[Y', U_1, \Phi, V].$

By the assumption we have

$$G_i := in_{\mathfrak{m}'}(g_i) \equiv F_i(Y') \mod \langle U_1, V \rangle \quad \text{for } i = 1, \ldots, \ell.$$

Since $(F_1, \ldots, F_m) \subset k[Y]$ is normalized (cf. Definition 1.3), $(G_1, \ldots, G_\ell)$ is normalized in $\text{gr}_{\mathfrak{m}'}(R')$. Therefore (8.18) and Corollary 1.4 imply that there exist $H_i \in \text{gr}_{\mathfrak{m}'}(R')$ homogeneous of degree $v_{\mathfrak{m}'}(f'_{\ell+1}) - n_i$ for $i = 1, \ldots, \ell$, such that

(8.22) $$in_{\mathfrak{m}'}(f'_{\ell+1}) = \sum_{1 \le i \le \ell} H_i G_i.$$

Setting

$$g^{(1)}_{\ell+1} = f'_{\ell+1} - \sum_{1 \le i \le \ell} \tilde{H}_i g_i \quad \text{with } \tilde{H}_i = \omega_{(y',u_1,\phi,v,\Gamma')}(H_i),$$

we have $v_{\mathfrak{m}'}(g^{(1)}_{\ell+1}) > v_{\mathfrak{m}'}(f'_{\ell+1})$. We claim

(8.23) $$g^{(1)}_{\ell+1} - \tilde{F}'_{n_{\ell+1},\Gamma'} \in \langle u_1, v \rangle + (\mathfrak{m}')^{n_{\ell+1}+1},$$

112

which completes the proof. Indeed (8.23) implies

$$v_{\mathfrak{m}'}(g^{(1)}_{\ell+1}) \le v_{\mathfrak{m}'}(\tilde{F}'_{n_{\ell+1},\Gamma'}) = n_{\ell+1}.$$

If $v_{\mathfrak{m}'}(g^{(1)}_{\ell+1}) = n_{\ell+1}$, (8.20) holds for $g^{(1)}_{\ell+1}$, which contradicts the maximality of ℓ. If $v_{\mathfrak{m}'}(g^{(1)}_{\ell+1}) < n_{\ell+1}$, we apply the same argument to $g^{(1)}_{\ell+1}$ instead of $f'_{\ell+1}$ and find $h_1, \ldots, h_\ell \in R'$ such that setting

$$g^{(2)}_{\ell+1} = g^{(1)}_{\ell+1} - \sum_{1 \le i \le \ell} h_i g_i, \qquad \text{we have}$$

$$v_{\mathfrak{m}'}(g^{(1)}_{\ell+1}) < v_{\mathfrak{m}'}(g^{(2)}_{\ell+1}) \le n_{\ell+1} \quad \text{and} \quad g^{(2)}_{\ell+1} - \tilde{F}'_{n_{\ell+1},\Gamma'} \in \langle u_1, v \rangle + (\mathfrak{m}')^{n_{\ell+1}+1}.$$

Repeating the process, we get $g_{\ell+1}$ for which (8.20) holds, which contradicts the maximality of ℓ. We now show claim (8.23). Noting

$$f'_{\ell+1} - \tilde{F}'_{n_{\ell+1},\Gamma'}, \ \ g_i - \tilde{F}'_{i,\Gamma'} \in \ \langle u_1, v \rangle + (\mathfrak{m}')^{n_i+1} \quad (i = 1, \ldots, \ell),$$

it suffices to show

$$\sum_{1 \le i \le \ell} \tilde{H}_i \tilde{F}'_{i,\Gamma'} \in \langle u_1, v \rangle.$$

For $i = 1, \ldots, \ell$, write $H_i = H_i^- + H_i^+$, where $H_i^- \in k'[Y', \Phi]$ and $H_i^+ \in \langle U_1, V \rangle$ and both are homogeneous of degree $v_{\mathfrak{m}'}(f'_{\ell+1}) - n_i$. Similarly we write $G_i = F_i(Y') + G_i^+$ with $G_i^+ \in \langle U_1, V \rangle$. Then (8.21) and (8.22) imply

$$\sum_{1 \le i \le \ell} H_i^- F_i(Y') = 0.$$

Let $\tilde{H}_i^\pm = \omega_{(y', u_1, \phi, v, \Gamma')}(H_i^\pm)$. Noting $\mathrm{char}(k) \in \mathfrak{m}R' = \langle u_1, v \rangle$, property (6.7) implies

$$\sum_{1 \le i \le \ell} \tilde{H}_i^- \tilde{F}'_{i,\Gamma'}, \ \tilde{H}_i^+, \ \tilde{H}_i - (\tilde{H}_i^+ + \tilde{H}_i^-) \ \in \ \langle u_1, v \rangle,$$

which shows the desired assertion. $\square$

Proof of Theorem 8.4 The second assertion follows at once from the first one, and we show the first. By Proposition 8.5 there exist homogeneous $G_1, \ldots, G_m \in \mathrm{gr}_{\mathfrak{m}'}(R')$ such that

(8.24)
$$\begin{aligned} & G_j \equiv F'_j := F_j(Y') \mod \langle U_1, V \rangle \quad \text{for all } j = 1, \ldots, m, \\ & In_{\mathfrak{m}'}(J')_K = \langle G_1, \ldots, G_m \rangle \subset \mathrm{gr}_{\mathfrak{m}'}(R')_K = K[Y', U_1, \Phi, V]. \end{aligned}$$

Let $W' = I\mathrm{Dir}(R'/J')_{/K} \cap \mathrm{gr}^1_{\mathfrak{m}'}(R')$, then there exist $H_1, \ldots, H_n \in K[W'] \cap In_{\mathfrak{m}'}(J')_K$ such that

$$G_i = \sum_{1 \le j \le n} h_{i,j} H_j \quad \text{for some homogeneous } h_{i,j} \in \mathrm{gr}_{\mathfrak{m}'}(R').$$

On the other hand, (8.24) implies

$$H_i = \sum_{1 \le j \le m} g_{i,j} G_j \quad \text{for some homogeneous } g_{i,j} \in \mathrm{gr}_{\mathfrak{m}'}(R').$$

Regarding everything $\mod \langle U_1, V \rangle$, we get

$$F'_i = \sum_{1 \le j \le n} \overline{h}_{i,j} \overline{H}_j \quad \text{and} \quad \overline{H}_i = \sum_{1 \le j \le m} \overline{g}_{i,j} F'_j \ \text{ in } K[Y', \Phi] = \text{gr}_{\mathfrak{m}'}(R')/\langle U_1, V \rangle.$$

The second equality implies

$$\overline{H}_i \in K[\overline{W'}] \cap \langle F'_1, \dots, F'_m \rangle \subset K[Y', \Phi] \quad \text{where } \overline{W'} = W' \mod \langle U_1, V \rangle,$$

so that the first equality implies

(8.25) $$\left(K[\overline{W'}] \cap \langle F'_1, \dots, F'_m \rangle \right) \cdot K[Y', \Phi] = \langle F'_1, \dots, F'_m \rangle.$$

Since $W = I\text{Dir}(R/J)_{/K} \cap \text{gr}^1_{\mathfrak{m}}(R)$ is the minimal subspace of $\bigoplus_{1 \le i \le r} K \cdot Y_i$ such that

$$\left(K[W] \cap \langle F_1(Y), \dots, F_m(Y) \rangle \right) \cdot K[Y] = \langle F_1(Y), \dots, F_m(Y) \rangle,$$

(8.25) implies $\psi(W) \subset \overline{W'}$, which is the desired assertion. □

We conclude this section with the following useful criteria for the nearness and the very nearness of x' and x. We keep the assumptions and notations of Setup B.

Theorem 8.6 *(1) If $\delta(f', y', (u_1, \phi, v)) \ge 1$, x' is near to x. The converse holds if $(f', y', (u_1, \phi, v))$ is prepared at any vertex lying in $\{A \in \mathbb{R}^{s'+t} | \ |A| \le 1\}$.*

(2) Assume $e_x(X) = e_x(X)_{k(x')}$. If $\delta(f', y', (u_1, \phi, v)) > 1$, x' is very near to x. The converse holds under the same assumption as in (1).

Proof Write $k' = k(x')$. Assume $\delta(f', y', (u_1, \phi, v)) \ge 1$. By Lemma 7.4 (1) and (8.8), $v_{\mathfrak{m}'}(f'_i) = n_i$ and we can write

$$in_{\mathfrak{m}'}(f'_i) = F(Y') + \sum_{|B| < n_i} Y'^B P_B(U_1, \Phi, V), \quad P_B(U_1, \Phi, V) \in k'[U_1, \Phi, V].$$

Put $I = \langle in_{\mathfrak{m}'}(f'_1), \dots, in_{\mathfrak{m}'}(f'_m) \rangle \subset \text{gr}_{\mathfrak{m}'}(R')$. We have

$$\nu^*(J) \ge \nu^*(J') = \nu^*(In_{\mathfrak{m}'}(J')) \ge \nu^*(I),$$

where the first inequality follows from Theorem 2.10 and the last from [H1] Lemma Ch.II Lemma 3. By the assumption $(F_1(Y), \dots, F_m(Y))$ is weakly normalized (cf. Definition 1.3), which implies that $(in_{\mathfrak{m}'}(f'_1), \dots, in_{\mathfrak{m}'}(f'_m))$ is weakly normalized so that $\nu^*(I) = \nu^*(J)$. Therefore we get $\nu^*(J) = \nu^*(J')$ and x' is near to x. We also get $\nu^*(In_{\mathfrak{m}'}(J')) = \nu^*(I)$, which implies $In_{\mathfrak{m}'}(J') = I$ by loc.cit.. Now assume x' is near to x. Let $g = (g_1, \dots, g_m)$ be as in Proposition 8.5. By (8.17) we have $\delta(g, y', (u_1, \phi, v)) \ge 1$. We claim that g is a (u_1, ϕ, v)-standard basis of J'. Indeed f' is a (u_1, ϕ, v)-standard basis of J' by Theorem 8.1, so the claim follows from the fact that $in_0(f') = in_0(g)$ by using Corollary 6.10. By the claim $\Delta(J', (u_1, \phi, v)) \subset \Delta(g, y', (u_1, \phi, v))$ so that there exists no vertex w of $\Delta(J', (u_1, \phi, v))$ such that $|w| < 1$. If $\Delta(f', y', (u_1, \phi, v))$ is prepared at any vertex w with $|w| \le 1$, we obtain $\delta(f', y', (u_1, \phi, v)) \ge 1$ from Theorem 7.16.

114

Assume $\delta(f', y', (u_1, \phi, v)) > 1$. By Lemma 7.4 (1) and (8.8), $in_{\mathfrak{m}'}(f'_i) = F(Y')$ so that

$$(8.26) \qquad\qquad in_{\mathfrak{m}'}(J') = I = \langle F_1(Y'), \ldots, F_m(Y') \rangle.$$

By the assumed equality $e_x(X) = e_x(X)_{k'}$, we have $I\mathrm{Dir}_{k'}(R/J) = \langle Y_1, \ldots, Y_r \rangle$ (cf. Remark 1.19). Since $In_{\mathfrak{m}}(J) = \langle F_1(Y), \ldots, F_m(Y) \rangle$, this implies $I\mathrm{Dir}(R'/J') = \langle Y'_1, \ldots, Y'_r \rangle$ by (8.26) so that x' is very near to x. Finally assume x' very near to x. By Theorem 8.1, f' is (u_1, ϕ, v)-standard basis of J'. By Theorem 8.3, (u_1, ϕ, v) is admissible for J'. Thus Corollary 7.17 implies $\delta(f', y', (u_1, \phi, v)) > 1$. This completes the proof of Theorem 8.6. $\square$

Remark 8.7 *By the above theorem it is important to compute $\delta(f', y', (u_1, \phi, v))$ from $\Delta(f, y, (u, v))$. It is also important to see if the well-preparedness of $\Delta(f, y, (u, v))$ implies that of $\Delta(f', y', (u_1, \phi, v))$. These issues are discussed later in this paper in various situations (e.g., see Lemma 9.3).*

9 Termination of the fundamental sequences of B-permissible blowups, and the case $e_x(X) = 1$

In this section we prove the Key Theorem 5.35 in §5, by deducing it from a stronger result, Theorem 9.2 below. Moreover we will give an explicit bound on the length of the fundamental sequence, by the δ-invariant of the polyhedron at the beginning. First we introduce a basic setup.

Setup C: Let Z be an excellent regular scheme, let $X \subset Z$ be a closed subscheme and take a point $x \in X$. Let $R = \mathcal{O}_{Z,x}$ with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m} = k(x)$, and write $X \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/J)$ with an ideal $J \subset \mathfrak{m}$. Define the integers $n_1 \leq n_2 \leq \cdots \leq n_m$ by

$$\nu_x^*(X, Z) = \nu^*(J) = (n_1, \ldots, n_m, \infty, \infty, \ldots).$$

We also assume given a simple normal crossing boundary $\mathcal{B}$ on Z and a history function $O : X \to \{\text{subsets of } \mathcal{B}\}$ for $\mathcal{B}$ on X (Definition 3.6). Note that $\mathcal{B}$ may be empty.

We introduce some notations.

Definition 9.1 *(1) A prelabel of (X, Z) at x is*

$$(f, y, u) = (f_1 \ldots, f_m, \ y_1, \ldots, y_r, u_1, \ldots, u_e),$$

where (y, u) is a system of regular parameters of R such that (u) is admissible for J (cf. Definition 6.1) and f is a (u)-standard base of J. By Lemma 6.13 we have $n_{(u)}(f_i) = n_i$ for $i = 1, \ldots, N$.

(2) A prelabel (f, y, u) is a label of (X, Z) at x if $\delta(f, y, u) > 1$. By Corollary 7.17, this means that (y, u) is strictly admissible for J and f is a standard base of J such that (f, y, u) is admissible. By Lemma 6.13 we have

$$\begin{aligned}
&v_{\mathfrak{m}}(f_i) = n_i \quad \text{and} \quad in_{\mathfrak{m}}(f_i) = in_0(f_i) \quad \text{for } i = 1, \ldots, N, \\
&\langle Y_1, \ldots, Y_r \rangle = I\mathrm{Dir}(R/J) \ \subset \mathrm{gr}_{\mathfrak{m}}(R), \\
&In_{\mathfrak{m}}(J) = \langle F_1(Y), \ldots, F_N(Y) \rangle \quad \text{with } F_i(Y) = in_0(f_i) \in k[Y],
\end{aligned}$$

where $k[Y] = k[Y_1 \ldots, Y_r] \subset \mathrm{gr}_{\mathfrak{m}}(R)$ with $Y_i = in_{\mathfrak{m}}(y_i) \in \mathrm{gr}_{\mathfrak{m}}(R)$.

(3) A label $(f, (y, u))$ is well-prepared (resp. totally prepared) if so is (f, y, u) in the sense of Definition 7.15 (resp. Remark 7.25). By Theorem 7.16, for a well-prepared label (f, y, u), we have $\Delta(J, u) = \Delta(f, y, u)$.

For each $B \in \mathcal{B}$ choose an element $l_B \in R$ such that $B \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle l_B \rangle)$. For a positive linear form $L : \mathbb{R}^e \to \mathbb{R}$ let $\delta_L(l_B, y, u)$ be as in Definition 7.2 (3). Writing

$$(9.1) \qquad l_B = \sum_{1 \leq i \leq r} a_i y_i + \sum_{A \in \mathbb{Z}_{\geq 0}^e} b_A u^A + g \quad (a_i, \ b_A \in R^\times \cup \{0\}),$$

where $g \in \langle y_1, \ldots, y_r \rangle^2$, we have

$$\delta_L(l_B, y, u) = \min\{L(A) \mid b_A \notin \mathfrak{m}\}.$$

It is easy to see that $\delta_L(l_B, y, u)$ depends only on B and not on the choice of l_B. We define

$$\delta_L^O(y, u) = \min\{\delta_L(l_B, y, u) \mid B \text{ irreducible component of } O(x)\}.$$

Theorem 9.2 *Assume* $\operatorname{char}(k(x)) = 0$ *or* $\operatorname{char}(k(x)) \geq \dim(X)/2 + 1$, *and assume there is a fundamental sequence of length* $\geq m$ *starting with* $(X, \mathcal{B}_X, O)$ *and* x *as in (5.23) of Definition 5.34, where* $m \geq 1$. *Let* (f, y, u) *be a label of* (X, Z) *at* x. *In case* $e_x(X) > e_x^O(X)$, *assume that* $u = (u_1, \ldots, u_e)$ ($e = e_x(X)$) *satisfies the following condition:*

There exist $B_j \in O(x)$ *for* $2 \leq j \leq s := e_x(X) - e_x^O(X) + 1$ *such that*

$$B_j \times_Z \operatorname{Spec}(R) = \operatorname{Spec}(R/\langle u_j \rangle) \quad and \quad \operatorname{Dir}_x^O(X) = \operatorname{Dir}_x(X) \cap \underset{2 \leq j \leq s}{\cap} T_x(B_j).$$

Assume further (f, y, u) *is prepared along the faces* E_{L_q} *for all* $q = 0, 1, \ldots, m - 1$, *where*

$$L_q : \mathbb{R}^e \to \mathbb{R} \; ; \; A = (a_1, \ldots, a_e) \to |A| + q \cdot \sum_{j=2}^{s} a_j.$$

Then we have

(9.2) $\qquad \delta_{L_q}(f, y, u) \geq q + 1 \quad and \quad \delta_{L_q}^O(y, u) \geq q + 1 \quad for \; q = 1, \ldots, m - 1.$

First we show how to deduce Theorem 5.35 from Theorem 9.2. It suffices to show that under the assumption of 5.35, there is no infinite fundamental sequence of $\mathcal{B}_X$-permissible blowups over x. Assume the contrary. As in (6.3) write

(9.3) $\qquad\qquad f_i = \sum_{(A,B)} C_{i,A,B} \, y^B u^A \quad \text{with} \quad C_{i,A,B} \in R^\times \cup \{0\}.$

Write $|A|_1 = \sum_{j=2}^{s} a_j$ for $A = (a_1, \ldots, a_e)$. By Definition 7.2 (3), we have

$$\delta_{L_q}(f, y, u) > q + 1 \Leftrightarrow |A| + q|A|_1 \geq (q+1)(n_i - |B|) \text{ if } |B| < n_i, \; C_{i,A,B} \neq 0$$

$$\Leftrightarrow |B| + |A|_1 \geq n_i - \frac{|A| - |A|_1}{q+1} \text{ if } |B| < n_i, \; C_{i,A,B} \neq 0.$$

Hence by (9.2) and the assumption we have the last statement for all $q \in \mathbb{N}$, which implies $|B| + |A|_1 \geq n_i$ for all A and B such that $C_{i,A,B} \neq 0$ in (9.3). Setting $\mathfrak{q} = \langle y_1, \ldots, y_r, u_2, \ldots, u_s \rangle \in \operatorname{Spec}(R)$, this implies $J \subset \mathfrak{q}$ and $v_\mathfrak{q}(f_i) = n_i = v_\mathfrak{m}(f_i)$ for all i. By Theorems 2.2 (iv) and 2.3 we conclude $H_X(\omega) = H_X(x)$, where $\omega \in X$ is the image of $\mathfrak{q} \in \operatorname{Spec}(R)$. By a similar argument, the second part of (9.2) implies $\omega \in B$ for all $B \in O(x)$ so that $H_X^O(\omega) = H_X^O(x)$. This contradicts the assumption of Theorem 5.35 since $\dim(R/\mathfrak{q}) = e - s + 1 = e_x^O(X)$. This completes the proof of Theorem 5.35. $\square$

Now we prepare for the proof of Theorem 9.2. Consider

$$\pi : Z' = B\ell_x(Z) \to Z \quad and \quad \pi : X' = B\ell_x(X) \to X.$$

By Theorem 2.14, any point of X' near to x is contained in $\mathbb{P}(\mathrm{Dir}_x(X)) \subset X'$. We now take a label (f, y, u) of (X, Z) at x as in Definition 9.1 (2). By (6.2) we have the identification determined by (u):

$$(9.4) \qquad \psi_{(u)} : \mathbb{P}(\mathrm{Dir}_x(X)) = \mathrm{Proj}(k[T]) = \mathbb{P}_k^{e-1} \quad (k[T] = k[T_1, \ldots, T_e])$$

Let $x' = (1 : 0 : \cdots : 0) \in \mathbb{P}(\mathrm{Dir}_x(X))$. If x' is near to x, Theorems 2.10 and 2.13 imply

$$\nu_{x'}^*(X', Z') = \nu_x^*(X, Z) \quad \text{and} \quad e_{x'}(X') \leq e.$$

Let $R' = \mathcal{O}_{Z', x'}$ with maximal ideal $\mathfrak{m}'$, and let $J' \subset R'$ be the ideal defining $X' \subset Z'$. Note that
$$\mathbb{P}(\mathrm{Dir}_x(X)) \times_{Z'} \mathrm{Spec}(R') = \mathrm{Spec}(R'/\langle u_1 \rangle),$$
and denote
$$y' = (y_1', \ldots, y_r'), \ u' = (u_2', \ldots, u_e'), \ f' = (f_1', \ldots, f_N') \,,$$
where $y_i' = y_i/u_1$, $u_i' = u_i/u_1$, $f_i' = f_i/u_1^{n_i}$. As is seen in Setup B in §8, (y', u_1, u') is a system of regular parameters of R' and $J' = \langle f_1', \ldots, f_N' \rangle$ and R' is the localization of $R[y_1', \ldots, y_r', u_2', \ldots, u_e']$ at (y', u_1, u'). We will use the usual identifications

$$\mathrm{gr}_{\mathfrak{m}'}(R') = k[Y', U_1, U_2', \ldots, U_e'], \quad Y_i' = in_{\mathfrak{m}'}(y_i'), \ U_j' = in_{\mathfrak{m}'}(u_j').$$

Moreover we will consider the maps:

$$\Psi : \mathbb{R}^e \to \mathbb{R}^e \ ; \ (a_1, a_2, \ldots, a_e) \mapsto (\sum_{i=1}^e a_i - 1, a_2, \ldots, a_e),$$

$$\Phi : \mathbb{R}^e \to \mathbb{R}^e \ ; \ (a_1, a_2, \ldots, a_e) \mapsto (\sum_{i=1}^e a_i, a_2, \ldots, a_e).$$

A semi-positive linear from $L : \mathbb{R}^e \to \mathbb{R}$ is called monic if $L(1, 0, \ldots, 0) = 1$.

Lemma 9.3 *(1) $\Delta(f', y', (u_1, u'))$ is the minimal F-subset containing $\Psi(\Delta(f, y, u))$.*

(2) For any monic semi-positive linear form $L : \mathbb{R}^e \to \mathbb{R}$ and $\widetilde{L} = L \circ \Phi$, we have

$$\delta_L(f', y', (u_1, u')) = \delta_{L \circ \Phi}(f, y, u) - 1,$$

$$U_1^{n_i} \cdot in_{E_L}(f_i')_{(y', (u_1, u'))} = in_{E_{\widetilde{L}}}(f_i)_{(y, u)|Y = U_1 Y', U_i = U_1 U_i' \ (2 \leq i \leq e)} \,,$$

$$in_{E_{\widetilde{L}}}(f_i)_{(y, u)} = U_1^{n_i} \cdot in_{E_L}(f_i)_{(y', (u_1, u'))|Y' = Y/U_1, U_i' = U_i/U_1 \ (2 \leq i \leq e)} \,.$$

If (f, y, u) is prepared along $E_{\widetilde{L}}$, then $(f', y', (u_1, u'))$ is prepared along E_L.

(3) Assume (f, y, u) prepared along $E_{\widetilde{L}_0}$ with $\widetilde{L}_0 = L_0 \circ \Phi$. Then $(f', y', (u_1, u'))$ is δ-prepared and x' is near (resp. very near) if and only if $\delta_{\widetilde{L}_0}(f, y, u) \geq 2$ (resp. $\delta_{\widetilde{L}_0}(f, y, u) > 2$). If (f, y, u) is totally prepared, so is $(f', y', (u_1, u'))$.

(4) Assume x' very near to x. Then $(f', y', (u_1, u'))$ is a prelabel of (X', Z') at x'. Assume further (f, y, u) prepared along $E_{\widetilde{L}_0}$ (resp. totally prepared), then $(f', y', (u_1, u'))$ is a δ-prepared (resp. totally prepared) label of (X', Z') at x'.

Proof From (9.3) we compute

$$(9.5) \qquad f'_i = f_i/u_1^{n_i} = \sum_{(A,B)} C_{i,A,B} \, (y')^B u'^A u_1^{|A|+(|B|-n_i)},$$

$$u'^A = u_2'^{\,a_2} \cdots u_e'^{\,a_e} \quad \text{for } A = (a_1, a_2, \ldots, a_e).$$

(1) follows at once from this. For a semi-positive linear form $L : \mathbb{R}^e \to \mathbb{R}$, we have

$$\delta_L(f', y', (u_1, u')) = \min\{ \frac{L(|A| + |B| - n_i, \ a_2, \ldots, a_e)}{n_i - |B|} \mid |B| < n_i, \ C_{i,A,B} \neq 0 \}$$

$$= \min\{ \frac{L \circ \Phi(A)}{n_i - |B|} - 1 \mid |B| < n_i, \ C_{i,A,B} \neq 0 \}$$

$$= \delta_{L \circ \Phi}(f, y, u) - 1.$$

From (9.5), we compute

$$in_{E_L}(f')_{(y',(u_1,u'))} = F_i(Y) + \sum_{B,A} \overline{C}_{i,A,B} Y'^B U'^A U_1^{|A|+|B|-n_i},$$

where the sum ranges over such B, A that $|B| < n_i$ and $L \circ \Phi(A) = \delta_{L \circ \Phi}(n_i - |B|)$. (2) follows easily from this. (3) follows from (2) and Theorem 8.6. The first assertion of (4) follows from Theorem 8.1 and Theorem 8.3. The other assertion of (4) follows from (3) and Corollary 7.17. $\square$

We now consider

$$C' := \mathbb{P}(\mathrm{Dir}_x^O(X)) \subset \mathbb{P}(\mathrm{Dir}_x(X)) = \mathrm{Proj}(k[T_1, \ldots, T_e]) \subseteq X'.$$

Let η' be the generic point of C', and let

$$t := e_x^O(X) \quad \text{and} \quad s = e - t + 1.$$

We assume $t \geq 1$ so that $1 \leq s \leq e$. By making a suitable choice of the coordinate $(u) = (u_1, \ldots, u_e)$, we may assume:

$$(9.6) \qquad \text{there exists } B_j \in O(x) \text{ for } 2 \leq j \leq s \text{ such that}$$

$$B_j \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle u_j \rangle) \quad \text{and} \quad \mathrm{Dir}_x^O(X) = \mathrm{Dir}_x(X) \cap \bigcap_{2 \leq j \leq s} T_x(B_j).$$

Then

$$C' \times_{Z'} \mathrm{Spec}(R') = \mathrm{Spec}(R'/\mathfrak{p}') \quad \text{with } \mathfrak{p}' = \langle y', u_1, u_2', \ldots, u_s' \rangle.$$

Note $\delta_{k(\eta')/k} := \mathrm{trdeg}_k(k(\eta')) = t - 1$. By Theorems 2.13 and 2.6, if η' is near x, we have $e_{\eta'}(X') \leq e - \delta_{k(\eta')/k} = s$. It also implies that $C' \subset X'$ is permissible by Theorems 1.33 and 2.3. By Theorem 3.17, if η' is very near x, we have $e_{\eta'}^O(X') \leq e_x^O(X) - \delta_{k(\eta')/k} = 1$.

Write $R'_{\eta'} = \mathcal{O}_{Z',\eta'}$ and let $\mathfrak{m}'_{\eta'}$ be its maximal ideal and $J'_{\eta'} = J' R'_{\eta'}$. Note that $R'_{\eta'}$ is the localization of R' at $\mathfrak{p}'$ and $(y', u_1, u_2', \ldots, u_s')$ is a system of regular parameters of $R'_{\eta'}$.

Lemma 9.4 *Let $\Delta(f', y', (u_1, u_2', \ldots, u_s'))$ be the characteristic polyhedron for $J'_{\eta'} \subset R'_{\eta'}$.*

(1) $\Delta(f', y', (u_1, u'_2, \ldots, u'_s))$ is the minimal F-subset containing $\pi \cdot \Psi(\Delta(f, y, u))$, where Ψ is as in Lemma 9.3 and

$$\pi : \mathbb{R}^e \to \mathbb{R}^s \; ; \; A = (a_1, a_2, \ldots, a_e) \mapsto (a_1, a_2, \ldots, a_s).$$

For any monic semi-positive linear form $L : \mathbb{R}^s \to \mathbb{R}$, we have

$$\delta_L(f', y', (u_1, u'_2, \ldots, u'_s)) = \delta_{L \circ \pi \circ \Phi}(f, y, u) - 1.$$

In particular,

$$\delta(f', y', (u_1, u'_2, \ldots, u'_s)) = \delta_{L_1}(f, y, u) - 1,$$

$$L_1 : \mathbb{R}^e \to \mathbb{R} \; ; \; A \mapsto |A| + \sum_{2 \leq i \leq s} a_i.$$

(2) If (f, y, u) is prepared along the face E_{L_1} of $\Delta(f, y, u)$, then $(f', y', (u_1, u'_2, \ldots, u'_s))$ is δ-prepared. If (f, y, u) is totally prepared, so is $(f', y', (u_1, u'_2, \ldots, u'_s))$.

(3) Assume (f, y, u) is prepared along the face E_{L_1} of $\Delta(f, y, u)$. Then η is near x if and only if $\delta_{L_1}(f, y, u) \geq 2$. if this holds, we have $v_{\mathfrak{m}'_\eta}(f'_i) = v_{\mathfrak{m}}(f_i) = n_i$ for $i = 1, \ldots, N$.

(4) Assume (f, y, u) is prepared along the face E_{L_1} of $\Delta(f, y, u)$. Then η is very near x if and only if $\delta_{L_1}(f, y, u) > 2$. if this holds, then $(f', (y', (u_1, u'_2, \ldots, u'_s)))$ is a δ-prepared label of (X', Z') at η.

Proof (1) and (2) follows from Lemma 9.3 and Theorem 7.27 applied to $(f', y', (u_1, u'))$ and $\mathfrak{p}' = (u_1, u'_2, \ldots, u'_s)$ and (9.5) in place of (f, y, u) and $\mathfrak{p}_s = (u_1, u_2 \ldots, u_s)$ and (7.5). We need to check the conditions $(P0)$ and $(P2)$ as well as Theorem 7.27 $(3)(i)$ and (ii) for the replacement. $(P0)$ holds since $k = R/\mathfrak{m} \hookrightarrow R'/\langle u_1 \rangle$. In view of the presentation (9.5), $(P2)$ holds by the fact that for fixed B and $a \in \mathbb{Z}$, there are only finitely many $A \in \mathbb{Z}^e_{\geq 0}$ such that $|A| + |B| - n_i = a$. Theorem 7.27 $(3)(i)$ is a consequence of the assumption that (f, y, u) is a label of (X, Z) at x (cf. Definition 9.1 (2)). Finally Theorem 7.27 $(3)(ii)$ follows from Lemma (9.3)(2). (3) and (4) are consequences of (2) and Theorems 8.6 and 8.1 and 8.3 and Lemma 7.4. This completes the proof of Lemma 9.4. $\square$

Proof of Theorem 9.2. Write $Z_1 = Z'$, $X_1 = X'$, $C_1 = C'$ and assume $\pi : Z' \to Z$ extends to a sequence (5.23). Let (f, y, u) be a label of (X, Z) at x. For $1 \leq q \leq m$, write $R_{\eta_q} = \mathcal{O}_{Z_q, \eta_q}$ with the maximal ideal $\mathfrak{m}_{\eta_q}$, and let $J_{\eta_q} \subset R_{\eta_q}$ be the ideal defining $X_q \subset Z_q$ at η_q. Write

$$f_i^{(q)} = f_i/u_1^{qn_i}, \; y_i^{(q)} = y_i/u_1^q, \; u_i^{(q)} = u_i/u_1^q \; (2 \leq i \leq s), \; u'_i = u_i/u_1 \; (s + 1 \leq i \leq e).$$

Claim 9.5 Let (f, y, u) be a label of (X, Z) at x prepared along the faces E_{L_q} for all $q = 1, \ldots, m - 1$. Then, for $q = 1, \ldots, m$, R_{η_q} is the localization of

$$R[y_1^{(q)}, \ldots, y_r^{(q)}, u_2^{(q)}, \ldots, u_s^{(q)}, u'_{s+1}, \ldots, u'_e] \quad \text{at } \langle y_1^{(q)}, \ldots, y_r^{(q)}, u_1, u_2^{(q)}, \ldots, u_s^{(q)} \rangle,$$

and $J_{\eta_q} = \langle f_1^{(q)}, \ldots, f_N^{(q)} \rangle$, and $(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)}, \ldots, u_s^{(q)}))$ is δ-prepared and

$$\delta(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)}, \ldots, u_s^{(q)})) = \delta_{L_q}(f, y, u) - q.$$

For $q \leq m - 1$, $(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)}, \ldots, u_s^{(q)}))$ is a δ-prepared label of (X_q, Z_q) at η_q.

120

Proof For $q = 1$ the claim follows from Lemma 9.4. For $q > 1$, by induction it follows from loc.cit. applied to

$$\text{Spec}(\mathcal{O}_{X_{q-1},\eta_{q-1}}) \leftarrow X_q \times_{X_{q-1}} \text{Spec}(\mathcal{O}_{X_{q-1},\eta_{q-1}})$$

and $(f^{(q-1)}, y^{(q-1)}, (u_1, u_2^{(q-1)}, \ldots, u_s^{(q-1)}))$ in place of $X \leftarrow X_1$ and (f, y, u) (note that the condition (9.6) is satisfied for $\text{Spec}(\mathcal{O}_{X_{q-1},\eta_{q-1}})$ by Lemma 9.4 (5)). $\square$

Recalling now that η_q is near to η_{q-1} for $1 \leq q \leq m-1$ ($\eta_0 = x$ by convention), we get $\delta_{L_q}(f, y, u) - q \geq 1$ for these q by Theorem 8.6 and Claim 9.5. It remains to show $\delta_{L_q}^O(y, u) \geq q + 1$. For this we rewrite (9.1) in R_{η_q} as follows:

$$(9.7) \qquad l_B = u_1^q \sum_{1 \leq i \leq r} a_i y_i^{(q)} + \sum_{C=(c,a_2,\ldots,a_s) \in \mathbb{Z}_{\geq 0}^s} P_C \cdot u_1^c (u_2^{(q)})^{a_2} \cdots (u_s^{(q)})^{a_s} + u_1^{2q} g',$$

where $g' = g/u_1^{2q} \in \langle y_1^{(q)}, \ldots, y_r^{(q)} \rangle^2$ and

$$P_C = \sum_{\Omega(A)=C} b_A \cdot u'^{a_{s+1}}_{s+1} \cdots u'^{a_e}_e,$$

for the map $\Omega : \mathbb{R}^e \to \mathbb{R}^s$; $A = (a_1, \ldots, a_e) \to (L_{q-1}(A), a_2, \ldots, a_s)$. We easily see

$$P_C \in \mathfrak{m}_{\eta_q} = \langle y^{(q)}, u_1, u_2^{(q)}, \ldots, u_s^{(q)} \rangle \iff b_A \in \mathfrak{m} \text{ for all } A \text{ such that } \Omega(A) = C,$$

by noting that $R \to R_{\eta_q}/\mathfrak{m}_{\eta_q}$ factors through $R \to k = R/\mathfrak{m}$ and $k[u'_{s+1}, \ldots, u'_e] \hookrightarrow k(\eta_q)$. The strict transform $\tilde{B}_q$ of B in $\text{Spec}(R_{\eta_q})$ is defined by

$$l'_B = l_B/u_1^\gamma \quad \text{ with } \gamma = v_{\langle u_1 \rangle}(l_B),$$

where $v_{\langle u_1 \rangle}$ is the valuation of R_{η_q} defined by the ideal $\langle u_1 \rangle \subset R_{\eta_q}$. From (9.7), we see

$$l'_B \notin \mathfrak{m}_{\eta_q} \iff P_C \notin \mathfrak{m}_{\eta_q} \text{ and } c \leq q \text{ for some } C = (c, 0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^s$$
$$\iff b_A \notin \mathfrak{m} \text{ and } |A| \leq q \text{ for some } A = (c', 0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^s.$$

For $A \in \mathbb{Z}_{\geq 0}^e - \{0\}$, we have

$$L_q(A) = |A| + \sum_{2 \leq j \leq s} a_j \geq q + 1 \iff (a_2, \ldots, a_s) \neq (0, \ldots, 0) \text{ or } |A| \geq q + 1.$$

Hence we get

$$l'_B \notin \mathfrak{m}_{\eta_q} \iff L(A) < q + 1 \text{ for some } A \in \mathbb{Z}_{\geq 0}^s - \{0\}. \iff \delta_{L_q}(l_B, y, u) < q + 1.$$

It implies $\delta_{L_q}^O(y, u) = \min\{\delta(l_B, y, u) \mid B \in O(x)\} \geq q + 1$ since $\eta_q \in \tilde{B}_q$ for $q \leq m-1$ by the assumption. This shows the desired assertion and the proof of Theorem 9.2 is complete.

Corollary 9.6 *Let (f, y, u) be a δ-prepared label of (X, Z) at x. Assume $e_x(X) = e_x^O(X)$ (for example $\mathcal{B} = \emptyset$), and that the assumptions of Theorem 5.35 hold. Then for the length m of the fundamental unit (Definition 5.34) we have*

$$m = \lfloor \delta(f, y, u) \rfloor \quad (:= \text{ greatest integer } \leq \delta(f, y, u)).$$

Proof By Claim 9.5 in case $s = 1$ and $L_q = L_0$ together with Theorem 8.6 (1) and by the assumption that η_{m-1} is near to η_{m-2} and η_m is not near to η_{m-1}, we have

$$\delta(f^{(m-1)}, y^{(m-1)}, u_1) = \delta(f, y, u) - (m-1) \geq 1, \quad \delta(f^{(m)}, y^{(m)}, u_1) = \delta(f, y, u) - m < 1.$$

10 Additional invariants in the case $e_x(X) = 2$

In order to show key Theorem 5.40 in §5, we recall further invariants for singularities, which were defined by Hironaka ([H6]). The definition works for any dimension, as long as the directrix is 2-dimensional.

Definition 10.1 *For a polyhedron $\Delta \subset \mathbb{R}^2_{\geq 0}$ we define*

$$
\begin{aligned}
\alpha(\Delta) &:= \inf \{ v_1 \mid (v_1, v_2) \in \Delta \} \\
\beta(\Delta) &:= \inf \{ v_2 \mid (\alpha(\Delta), v_2) \in \Delta \} \\
\delta(\Delta) &:= \inf \{ v_1 + v_2 \mid (v_1, v_2) \in \Delta \} \\
\gamma^+(\Delta) &:= \sup \{ v_2 \mid (\delta(\Delta) - v_2, v_2) \in \Delta \} \\
\gamma^-(\Delta) &:= \inf \{ v_2 \mid (\delta(\Delta) - v_2, v_2) \in \Delta \} \\
\epsilon(\Delta) &:= \inf \{ v_2 \mid (v_1, v_2) \in \Delta \} \\
\zeta(\Delta) &:= \inf \{ v_1 \mid (v_1, \epsilon(\Delta)) \in \Delta \}
\end{aligned}
$$

The picture is as follows:
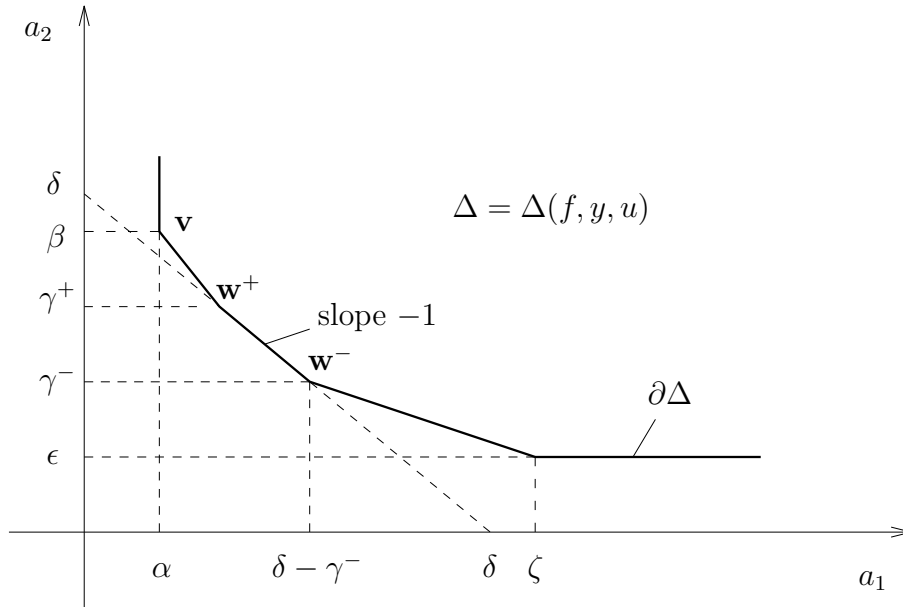
There are three vertices of $\Delta(f, y, u)$ which play crucial roles:

$$
\begin{aligned}
\mathbf{v} := \mathbf{v}(\Delta) &:= (\alpha(\Delta), \beta(\Delta)), \\
\mathbf{w}^+ := \mathbf{w}^+(\Delta) &:= (\delta(\Delta) - \gamma^+(\Delta), \gamma^+(\Delta)), \\
\mathbf{w}^- := \mathbf{w}^-(\Delta) &:= (\delta(\Delta) - \gamma^-(\Delta), \gamma^-(\Delta)).
\end{aligned}
$$

We have

(10.1) $$ \beta(\Delta) \geq \gamma^+(\Delta) \geq \gamma^-(\Delta). $$

Now let us consider the situation of Setup C in §9 and assume $e_x(X) = 2$.

Let $(f, y, u) = (f_1, \ldots, f_N, y_1, \ldots, y_r, u_1, u_2)$ be a prelabel of (X, Z) at x. Recall $n_{(u)}(f_i) = n_i$ for $i = 1, \ldots, N$. Write as (6.3):

$$f_i = \sum_{A,B} C_{i,A,B}\, y^B u^A\,, \quad \text{with } A = (a_1, a_2),\ B = (b_1, \ldots, b_r),\ C_{i,A,B} \in R^\times \cup \{0\}.$$

For $* = \alpha,\ \beta,\ \delta,\ \gamma^\pm,\ \mathbf{v},\ \mathbf{w}^\pm$, we write $*(f, y, u)$ for $*(\Delta(f, y, u))$. Then we see

$$\alpha(f, y, u) = \inf\left\{ \frac{a_1}{n_i - |B|} \mid 1 \le i \le m,\ C_{i,A,B} \ne 0 \right\}$$

$$\beta(f, y, u) = \inf\left\{ \frac{a_2}{n_i - |B|} \mid 1 \le i \le m,\ \frac{a_1}{n_i - |B|} = \alpha(f, y, u),\ C_{i,A,B} \ne 0 \right\}$$

$$\delta(f, y, u) = \inf\left\{ \frac{|A|}{n_i - |B|} \mid 1 \le i \le m,\ C_{i,A,B} \ne 0 \right\}$$

$$\gamma^+(f, y, u) = \sup\left\{ \frac{a_2}{n_i - |B|} \mid 1 \le i \le m,\ \frac{|A|}{n_i - |B|} = \delta,\ C_{i,A,B} \ne 0 \right\}$$

$$\gamma^-(f, y, u) = \inf\left\{ \frac{a_2}{n_i - |B|} \mid 1 \le i \le m,\ \frac{|A|}{n_i - |B|} = \delta,\ C_{i,A,B} \ne 0 \right\}$$

$$\epsilon(f, y, u) = \inf\left\{ \frac{a_2}{n_i - |B|} \mid 1 \le i \le m,\ C_{i,A,B} \ne 0 \right\}$$

$$\mathbf{v} := \mathbf{v}(f, y, u) = (\alpha(f, y, u), \beta(f, y, u)),$$
$$\mathbf{w}^+ := \mathbf{w}^+(f, y, u) = (\delta(f, y, u) - \gamma^+(f, y, u), \gamma^+(f, y, u)),$$
$$\mathbf{w}^- := \mathbf{w}^-(f, y, u) = (\delta(f, y, u) - \gamma^-(f, y, u), \gamma^-(f, y, u))\,.$$

Definition 10.2 *(1) A prelabel (f, y, u) (cf. Definition 9.1) is $\mathbf{v}$-prepared if (f, y, u) is prepared at $\mathbf{v}(f, y, u)$.*

(2) We say that (X, Z) is $\mathbf{v}$-admissible at x if there exists a $\mathbf{v}$-prepared prelabel (f, y, u) of (X, Z) at x. By Theorem 7.24, (X, Z) is $\mathbf{v}$-admissible at x if $R = \mathcal{O}_{Z,x}$ is complete.

We now extend the above definition to the situation where the old components of $\mathcal{B}$ at x are taken into account. We assume

(10.2) $\qquad\qquad\qquad \mathrm{Spec}(R/\langle u \rangle) \not\subset B \quad$ for any $B \in O\mathcal{B}(x)$.

Definition 10.3 *Let (y, u) be a system of regular parameters of R such that (u) is admissible for J.*

(1) For each $B \in O(x)$, choose $l_B \in R$ such that $B = \mathrm{Spec}(R/\langle l_B \rangle) \subset Z = \mathrm{Spec}(R)$. (10.2) implies $l_B \notin \langle u \rangle$ so that $\Delta(l_B, y, u)$ is well-defined (cf. Definition 7.2). We define

$$\Delta^O(y, u) = \text{ the minimal } F\text{-subset containg } \bigcup_{B \in O(x)} \Delta(l_B, y, u).$$

It is easy to see that $\Delta^O(y, u)$ is independent of the choice of l_B. For a prelabel (f, y, u) of (X, Z) at x, let

$$\Delta^O(f, y, u) = \text{ the minimal } F\text{-subset containg } \Delta(f, y, u) \cup \Delta^O(y, u),$$

$$*^O(f, y, u) = *(\Delta^O(f, y, u)) \quad for * = \mathbf{v}, \; \alpha, \; \beta, \; \gamma^{\pm}, \ldots.$$

Note that

(10.3) $\qquad \Delta^O(f, y, u) = \Delta(f^O, y, u) \quad$ *where* $f^O = (f, \; l_B \; (B \in O(x)))$.

(2) Assume that (X, Z) *is* $\mathbf{v}$-*admissible at* x. *Then we define*

$$\beta_x^O(X, Z) = \beta^O(J) := \min_{(f, y, u)} \{\beta^O(f, y, u)| \; (f, y, u) \; is \; \mathbf{v}\text{-prepared}\}.$$

(3) A prelabel (f, y, u) *of* (X, Z) *at* x *is called* O-*admissible if* (f, y, u) *is* $\mathbf{v}$-*prepared and* $\beta_x^O(X, Z) = \beta^O(f, y, u)$. *Such a prelabel exists if and only if* (X, Z) *is* $\mathbf{v}$-*admissible.*

By Lemma 7.6 and Lemma 6.13 (2), for $\mathbf{v}$-prepared (f, y, u) we have

(10.4) $$\beta^O(f, y, u) \in \frac{1}{n_N!} \mathbb{Z}_{\geq 0} \subseteq \mathbb{R} \;,$$

Lemma 10.4 *(1) Let* (f, y, u) *be a* $\mathbf{v}$-*prepared prelable of* (X, Z) *at* x. *Then, for a preparation* $(f, y, u) \to (g, z, u)$ *at a vertex* $v \in \Delta(f, y, u)$, *we have* $\beta^O(f, y, u) = \beta^O(g, z, u)$.

(2) Assume that (X, Z) *is* $\mathbf{v}$-*admissible at* x. *For any integer* $m \geq 1$, *there exists an* δ-*prepared and* O-*admissible label* (f, y, u) *of* (X, Z) *at* x. *Moreover, if* R *is complete, one can make* (f, y, u) *totally prepared.*

Proof (2) is a consequence of (1) in view of Corollary 7.17. We prove (1). Setting

$$\mathbf{v}^O(y, u) = \mathbf{v}(\Delta^O(y, u)) = (\alpha^O(y, u), \beta^O(y, u)),$$

we have

(10.5) $\quad \mathbf{v}^O(f, y, u) = (\alpha^O(f, y, u), \beta^O(f, y, u)) = \begin{cases} \mathbf{v}^O(y, u) & if \; \alpha^O(y, u) \leq \alpha(f, y, u) \\ \mathbf{v}(f, y, u) & if \; \alpha^O(y, u) > \alpha(f, y, u). \end{cases}$

By the $\mathbf{v}$-preparedness of (f, y, u), any vertex $v \in \Delta(f, y, u)$ which is not prepared lies in the range $\{(a_1, a_2) \in \mathbb{R}^2| \; a_1 > \alpha(f, y, u)\}$. Theorem 7.19 implies $\mathbf{v}^O(f, y, u) = \mathbf{v}^O(g, y, u)$ for the normalization $(f, y, u) \to (g, y, u)$ at such v. Thus it suffices to show $\mathbf{v}^O(f, y, u) = \mathbf{v}^O(f, z, u)$ for the dissolution $(f, y, u) \to (f, z, u)$ at v. Write $v = (a, b)$. By the above remark, we have $a > \alpha(f, y, u)$. The dissolution is given by a coordinate transformation:

$$y = (y_1, \ldots, y_r) \to z = (z_1, \ldots, z_r) \quad with \; z_i = y_i + \lambda_i u_1^a u_2^b \quad (\lambda_i \in R)$$

Write $\overline{\alpha} = \alpha^O(y, u)$ for simplicity. For each $B \in O(x)$ choose $l_B \in R$ such that $B = \mathrm{Spec}(R/\langle l_B \rangle)$. We may write

$$l_B = \Lambda_B(y) + u_1^{\overline{\alpha}} \phi_B \quad with \; \Lambda_B(y) = \sum_{1 \leq i \leq r} c_{B,i} \cdot y_i \quad (c_{B,i}, \; \phi_B \in R)$$

and $\phi_B \neq 0$ for some $B \in O(x)$. Then we get

(10.6) $\qquad l_B = \Lambda_B(z) + u_1^{\overline{\alpha}} \phi_B + u_1^a \psi_B \quad for \; some \; \psi_B \in R$

124

In case $\bar{\alpha} = \alpha^O(y, u) \leq \alpha(f, y, u)$, (10.5) implies

$$\mathbf{v}^O(f, y, u) = \mathbf{v}^O(y, u) = \mathbf{v}^O(z, u) = \mathbf{v}^O(f, z, u),$$

where the second equality follows from (10.6) because $a > \alpha(f, y, u) \geq \bar{\alpha}$, and the third follows from $\mathbf{v}(f, y, u) = \mathbf{v}(f, z, u)$ by the $\mathbf{v}$-preparedness of (f, y, u). In case $\bar{\alpha} = \alpha^O(y, u) > \alpha(f, y, u)$, (10.5) implies

$$\mathbf{v}^O(f, y, u) = \mathbf{v}(f, y, u) = \mathbf{v}(f, z, u) = \mathbf{v}^O(f, z, u),$$

where the second equality follows from the $\mathbf{v}$-preparedness of (f, y, u). The third equality holds since by (10.6), we have $\alpha^O(z, u) \geq a > \alpha(f, y, u)$ if $\psi_B \neq 0$ for some $B \in O(x)$, and $\alpha^O(z, u) = \bar{\alpha} > \alpha(f, y, u)$ if $\psi_B = 0$ for all $B \in O(x)$. This completes the proof of Lemma 10.4. $\square$

Lemma 10.5 *Let (f, y, u) be a prelabel of (X, Z) at x. Assume that there is no regular closed subscheme $D \subseteq \{\xi \in X \mid H_X^O(\xi) \geq H_X^O(x)\}$ of dimension 1 with $x \in D$. (In particular this holds if x is isolated in $\{\xi \in X \mid H_X^O(\xi) \geq H_X^O(x)\}$.) Then $\alpha^O(f, y, u) < 1$ and $\epsilon^O(f, y, u) < 1$.*

Proof By corollary 7.17, we prepare (f, y, u) at the vertices in $\{A \in \mathbb{R}^2 \mid |A| \leq 1\}$ to get a label (g, z, u) of (X, Z) at x. Then $\alpha^O(g, z, u) \geq \alpha^O(f, y, u)$ since $\Delta^O(g, z, u) \subset \Delta^O(f, y, u)$. Thus we may replace (f, y, u) with (g, z, u) to assume that f is a standard base of J.

Assume $\alpha^O(f, y, u) \geq 1$. Then, letting $\mathfrak{p} = (y_1, \ldots, y_r, u_1) \subset R$, we have $v_{\mathfrak{p}}(f_j) \geq n_j$ for $j = 1, \ldots, N$ (cf. (7.2)). Since $n_j = n_{(u)}(f_j) \geq v_{\mathfrak{m}}(f_j) \geq v_{\mathfrak{p}}(f_j)$, this implies $v_{\mathfrak{p}}(f_j) = v_{\mathfrak{m}}(f_j)$ for $j = 1, \ldots, N$. This implies by Theorems 2.2 (iv) and 2.3 that $\eta \in X$ and $H_X(\eta) = H_X(x)$, where $\eta \in Z$ is the point corresponding to $\mathfrak{p}$. By the same argument we prove $v_{\mathfrak{p}}(l_B) = 1 = v_{\mathfrak{m}}(l_B)$ for $B \in O(x)$ so that $H_X^O(\eta) = H_X^O(x)$. Thus $\overline{\{\eta\}} = \mathrm{Spec}(R/\mathfrak{p})$ is O-permissible, which contradicts the assumption of the lemma. The assertion $\epsilon^O(f, y, u) < 1$ is shown in the same way. $\square$

Lemma 10.6 *Let (y, u) be a system of regular parameters of R which is strictly admissible for J. Assume*

$$(*) \qquad e_x^O(X) = \dim_k \left(\mathrm{Dir}_x(X) \cap \bigcap_{B \in O(x)} T_x(B) \right) = 2.$$

Then we have $\delta^O(y, u) > 1$. Assume in addition that $\delta(f, y, u) > 1$ (so that (f, y, u) is a label). Then we have $\delta^O(f, y, u) > 1$.

Proof By the assumption on (y, u), we have $I\mathrm{Dir}_x(X) = \langle in_{\mathfrak{m}}(y_1), \ldots, in_{\mathfrak{m}}(y_r)\rangle$. Hence $(*)$ implies

$$l_B \in \langle y_1, \ldots, y_r\rangle + \mathfrak{m}^2 \quad \text{for } B \in O(x).$$

We then easily deduce the first assertion of the lemma. The second assertion is an obvious consequence of the first. $\square$

11 Proof in the case $e_x(X) = \overline{e}_x(X) = 2$, I: some key lemmas

In this section we prepare some key lemmas for the proof of Theorem 5.40.

Let the assumption be those of Setup C in §9. We assume

- $\mathrm{char}(k(x)) = 0$ or $\mathrm{char}(k(x)) \geq \dim(X)/2 + 1$.

- $e_x^O(X) = e_x(X) = \overline{e}_x(X) = 2$.

We fix a label (f, y, u) of (X, Z) at x and adopt the notations of Definition 9.1 (1) and (2). We recall

$$F_i(Y) = in_{\mathfrak{m}}(f_i) \in k[Y] = k[Y_1, \ldots, Y_r] \subset \mathrm{gr}_{\mathfrak{m}}(R). \quad (Y_j = in_{\mathfrak{m}}(y_j))$$

By Lemma 10.6, the assumption $e_x^O(X) = e_x(X) = 2$ implies:

$$(11.1) \qquad\qquad \delta^O(f, y, u) > 1.$$

It also implies that (10.2) is always satisfied so that $\Delta^O(f, y, u)$ is well-defined.

For each $B \in O(x)$, we choose $l_B \in R$ such that $B \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\langle l_B \rangle)$. We study two cases.

Case 1 (point blowup): Consider

$$\pi : Z' = B\ell_x(Z) \to Z \quad \text{and} \quad \pi : X' = B\ell_x(X) \to X.$$

Note that $x \hookrightarrow X$ is $\mathcal{B}$-permissible for trivial reasons. Let $(\mathcal{B}', O')$ be the complete transform of $(\mathcal{B}, O)$ in Z' (cf. Definition 3.15). In this case we have $\mathcal{B}' = \tilde{\mathcal{B}} \cup \{\pi^{-1}(x)\}$, where $\tilde{\mathcal{B}} = \{\tilde{B} | B \in \mathcal{B}\}$ with $\tilde{B}$, the strict transform of B in Z'.

By Theorem 2.14, any point of X' near to x is contained in

$$C' := \mathbb{P}(\mathrm{Dir}_x(X)) \subset \mathbb{P}(T_x(Z)) = \pi_Z^{-1}(x) \subset Z'.$$

Let T_1, T_2 be a pair of new variables over k and let

$$(11.2) \qquad\qquad \psi_{(u)} : C' = \mathbb{P}(\mathrm{Dir}(R/J)) \xrightarrow{\sim} \mathrm{Proj}(k[T_1, T_2]) = \mathbb{P}_k^1$$

be the isomorphism (6.2) which is determined by (u).

Take a closed point $x' \in C'$ near to x. By Theorems 2.10 and 2.13, we have

$$\nu_{x'}^*(X', Z') = \nu_x^*(X, Z) \quad \text{and} \quad e_{x'}(X') \leq 2.$$

Put $R' = \mathcal{O}_{Z', x'}$ with the maximal ideal $\mathfrak{m}'$. Let J' and $\mathfrak{p}'$ be the ideals of R' such that

$$(11.3) \qquad X' \times_{Z'} \mathrm{Spec}(R') = \mathrm{Spec}(R'/J') \quad \text{and} \quad C' \times_{Z'} \mathrm{Spec}(R') = \mathrm{Spec}(R'/\mathfrak{p}').$$

Lemma 11.1 *Assume $\psi_{(u)}(x') = (1 : 0) \in \mathrm{Proj}(k[T_1, T_2])$. Let*

$$y' = (y_1', \ldots, y_r') \text{ with } y_i' = y_i/u_1, \ u_2' = u_2/u_1, \ f' = (f_1', \ldots, f_N') \text{ with } f_i' = f_i/u_1^{n_i}.$$
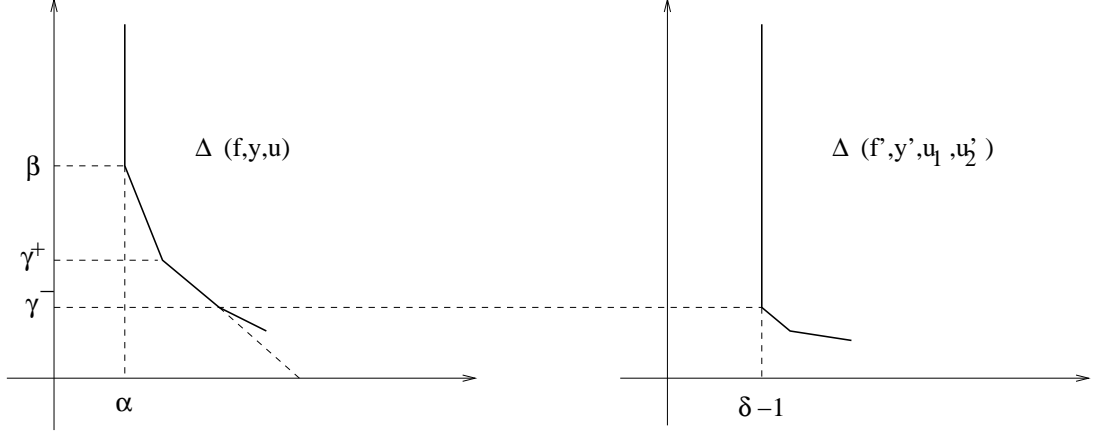
(1) (y', u_1, u_2') is a system of regular parameters of R' such that $\mathfrak{p}' = (y', u_1)$, and $J' = \langle f_1', \ldots, f_N' \rangle$.

(2) If x' is very near to x, then $(f', y', (u_1, u_2'))$ is a prelabel of (X', Z') at x'.

(3) $\Delta(f', y', (u_1, u_2'))$ is the minimal F-subset containg $\Psi(\Delta(f, y, u))$, where

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2, (a_1, a_2) \mapsto (a_1 + a_2 - 1, a_2)$$

for which all vertices move horizontally. We have

$$\beta(f', y', (u_1, u_2')) = \gamma^-(f, y, u) \le \beta(f, y, u).$$

$$\alpha(f', y', (u_1, u_2')) = \delta(f, y, u) - 1.$$

(4) If (f, y, u) is prepared at $\mathbf{w}^-(f, y, u)$, then $(f', y', (u_1, u_2'))$ is $\mathbf{v}$-prepared. If (f, y, u) is prepared along the face E_L, then $(f', y', (u_1, u_2'))$ is δ-prepared, where

$$L : \mathbb{R}^2 \to \mathbb{R} ;\ (a_1, a_2) \to a_1 + 2a_2.$$

If (f, y, u) is totally prepared, so is $(f', y', (u_1, u_2'))$.

(5) Assume that x' is $\mathcal{B}$-near to x. Putting $l_B' = l_B/u_1 \in R'$ for $B \in O(x)$, we have

$$\Delta^O(f', y', (u_1, u_2')) = \Delta(f'^O, y, (u_1, u_2')) \quad \text{with } f'^O = (f', l_B'\ (B \in O(x))).$$

The same assertions as (3) hold replacing Δ by Δ^O and $*$ by $*^{OB}$ for $* = \alpha,\ \beta,\ \gamma^-,\ \delta$.

Proof By Definition 9.1, (y, u) is strictly admissible for J and f is a standard base of J which is admissible for (y, u). Hence (1) has been seen in Setup B in §8. (2) follows from Theorem 8.1 and Theorem 8.3. (3) and (4) follow from Lemma 9.3. As for (5), the assumption implies $O'(x') = \{B' \mid B \in O(x)\}$ with B', the strict transform of B in Z' and we have

$$B' \times_{Z'} \operatorname{Spec}(R') = \operatorname{Spec}(R'/\langle l_B' \rangle) \subset \operatorname{Spec}(R').$$

This implies the first assertion of (5) by (10.3). The second assertion of (5) then follows from the first. $\square$

The following lemma is shown in the same way as the previous lemma except that the last assertion of (5) follows from Lemma 10.5.

Lemma 11.2 *Assume $\psi_{(u)}(x') = (0 : 1) \in \mathrm{Proj}(k[T_1, T_2])$ and put*

$$z' = (z'_1, \ldots, z'_r) \text{ with } z'_i = y_i/u_2, \ u'_1 = u_1/u_2, \ g' = (g'_1, \ldots, g'_N) \text{ with } f'_i = f_i/u_2^{n_i}.$$

(1) *(z', u'_1, u_2) is a system of regular parameters of R' such that $\mathfrak{p} = (z', u_2)$, and $J' = \langle g'_1, \ldots, g'_N \rangle$.*

(2) *If x' is very near to x, then $(g', z', (u'_1, u_2))$ is a prelabel of (X', Z') at x'.*

(3) *$\Delta(g', z', (u'_1, u_2))$ is the minimal F-subset containg $\Psi(\Delta(f, y, u))$, where*

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^2, (a_1, a_2) \mapsto (a_1, a_1 + a_2 - 1)$$

We have

$$\alpha(g', z', (u'_1, u_2)) = \alpha(f, y, u).$$
$$\beta(g', z', (u'_1, u_2)) \leq \beta(f, y, u) + \alpha(f, y, u) - 1.$$

(4) *If (f, y, u) is prepared at $\mathbf{w}^+(f, y, u)$, then $(g', z', (u'_1, u_2))$ is $\mathbf{v}$-prepared. If (f, y, u) is prepared along the face $E_{L'}$, then $(f', y', (u_1, u'_2))$ is δ-prepared, where*

$$L' : \mathbb{R}^2 \to \mathbb{R} \ ; \ (a_1, a_2) \to 2a_1 + a_2.$$

If (f, y, u) is totally prepared, so is $(g', z', (u'_1, u_2))$.

(5) *Assume that x' is $\mathcal{B}$-near to x. Then*

$$\beta^O(g', z', (u'_1, u_2)) \leq \beta^O(f, y, u) + \alpha^O(f, y, u) - 1.$$

If there is no regular closed subscheme $D \subseteq \{\xi \in X |\ H_X^O(\xi) \geq H_X^O(x)\}$ of dimension 1 with $x \in D$, then $\alpha^O(f, y, u) < 1$ so that

$$\beta^O(g', z', (u_1, u'_2)) < \beta^O(f, y, u).$$

Now let η' be the generic point of C'. By Theorem 2.13 and Theorem 2.6, if η' is near to x, we have $e_{\eta'}(X') \leq 1$. Write $R'_{\eta'} = \mathcal{O}_{X', \eta'}$ with the maximal ideal $\mathfrak{m}'_{\eta'}$. Note $(y', u_1) = (y'_1, \ldots, y'_r, u_1)$ is a system of regular parameters of $R'_{\eta'}$.

Lemma 11.3 *Let $\Delta(f', y', u_1)$ be the characteristic polyhedron for $(R'_{\eta'}, JR'_{\eta'})$.*

(1) *$\Delta(f', y', u_1) = [\delta(f, y, u) - 1, \infty) \subset \mathbb{R}_{\geq 0}$.*

Assume (f, y, u) is δ-prepared.

(2) *$\Delta(f', y', u_1)$ is well prepared.*

(3) *η' is near to x if and only if $\delta(f, y, u) \geq 2$. If η' is near to x, we have*

$$v_{\mathfrak{m}'_{\eta'}}(f'_i) = v_\mathfrak{m}(f_i) = n_i \quad \text{for } i = 1, \ldots, N.$$

(4) η' *is very near to x if and only if $\delta(f, y, u) > 2$. If η' is very near to x, then $(f', (y', u_1))$ is a well-prepared label of (X', Z') at η'.*

Proof   The lemma is a special case of Lemma 9.4. $\square$

Case 2 (curve blowup): Let the assumption be as in the beginning of this section. Assume given a regular curve $C \subset X$ containing x which is $\mathcal{B}$-permissible (cf. Definition 3.5) and such that

$$C \times_Z \mathrm{Spec}(R) = \mathrm{Spec}(R/\mathfrak{p}) \quad \text{with } \mathfrak{p} = (y, u_1) = (y_1, \ldots, y_r, u_1).$$

By Theorem 2.6, the assumption $e_x(X) = 2$ implies

$$e_\eta(X) \leq 1 \quad \text{where } \eta \text{ is the generic point of } C.$$

Consider
$$\pi : Z' = B\ell_C(Z) \to Z \quad \text{and} \quad \pi : X' = B\ell_C(X) \to X.$$

Let $(\mathcal{B}', O')$ be the complete transform of $(\mathcal{B}, OI)$ in Z' (cf. Definition 3.15). In this case we have $\mathcal{B}' = \tilde{\mathcal{B}} \cup \{\pi^{-1}(C)\}$, where $\tilde{\mathcal{B}} = \{\tilde{B} | B \in \mathcal{B}\}$ with $\tilde{B}$, the strict transform of B in Z'. By Theorem 2.14, there is the unique point $x' \in \pi^{-1}(x)$ possibly near to x, given by

$$x' := \mathbb{P}(\mathrm{Dir}_x(X)/T_x(C)) \subset \mathrm{Proj}(T_x(Z)/T_x(C))) = \pi_Z^{-1}(x) \subset Z'.$$

In what follows we assume x' near to x. By Theorems 2.10 and 2.13, we have

$$\nu^*_{x'}(X', Z') = \nu^*_x(X, Z) \quad \text{and} \quad e_{x'}(X') \leq 2.$$

Let $R' = \mathcal{O}_{Z',x'}$ with the maximal ideal $\mathfrak{m}'$, and let $J' \subset R'$ be the ideal such that

$$X' \times_{Z'} \mathrm{Spec}(R') = \mathrm{Spec}(R'/J').$$

As is seen in Setup B in §8,

$$(y', u) = (y'_1, \ldots, y'_r, u_1, u_2) = (y_1/u_1, \ldots, y_r/u_1, u_1, u_2)$$

is a system of regular parameters of R'.

Lemma 11.4   *(1) If (f, y, u) is $\mathbf{v}$-prepared, then $v_\mathfrak{p}(f_i) = v_\mathfrak{m}(f_i) = n_{(u)}(f_i)$ and*

$$J' = \langle f'_1, \ldots, f'_N \rangle \quad \text{with } f'_i := f_i/u_1^{n_i} \in R'.$$

(2) If x' is very near to x, then (f', y', u) is a prelabel of (X', Z') at x'.

(3) If (f, y, u) is $\mathbf{v}$-prepared (resp. δ-prepared, resp. totally prepared), so is (f', y', u).

(4) We have

$$\Delta(f', y', u) = \Psi(\Delta(f, y, u)) \quad \text{with } \Psi : \mathbb{R}^2 \to \mathbb{R}^2, (a_1, a_2) \mapsto (a_1 - 1, a_2),$$

$$\beta(f', y', u) = \beta(f, y, u) \quad \text{and} \quad \alpha(f', y', u) = \alpha(f, y, u) - 1.$$

If x' is $\mathcal{B}$-near to x, the same assertions hold replacing Δ by Δ^O and $*$ by $*^{OB}$ for $* = \alpha, \beta$.

Proof The first assertion of (1) follows from Theorem 7.18 and the second from the first (cf. Setup B in §8). (2) follows from Theorems 8.1 and 8.3. To show (3) and (4), write, as in (6.3):

$$(11.4) \qquad f_i = \sum_{(A,B)} C_{i,A,B}\, y^B u^A \quad \text{with} \quad C_{i,A,B} \in R^\times \cup \{0\}$$

We compute

$$(11.5) \qquad f_i' = f_i/u_1^{n_i} = \sum_{(A,B)} C_{i,A,B}\, (y')^B u_2^{a_2} u_1^{a_1 + (|B| - n_i)} \quad \text{with } A = (a_1, a_2).$$

This immediately implies the first assertion of (4). Then (3) is shown in the same way as Lemma 9.3 (2). Finally the last assertion of (4) is shown in the same way as Lemma 11.1 (5). This completes the proof of Lemma 11.4. $\square$

Lemma 11.5 *Assume that $(f, (y, u_1))$ is a well-prepared label of (X, Z) at η (note that this implies $e_\eta(X) = 1$). Let $C' = \mathrm{Spec}(R'/\mathfrak{p}')$ with $\mathfrak{p}' = (y', u_1) \subset R'$ and let η' be the generic point of C'. Then $C' \subset X'$ and η' is the unique point of X' possibly near to η. If η' is very near to η, then $(f', (y', u_1))$ is a well-prepared label of (X', Z') at η'.*

Proof The first assertion is a direct consequence of Theorem 2.14. The second assertion follows from Lemma 9.3 applied to the base change via $\eta \to C$ of the diagram

$$
\begin{array}{ccccc}
C' & \hookrightarrow & X' & \hookrightarrow & Z' \\
\downarrow & & \downarrow \pi & & \downarrow \pi \\
C & \hookrightarrow & X & \hookrightarrow & Z
\end{array}
$$

$\square$

12 Proof in the case $e_x(X) = \overline{e}_x(X) = 2$, II: separable residue extensions

In this section we prove Theorem 12.7 below, which implies Key Theorem 5.40 under the assumption that the residue fields of the initial points of $\mathcal{X}_n$ are separably algebraic over that of $\mathcal{X}_1$. The proof is divided into two steps.

Step 1 (one fundamental unit): Let the assumptions and notations be as in the beginning of the previous section. Assume given a fundamental unit of $\mathcal{B}$-permissible blowups as in Definition 5.34:

(12.1)
$$
\begin{array}{ccccccccc}
\mathcal{B} = & \mathcal{B}_0 & & \mathcal{B}_1 & & \mathcal{B}_2 & & \mathcal{B}_{m-1} & & \mathcal{B}_m \\[4pt]
Z = & Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \leftarrow & Z_{m-1} & \xleftarrow{\pi_m} & Z_m \\
& \cup & & \cup & & \cup & & \cup & & \cup \\
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{m-1} & \xleftarrow{\pi_m} & X_m \\
& \uparrow & & \cup & & \cup & & \cup & & \uparrow \\
x = & x_0 & \leftarrow & C_1 & \xleftarrow{\sim} & C_2 & \xleftarrow{\sim} \ldots \xleftarrow{\sim} & C_{m-1} & \leftarrow & x_m
\end{array}
$$

We denoted it by $(\mathcal{X}, \mathcal{B})$. For $2 \leq q \leq m-1$, let η_q be the generic point of C_q and let $x_q \in C_q$ be the image of x_m. By definition the following conditions hold:

- For $1 \leq q \leq m$, x_q is near to x_{q-1} and $k(x_{q-1}) \simeq k(x_q)$.

- For $1 \leq q \leq m-1$, $C_q = \{\xi \in \phi_q^{-1}(x)|\ H_{X_q}^O(\xi) = H_X^O(x)\}$ with $\phi_q : X_q \to X$.

- For $1 \leq q \leq m$, $H_{X_q}^O(x_q) = H_X^O(x)$ and $e_{x_q}^O(X_q) = e_{x_q}(X_q) = 2$.

- For $1 \leq q \leq m-1$, $H_{X_q}^O(\eta_q) = H_X^O(x)$.

- For $1 \leq q \leq m-2$, $e_{\eta_q}^O(X_q) = e_{\eta_q}(X_q) = 1$.

Let $R_q = \mathcal{O}_{Z_q, x_q}$ with the maximal ideal $\mathfrak{m}_q$, and let J_q and $\mathfrak{p}_q$ be the ideals of R_q such that

(12.2) $\qquad X_q \times_{Z_q} \operatorname{Spec}(R_q) = \operatorname{Spec}(R_q/J_q) \quad$ and $\quad C_q \times_{Z_q} \operatorname{Spec}(R_q) = \operatorname{Spec}(R_q/\mathfrak{p}_q)$.

Let $R_{\mathfrak{p}_q}$ be the localization of R_q at $\mathfrak{p}_q$ and $J_{\mathfrak{p}_q} = J_q R_{\mathfrak{p}_q}$. Let T_1, T_2 be a pair of new variables over k and consider the isomorphism (11.2):

(12.3) $\qquad\qquad \psi_{(u)} : C_1 = \mathbb{P}(\operatorname{Dir}_x(X)) \xrightarrow{\sim} \operatorname{Proj}(k[T_1, T_2]) = \mathbb{P}_k^1$

Definition 12.1 *(1) A prelabel (resp. label) Λ of $(\mathcal{X}, \mathcal{B})$ is a prelabel (resp. label) $\Lambda = (f, y, u)$ of (X, Z) at x. When x_1 is a k-rational point of C_1, the homogeneous coordinates of $\psi_{(u)}(x_1) \in \operatorname{Proj}(k[T_1, T_2])$ are called the coordinates of $(\mathcal{X}, \Lambda)$.*

(2) We say that $(\mathcal{X}, \mathcal{B})$ is $\mathbf{v}$-admissible if (X, Z) is $\mathbf{v}$-admissible in the sense of Definition 10.2 (2). A prelabel (f, y, u) of $(\mathcal{X}, \mathcal{B})$ is O-admissible if it is O-admissible as a prelabel of (X, Z) at x (cf. Definition 10.3 (3)).

We remark that the coordinates of $(\mathcal{X}, \mathcal{B}; \Lambda)$ depend only on (u), not on (f, y).

Lemma 12.2 *Let $\Lambda = (f, y, u)$ be a label of $(\mathcal{X}, \mathcal{B})$ which is $\mathbf{v}$-prepared and δ-prepared and prepared along the face E_L, where $L : \mathbb{R}^2 \to \mathbb{R}$; $(a_1, a_2) \to a_1 + 2a_2$. Assume the coordinates of $(\mathcal{X}, \mathcal{B}; \Lambda)$ are $(1 : 0)$. Let*

$$u_2' = u_2/u_1, \; y^{(q)} = (y_1^{(q)}, \dots, y_r^{(q)}) \; (y_i^{(q)} = y_i/u_1^q), \; f^{(q)} = (f_1^{(q)}, \dots, f_p^{(q)}) \; (f_i^{(q)} = f_i/u_1^{qn_i}).$$

For $1 \le q \le m$, the following holds:

(1) $(f^{(q)}, (y^{(q)}, (u_1, u_2')))$ is a $\mathbf{v}$-prepared and δ-prepared label of (X_q, Z_q) at x_q.

(2) $\Delta^O(f^{(q)}, y^{(q)}, (u_1, u_2'))$ is the minimal F-subset containg $\Psi_q(\Delta^O(f, y, u))$, where
$$\Psi_q : \mathbb{R}^2 \to \mathbb{R}^2 , \quad (a_1, a_2) \mapsto (a_1 + a_2 - q, a_2)$$

   We have
$$\beta^O(f^{(q)}, y^{(q)}, (u_1, u_2')) = \gamma^{-O}(f, y, u) \le \beta^O(f, y, u).$$
$$\alpha^O(f^{(q)}, y^{(q)}, (u_1, u_2')) = \delta^O(f, y, u) - q.$$

(3) For $q \le m - 1$, $\mathfrak{p}_q = (y^{(q)}, u_1) = (y_1^{(q)}, \dots, y_r^{(q)}, u_1)$ and $v_{\mathfrak{p}_q}(f_i^{(q)}) = n_i$ for $i = 1, \dots, N$. For $q \le m - 2$, $(f^{(q)}, y^{(q)}, u_1)$ is a well prepared label of (X_q, Z_q) at η_q.

Proof By Lemma 11.1, $(f^{(1)}, (y^{(1)}, (u_1, u_2')))$ is a label of (X_1, Z_1) at x_1 which is $\mathbf{v}$-prepared and δ-prepared. By Lemma 11.3, $(f^{(1)}, (y^{(1)}, (u_1, u_2')))$ is a well prepared label of (X_1, Z_1) at η_1 if η_1 is very near to x_1. Then the lemma follows from Lemmas 11.4 and 11.5, applied to $X_q \leftarrow X_{q+1}$ in place of $X \leftarrow X'$. $\square$

Definition 12.3 *Call x or $(\mathcal{X}, \mathcal{B})$ quasi-isolated if there is no regular closed subscheme*
$$D \subseteq \{\xi \in X | \; H_X^O(\xi) \ge H_X^O(x)\}$$
of dimension 1 with $x \in D$.

We note that this holds in particular if x is isolated in $\{\xi \in X | \; H_X^O(\xi) \ge H_X^O(x)\}$, and especially if x lies in the O-Hilbert-Samuel locus and is isolated in it.

Lemma 12.4 *Let $\Lambda = (f, (y, u))$ be a label of $(\mathcal{X}, \mathcal{B})$ which is $\mathbf{v}$-prepared and δ-prepared and prepared along the face $E_{L'}$, where $L' : \mathbb{R}^2 \to \mathbb{R}$; $(a_1, a_2) \to 2a_1 + a_2$. Assume the coordinates of $(\mathcal{X}, \mathcal{B}; \Lambda)$ are $(0 : 1)$. Set*

$$u_1' = u_1/u_2, \; z^{(q)} = (z_1^{(q)}, \dots, z_r^{(q)}) \; (z_i^{(q)} = y_i/u_2^q), \; g^{(q)} = (g_1^{(q)}, \dots, g_N^{(q)}) \; (g_i^{(q)} = f_i/u_2^{qn_i}).$$

For $1 \le q \le m$, the following hold:

(1) $(g^{(q)}, (z^{(q)}, (u_1', u_2)))$ is a $\mathbf{v}$-prepared and δ-prepared label of (X_q, Z_q) at x_q.

(2) $\Delta^O(g^{(q)}, z^{(q)}, (u_1', u_2))$ is the minimal F-subset containg $\Phi_q(\Delta^O(f, y, u))$, where
$$\Phi_q : \mathbb{R}^2 \to \mathbb{R}^2, (a_1, a_2) \mapsto (a_1, a_1 + a_2 - q)$$

   and we have
$$\alpha^O(g^{(q)}, z^{(q)}, (u_1', u_2)) = \alpha^O(f, y, u),$$
$$\beta^O(g^{(q)}, z^{(q)}, (u_1', u_2)) \le \beta^O(f, y, u) + \alpha^O(f, y, u) - q.$$

(3) If x is quasi-isolated, then $\alpha^O(f, y, u) < 1$ so that

$$\beta^O(g^{(q)}, z^{(q)}, (u'_1, u_2)) < \beta^O(f, y, u).$$

Proof This is shown in the same way as Lemma 12.2 using Lemma 11.2 instead of 11.1.
□

Proposition 12.5 *Assume that $(\mathcal{X}, \mathcal{B})$ is $\mathbf{v}$-admissible and quasi-isolated, and that $k(x) = k(x_1)$. Then, for all $q = 1, \ldots, m$, (X_q, Z_q) is $\mathbf{v}$-admissible at x_q and*

$$\beta^O_{x_q}(X_q, Z_q) \leq \beta^O_x(X, Z).$$

Proof By the assumption we can take an O-admissible prelabel $\Lambda = (f, y, u)$ of (X, Z), and the coordinates of $(\mathcal{X}, \Lambda)$ are either $(1 : 0)$, or $(0 : 1)$, or $(1 : \lambda)$ for some $\lambda \in k$. Assume we are in the first case. By Lemma 10.4, after preparation we may assume that (f, y, u) is prepared along the δ-face and the face E_L in Lemma 12.2. By Lemmas 12.2 and 12.4, applied to $X_q \leftarrow X_{q+1}$ for $q = 1, \ldots, m$ in place of $X \leftarrow X'$, we get a label $\Lambda_q := (f^{(q)}, (y^{(q)}, (u_1, u'_2)))$ of (X_q, Z_q) at x_q which is $\mathbf{v}$-prepared and δ-prepared and deduce

$$\beta^O_{x_q}(X_q, Z_q) \leq \beta^O(f^{(q)}, y^{(q)}, (u_1, u'_2)) = \beta^O(f^{(1)}, y^{(1)}, (u_1, u'_2))$$
$$= \gamma^{-O}(f, y, u) \leq \beta^O(f, y, u) = \beta^O_x(X, Z).$$

where the first inequality comes from $\mathbf{v}$-preparedness of Λ_q. The case that the coordinates of $(\mathcal{X}, \Lambda)$ are $(0 : 1)$ is shown in the same way by using Lemma 12.4 instead of Lemma 12.2.

Now assume that the coordinates of $(\mathcal{X}, \Lambda)$ are $(1 : \lambda)$. Let $\tilde{u}_2 := u_2 - \phi u_1$ for some $\phi \in R = \mathcal{O}_{Z,x}$ such that $\phi \mod \mathfrak{m} = \lambda$. Then $(f, y, (u_1, \tilde{u}_2))$ is a prelabel of (X, Z) at x and $\psi_{(u_1, \tilde{u}_2)}(x_1) = (1 : 0) \in \mathrm{Proj}(k[T_1, T_2])$ so that the coordinate of $(f, y, (u_1, \tilde{u}_2))$ is $(1 : 0)$. Hence the proof is reduced to the first case in view of the following lemma.

Lemma 12.6 *Let $\tilde{u}_2 = u_2 - \phi u_1$ for $\phi \in R = \mathcal{O}_{Z,x}$. Then $\mathbf{v}(f, y, u) = \mathbf{v}(f, y, (u_1, \tilde{u}_2))$. For $v = \mathbf{v}(f, y, u)$ we have*

$$in_v(f)_{(y, (u_1, \tilde{u}_2))} = in_v(f)_{(y,u)|U_2 = \tilde{U}_2} \in k[Y, U_1, \tilde{U}_2] = \mathrm{gr}_{\mathfrak{m}}(R),$$

where $\tilde{U}_2 = in_{\mathfrak{m}}(\tilde{u}_2) \in \mathrm{gr}_{\mathfrak{m}}(R)$. Hence, if (f, y, u) is $\mathbf{v}$-prepared, then so is $(f, y, (u_1, \tilde{u}_2))$. We also have $\mathbf{v}^O(f, y, u) = \mathbf{v}^O(f, y, (u_1, \tilde{u}_2))$ and hence if (f, y, u) is O-admissible, then so is $(f, y, (u_1, \tilde{u}_2))$. .

Proof We compute

$$
\begin{aligned}
u_1^{a_1} u_2^{a_2} &= u_1^{a_1} (\tilde{u}_2 + u_1 \phi)^{a_2} \\
&= u_1^{a_1} \sum_{m=0}^{a_2} \binom{a_2}{m} u_1^m \tilde{u}_2^{a_2-m} \phi^m \\
&= u_1^{a_1} \tilde{u}_2^{a_2} + \sum_{m=1}^{a_2} \binom{a_2}{m} u_1^{a_1+m} \tilde{u}_2^{a_2-m} \phi^m.
\end{aligned}
$$

133

This implies that the vertices on the line $\{(a_1, a_2)| \ a_1 = \alpha(f, y, u)\}$ together with the initial forms at it, are not affected by the transformation $(f, y, u) \to (f, y, (u_1, \tilde{u}_2))$. Thus the first assertion follows. The last assertion is shown by the same argument applied to f^O instead of f (cf. (10.3)) . $\square$

Step 2 (a chain of fundamental units): In this step we consider the following situation: Assume given a chain of fundamental units of $\mathcal{B}$-permissible blowups (cf. Definition 5.39):

$$(12.4) \qquad\qquad (\mathcal{X}_0, \mathcal{B}_0) \leftarrow (\mathcal{X}_1, \mathcal{B}_1) \leftarrow (\mathcal{X}_2, \mathcal{B}_2) \leftarrow \dots$$

where each $(\mathcal{X}_q, \mathcal{B}_q)$ is as (12.1). For $q \geq 0$, let $(x^{(q)}, X^{(q)}, Z^{(q)}, \mathcal{B}^{(q)})$ be the initial part of $(\mathcal{X}_q, \mathcal{B}_q)$ and m_q be the length of $\mathcal{X}_q$. Let $R_q = \mathcal{O}_{Z^{(q)}, x^{(q)}}$ with the maximal ideal $\mathfrak{m}_q$ and

$$X^{(q)} \times_{Z^{(q)}} \operatorname{Spec}(R_q) = \operatorname{Spec}(R_q/J_q) \quad \text{for an ideal } J_q \subset R_q.$$

Theorem 12.7 *Assume that for all $q \geq 0$, $k(x^{(q)})$ is separably algebraic over $k(x^{(0)})$ and x_q is quasi-isolated (see Definition 12.3). Then the sequence (12.4) stops after finitely many steps.*

Proof By Lemmas 1.27 and 1.37, it suffices to show the claim after replacing each $\mathcal{X}_q$ by its base changes via $\operatorname{Spec}(\hat{R}^{ur}) \to Z^{(0)}$, where $\hat{R}^{ur}$ is the maximal unramified extension of the completion of $R = R_0$ (here we use that Z is excellent). Thus we may assume that $k(x^{(q)}) = k(x)$ and that R_0 is complete and hence that $(\mathcal{X}_0, \mathcal{B}_0)$ is **v**-admissible. Proposition 12.5 implies that for all $q \geq 0$, $(\mathcal{X}_q, \mathcal{B}_q)$ is **v**-admissible so that $\beta_q := \beta_{x^{(q)}}^O(X^{(q)}, Z^{(q)})$ is well-defined and

$$(12.5) \qquad\qquad \beta_{q+1} \leq \beta_q \quad \text{for all } q \geq 0.$$

Note $\beta_q \in 1/n_N! \cdot \mathbb{Z}^2 \subset \mathbb{R}^2$, where $\nu^*(J_0) = \nu^*(J_q) = (n_1, \dots, n_N, \infty, \dots)$ (cf. Lemma 7.6, Lemma 6.13 and Theorem 2.10 (3)). Hence the strict inequality may occur in (12.5) only for finitely many q. Hence we may assume $\beta_q = \beta_0$ for all $q \geq 0$.

In what follows, for a prelabel $(g, (z, u))$ of $\mathcal{X}_0$ with $g = (g_1, \dots, g_N)$ and $z = (z_1, \dots, z_r))$ and for $q \geq 0$, we write

$$g^{(q)} = (g_1^{(q)}, \dots, g_N^{(q)}) \quad \text{with } g_i^{(q)} = \begin{cases} g_i/u_1^{n_i(m_1 + \dots + m_q)} & , \quad q \geq 1, \\ g_i & , \quad q = 0. \end{cases}$$

$$z^{(q)} = (z_1^{(q)}, \dots, z_r^{(q)}) \quad \text{with } z_i^{(q)} = \begin{cases} z_i/u_1^{m_0 + \dots + m_{q-1}} & , \quad q \geq 1, \\ z_i & , \quad q = 0. \end{cases}$$

By the completeness of R_0, we can choose a totally prepared and O-admissible label $\Lambda_0 = (f, y, u))$ of $(\mathcal{X}_0, \mathcal{B}_0)$. By the assumption (a), $\beta_0 = \beta_1$ implies by Lemma 12.4 (2) that the coordinate of $(\mathcal{X}_0, \Lambda_0)$ must be $(1 : -\lambda_0)$ for some $\lambda_0 \in k$. Put $v_1 = u_2 + \phi_0 u_1$ for a lift $\phi_0 \in R$ of λ_0 and prepare $(f, y, (u_1, v_1))$ to get a totally prepared label $\Lambda_0' = (g, z, (u_1, v_1))$ of $(\mathcal{X}_0, \mathcal{B}_0)$. By Lemma 12.6, Λ_0' is O-admissible and the coordinate of $(\mathcal{X}_0, \Lambda_0')$ is $(1 : 0)$. Lemma 12.2 (1) implies that $\Lambda_1 = (g^{(1)}, z^{(1)}, (u_1, v_1/u_1))$ is a totally prepared label of $\mathcal{X}_1$. By the assumption (a), $\beta_0 = \beta_1 = \beta_2$ implies by Lemma 12.4 (2) that the coordinate of $(\mathcal{X}_1, \Lambda_1)$ is $(1 : -\lambda_1)$ for some $\lambda_1 \in k$. Put $v_2 = v_1 + \phi_1 u_1^2 = $

134

134

$u_2 + \phi_0 u_1 + \phi_1 u_1^2$ for a lift $\phi_1 \in R$ of λ_1. Prepare $(f, y, (u_1, v_2))$ to get a totally prepared label $\Lambda_0'' = (h, w, (u_1, v_2))$ of $(\mathcal{X}_0, \Lambda_0')$. Then Λ_0'' is O-admissible by Lemma 12.6 and $\Lambda_1' = (h^{(1)}, (w^{(1)}, (u_1, v_2/u_1)))$ is a totally prepared label of $\mathcal{X}_1$ by Lemma 12.2 (1). Moreover, the coordinate of $(\mathcal{X}_0, \Lambda_0'')$ and that of $(\mathcal{X}_1, \Lambda_1')$ are both $(1:0)$. Lemma 12.2 (1) then implies that $\Lambda_2 = (h^{(2)}, w^{(2)}, (u_1, v_2/u_1^2))$ is a totally prepared label of $\mathcal{X}_2$. The same argument repeats itself to imply the following:

Claim 12.8 *Assume that the sequence* (12.4) *proceeds in infinitely many steps and that $\beta_q = \beta_0$ for all $q \geq 0$. Then there exists a sequence $\phi_0, \phi_1, \phi_2, \dots$ of elements in R for which the following holds: Recalling that R is complete, set*

$$v = \lim_{q \to \infty} \left(u_2 + \phi_0 u_1 + \phi_1 u_1^2 + \dots + \phi_{q-1} u_1^q \right) \; \in \; R \, .$$

Prepare $(f, y, (u_1, v)))$ to get a totally prepared label $(\hat{f}, \hat{y}, (u_1, v))$ of $(\mathcal{X}_0, \mathcal{B}_0)$. Then $(\hat{f}, \hat{y}, (u_1, v))$ is O-admissible and $\hat{\Lambda}_q = (\hat{f}^{(q)}, \hat{y}^{(q)}, (u_1, v^{(q)}))$ is a totally prepared label of $(\mathcal{X}_q, \mathcal{B}_q)$ and the coordinate of $(\mathcal{X}_q, \hat{\Lambda}_q)$ is $(1:0)$ for all $q \geq 0$. Here $v^{(q)} = v/u_1^q$.

We now write $(f, y, (u_1, u_2))$ for $(\hat{f}, \hat{y}, (u_1, v))$. Lemma 12.2 implies that for all $q \geq 0$, $(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)}))$ is a totally prepared label of $(\mathcal{X}_q, \mathcal{B}_q)$. Moreover $\Delta^O(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)}))$ is the minimal F-subset of $\mathbb{R}^2$ containing

$$T_q(\Delta^O(f^{(q-1)}, y^{(q-1)}, (u_1, u_2^{(q-1))}))), \quad T_q : \mathbb{R}^2 \to \mathbb{R}^2 \; ; \; (a_1, a_2) \to (a_1 + a_2 - m_{q-1}, a_2),$$

where m_{q-1} is the length of $\mathcal{X}_{q-1}$. This implies that $\epsilon^O(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)})) = \epsilon^O(f, y, u) =: \epsilon^O$ for all $q \geq 0$ (see Definition 10.1), and that

$$
\begin{aligned}
\zeta^O(f^{(q)}, y^{(q)}, (u_1, u_2^{(q)})) &= \zeta^O(f^{(q-1)}, y^{(q-1)}, (u_1, u_2^{(q-1)})) + \epsilon^O - m_{q-1} \\
&< \zeta^O(f^{(q-1)}, y^{(q-1)}, (u_1, u_2^{(q-1)})) \, ,
\end{aligned}
$$

for all $q \geq 1$, because $m_{q-1} \geq 1$, and $\epsilon^O < 1$ by Lemma 10.5 and the assumption (a) of Theorem 12.7. This implies that the sequence must stop after finitely many steps as claimed.

13 Proof in the case $e_x(X) = \bar{e}_x(X) = 2$, III: inseparable residue extensions

In this section we complete the proof of key Theorem 5.40 (see Theorem 13.4 below).

Let the assumptions and notations be as in the beginning of Case 1 of §11. Let $\Phi(T_1, T_2) \in k[T_1, T_2]$ be an irreducible homogeneous polynomial corresponding to $x' \in C = \mathrm{Proj}(k[T_1, T_2])$ (cf. (11.2)). We set

$$\Phi = \Phi(U_1, U_2) \in k[U_1, U_2] \subset k[Y, U_1, U_2] = \mathrm{gr}_{\mathfrak{m}}(R).$$

We assume $x' \neq (0 : 1)$ so that U_1 does not divide Φ. Let

$$d = \deg \Phi = [k(x') : k(x)].$$

Choosing a lift $\tilde{\Phi}(U_1, U_2) \in R[U_1, U_2]$ of $\Phi \in k[U_1, U_2]$, set

$$\phi = \tilde{\Phi}(u_1, u_2) \in R \quad \text{and} \quad \phi' = \phi/u_1^{deg\Phi} \in R' \ .$$

The following two lemmas are shown in the same way as Lemma 11.1 (1), (2) and (5).

Lemma 13.1 *Let*

$$y' = (y_1', \ldots, y_r') \ \text{with} \ y_i' = y_i/u_1, \ f' = (f_1', \ldots, f_N') \ \text{with} \ f_i' = f_i/u_1^{n_i}.$$

(1) $(y', (u_1, \phi'))$ *is a system of regular parameters of R' such that $\mathfrak{p} = (y', u_1)$ (cf. (11.3)), and $J' = \langle f_1', \ldots, f_N' \rangle$.*

(2) If x' is very near to x, then $(f', y', (u_1, \phi'))$ is a prelabel of (X', Z') at x'.

Lemma 13.2 *Assume x' is O-near to x. Setting $l_B' = l_B/u_1 \in R'$ for $B \in O(x)$, we have*

$$\Delta^O(f', y', (u_1, \phi')) = \Delta(f'^O, y, (u_1, \phi')) \quad \text{with} \ f'^O = (f', \ l_B' \ (B \in O(x))).$$

Now we state the main result of this section.

Proposition 13.3 *Let $\delta = \delta(f, y, u)$. Assume the following conditions:*

(a) $d \geq 2$ and there is no regular closed subscheme $D \subseteq X_{max}^O$ of dimension 1 with $x \in D$,

(b) (f, y, u) is $\mathbf{v}$-prepared and prepared at $\mathbf{w}^+(f, y, u)$,

(c) $in_\delta(f)_{(y,u)}$ is normalized.

Then there exists a part of a system of regular parameters $z' = (z_1', \ldots, z_r') \subset \mathfrak{m}'$ such that the following holds:

(1) $(f', z', (u_1, \phi'))$ is a $\mathbf{v}$-prepared prelabel of (X', Z') at x',

(2) $\beta^O(f', z', (u_1, \phi')) < \beta^O(f, y, u), \ \alpha(f', z', (u_1, \phi')) = \alpha(f', y', (u_1, \phi')) = \delta - 1,$

(3) $z' = y'$ unless $\delta \in \mathbb{Z}$ and $\delta \geq 2$. In the latter case we have $z_i' - y_i' \in \langle u_1^{\delta-1} \rangle$ for $i = 1, \ldots, r$. In particular $\langle y', u_1 \rangle = \langle z', u_1 \rangle$.

Before proving Proposition 13.3, we now complete the proof of Theorem 5.40. In view of Proposition 12.5 and Theorem 12.7 (and its proof), Theorem 5.40 is an obvious consequence of the following.

Theorem 13.4 *Assume given a fundamental unit of $\mathcal{B}$-permissible blowups (12.1). Assume that $k(x_1) \neq k(x)$ and that there is no regular closed subscheme $D \subseteq X_{max}^O$ of dimension 1 with $x \in D$. (E.g., this holds if x is isolated in X_{max}^O). Assume that (X, Z) is $\mathbf{v}$-admissible at x (cf. Definition 10.3 (3)). Then, for $q = 1, \ldots, m$, (X_q, Z_q) is $\mathbf{v}$-admissible at x_q and*
$$\beta_{x_q}^O(X_q, Z_q) < \beta_x^O(X, Z).$$

Proof By the assumption we can take an O-admissible prelabel $\Lambda = (f, y, u)$. By Lemma 10.4, after preparation we may assume that (f, y, u) is $\mathbf{v}$-prepared and δ-prepared. Let $(f', y', (u_1, \phi'))$ and $(f', z', (u_1, \phi'))$ be as in Lemma 13.1 and Proposition 13.3, applied to $X \leftarrow X_1$ in place of $X \leftarrow X'$. By Claim 9.5, we have (cf. (12.2))
$$\mathfrak{p}_q = \langle y_1'/u_1^{q-1}, \ldots, y_r'/u_1^{q-1}, u_1 \rangle \quad \text{for } q = 1, \ldots, m-1.$$

By Proposition 13.3 (3) and since $\delta(f, y, u) \geq m$ by Corollary 9.6, this implies

(13.1) $$\mathfrak{p}_q = \langle z_1'/u_1^{q-1}, \ldots, z_r'/u_1^{q-1}, u_1 \rangle \quad \text{for } q = 1, \ldots, m-1.$$

We prepare $(f', z', (u_1, \phi'))$ at all vertices and the faces lying in
$$\{A \in \mathbb{R}^2 \mid |A| \leq |\mathbf{v}(f', z', (u_1, \phi'))|\}$$

to get a $\mathbf{v}$-prepared and δ-prepared label $(g, w, (u_1, \phi'))$ of (X_1, Z_1) at x_1. Note that

(13.2) $$w_i - z_i' \in \langle u_1^\gamma \rangle \ (1 \leq i \leq r) \quad \text{for } \gamma \in \mathbb{Z}_{\geq 0}, \ \gamma > \alpha(f', z', (u_1, \phi')) = \delta - 1.$$

By Lemma 10.4, we have
$$\beta^O(g, w, (u_1, \phi')) = \beta^O(f', z', (u_1, \phi')).$$

For $q = 1, \ldots, m$, let
$$g^{(q)} = (g_1^{(q)}, \ldots, g_N^{(q)}) \ (g_i^{(q)} = g/u_1^{n_i(q-1)}), \quad w^{(q)} = (w_1^{(q)}, \ldots, w_N^{(q)}) \ (w_i^{(q)} = w/u_1^{(q-1)}).$$

Then (13.1) and (13.2) imply $\mathfrak{p}_q = \langle w_1^{(q)}, \ldots, w_r^{(q)}, u_1 \rangle$ for $q = 1, \ldots, m-1$. For $q \geq 1$, Lemma 11.4, applied to $X_q \leftarrow X_{q+1}$ in place of $X \leftarrow X'$, implies that $\Lambda_q := (g^{(q)}, w^{(q)}, (u_1, \phi'))$ is a label of (X_q, Z_q) at x_q which is $\mathbf{v}$-prepared and δ-prepared. Then we get
$$\beta_{x_q}^O(X_q, Z_q) \leq \beta^O(g^{(q)}, w^{(q)}, (u_1, \phi')) = \beta^O(g, w, (u_1, \phi')) = \beta^O(f', z', (u_1, \phi'))$$
$$< \beta^O(f, y, u) = \beta_x^O(X, Z),$$

where the first inequality (resp. equality) comes from $\mathbf{v}$-preparedness of Λ_q (resp. Lemma 11.4 (4)). This completes the proof of the theorem. $\square$

Now we start the proof of Proposition 13.3. We may write

$$(13.3) \qquad in_\delta(f_i)_{(y,u)} = F_i(Y) + \sum_{|B|<n_i} P_{i,B}(U) \cdot Y^B \quad \in k[Y, U_1, U_2],$$

where $P_{i,B}(U) \in k[U_1, U_2]$, for $B \in \mathbb{Z}_{\geq 0}^r$ with $|B| < n_i$, is either 0 or homogeneous of degree $\delta(n_i - |B|)$. Write

$$(13.4) \qquad P_{i,B}(U) = \Phi^{e_i(B)} \cdot Q_{i,B}(U),$$

where $e_i(B) \in \mathbb{Z}_{\geq 0}$ and $Q_{i,B}(U) \in k[U_1, U_2]$ is either 0, or homogeneous of degree $(\delta - d \cdot e_i(B))(n_i - |B|)$ and not divisible by Φ. Then we get

$$(13.5) \qquad in_\delta(f_i)_{(y,u)} = F_i(Y) + \sum Q_{i,B}(U) Y^B \Phi^{e_i(B)} \quad \in k[Y, U_1, U_2],$$

From this we compute

$$(13.6) \qquad \begin{aligned} \gamma^+(f, y, u) &= \sup \left\{ \frac{\deg_{U_2} P_{i,B}(1, U_2)}{n_i - |B|} \,\Big|\, 1 \leq i \leq N, \ P_{i,B}(U) \not\equiv 0 \right\}. \\ &= \sup \left\{ \frac{d \cdot e_i(B) + \deg_{U_2} Q_{i,B}(1, U_2)}{n_i - |B|} \,\Big|\, 1 \leq i \leq N, \ Q_{i,B}(U) \not\equiv 0 \right\}. \\ &\geq d \cdot \sup \left\{ \frac{e_i(B)}{n_i - |B|} \,\Big|\, 1 \leq i \leq N, \ Q_{i,B}(U) \not\equiv 0 \right\}. \end{aligned}$$

where $\deg_{U_2}$ denotes the degree of a polynomial in $k[U_2]$. Set

$$q_{i,B} = \tilde{Q}_{i,B}(u_1, u_2) \in R \quad \text{and} \quad q'_{i,B} = q_{i,B}/u_1^{\deg Q_{i,B}} \in R',$$

where $\tilde{Q}_{i,B}(U_1, U_2) \in R[U_1, U_2]$ is a lift of $Q_{i,B}(U) \in k[U_1, U_2]$. Letting $\tilde{F}(Y) \in R[Y]$ be a lift of $F(Y) \in k[Y]$, (13.3) and (13.4) imply

$$(13.7) \qquad f_i = \tilde{F}_i(y) + \sum_{|B|<n_i} y^B u_1^{a_1}(\phi)^{e_i(B)} q_{i,B} + g \quad \text{with } v_L(g)_{(y,u)} > v_L(f_i)_{(y,u)} = n_i,$$

By Lemma 8.2(2) this implies
$$(13.8)$$
$$f'_i = f_i/u_1^{n_i} = \tilde{F}_i(y') + \sum_{|B|<n_i} y'^B u_1^{(\delta-1)(n_i-|B|)}(\phi')^{e_i(B)} q'_{i,B} + g' \quad \text{with } v_\Lambda(g')_{(y',(u_1,\phi'))} > n_i,$$

where $\Lambda : \mathbb{R}^2 \to \mathbb{R}$; $(a_1, a_2) \to \dfrac{a_1}{\delta - 1}$ and $g' = g/u_1^{n_i}$. Noting that

$$(13.9) \qquad q'_{i,B} \in (R')^\times \Leftrightarrow Q_{i,B}(U) \not\equiv 0 \in k[U_1, U_2],$$

we get

$$(13.10) \qquad \begin{aligned} \alpha' &:= \alpha(f', y', u_1, \phi') = \delta - 1 \quad \text{where } \delta := \delta(f, y, u), \\ \beta' &:= \beta(f', y', u_1, \phi') = \inf \left\{ \frac{e_i(B)}{n_i - |B|} \,\Big|\, 1 \leq i \leq N, \ Q_{i,B}(U) \not\equiv 0 \right\}. \end{aligned}$$

138

The same argument, applied to l_B with $B \in O(x)$, shows

$$(13.11) \qquad \alpha^O(f', y', u_1, \phi') = \delta^O(f, y, u) - 1.$$

Let $v' = \mathbf{v}(f', y', (u_1, \phi'))$. Under the identification

$$\mathrm{gr}_{\mathfrak{m}'}(R') = k'[Y', U_1, \Phi'] \quad \left(Y_i' = in_{\mathfrak{m}'}(y_i'), \ U_1 = in_{\mathfrak{m}'}(u_1), \ \Phi' = in_{\mathfrak{m}'}(\phi')\right),$$

we get

$$(13.12) \qquad in_{v'}(f_i') = F_i(Y') + \sum_{e_i(B) = \beta'(n_i - |B|)} \overline{q_{i,B}'} \cdot Y'^B (U_1^{\alpha'} \Phi'^{\beta'})^{n_i - |B|} \ \in k'[Y', U_1, \Phi']$$

where $\overline{q_{i,B}'} \in k' := k(x')$ is the residue class of $q_{i,B}' \in R'$.

Lemma 13.5 *We have*

$$\beta(f, y, u) \geq \gamma^+(f, y, u) \geq d \cdot \beta(f', y', (u_1, \phi')),$$
$$\beta^O(f, y, u) \geq \gamma^{+O}(f, y, u) \geq d \cdot \beta^O(f', y', (u_1, \phi')).$$

Proof The first inequality holds in general (cf. the picture below Definition 10.1) and the second follows from (13.10) and (13.6). This proves the first assertion. The second assertion follows by applying the same argument to l_B for $B \in O(x)$ in view of Lemma 13.2. This completes the proof. $\square$

Corollary 13.6 *If $(f', y', (u_1, \phi'))$ is not solvable at v', Proposition 13.3 holds.*

Proof Indeed, it suffices to take $z' = y'$ in this case. Proposition 13.3 (3) follows from Lemma 11.1 (1). As for (1), Proposition 13.3 (c) implies, in view of (13.5), (13.12), and (13.9), that $(f', y', (u_1, \phi'))$ is normalized at v'. Hence the assumption implies $(f', y', (u_1, \phi'))$ is prepared at v'. It remains to show (2). By Lemma 13.5 it suffices to show $\beta^O(f, y, u) \neq 0$. By (11.1) we have

$$\beta^O(f, y, u) + \alpha^O(f, y, u) \geq \delta^O(f, y, u) > 1.$$

By the assumption (a) and Lemma 10.5, $\alpha^O(f, y, u) < 1$ and hence $\beta^O(f, y, u) > 0$. $\square$

So for the proof of Proposition 13.3, it remains to treat the case where $(f', y', (u_1, \phi'))$ is solvable at v'. Assume that we are now in this case. This implies

$$(13.13) \qquad \beta' := \beta(f', y', (u_1, \phi')) \in \mathbb{Z}_{\geq 0}, \quad \delta := \delta(f, y, u) \in \mathbb{Z}, \quad \delta \geq 2.$$

Indeed $\delta \in \mathbb{Z}$ since $\alpha' = \delta - 1 \in \mathbb{Z}$ by (13.10) and $\delta > 1$ by the assumption (d). It also implies that there exist $\lambda_1, \ldots, \lambda_r \in k'$ such that

$$(13.14) \qquad in_{v'}(f_i') = F_i(Y' + \lambda \cdot \Phi'^{\beta'} U_1^{\delta - 1}) \quad \text{for } i = 1, \ldots, N,$$

where $\lambda = (\lambda_1, \ldots, \lambda_r)$. For $1 \leq j \leq r$, let $A_j(U) = A_j(U_1, U_2) \in k[U_1, U_2]$ be homogeneous polynomials such that:

(C1) $\alpha_j := \deg A_j < d = \deg \Phi$,

(C2) A_j is not divisible by U_1 (which implies $\alpha_j = \deg_{U_2} A_j(1, U_2)$)),

(C3) $\lambda_j \equiv A_j(1, U_2) \mod \Phi(1, U_2)$ in $k' = k(x') \cong k[U_2]/\langle \Phi(1, U_2) \rangle$.

Choose a lift of $\tilde{A}_j(U_1, U_2) \in R[U_1, U_2]$ of $A_j(U) \in k[U_1, U_2]$ and set

$$a_j = \tilde{A}_j(u_1, u_2) \in R \quad \text{and} \quad a'_j = a_j/u_1^{\alpha_j} \in R'.$$

Then $\lambda_j \equiv a'_j \mod \mathfrak{m}'$. Define

$$(13.15) \qquad\qquad z_j := y_j + \phi^{\beta'} \cdot u_1^{\delta - (d\beta' + \alpha_j)} \cdot a_j \in R[1/u_1],$$

$$(13.16) \qquad\qquad z'_j := z_j/u_1 = y'_j + \phi'^{\beta'} \cdot u_1^{\delta - 1} \cdot a'_j \in R'.$$

Note that $z'_i - y'_i \in \langle u_1 \rangle$ for $i = 1, \ldots, r$ since $\delta \geq 2$ as noted in (13.13). Consider the following condition

$$(13.17) \qquad \gamma^+ := \gamma^+(f, y, u) \geq d(\beta' + 1) \quad (\beta' := \beta(f', y', (u_1, \phi'))).$$

Lemma 13.7 *Assume* (13.17) *holds. Then the following is true.*

(1) *We have $z_j \in R$ for $j = 1, \ldots, r$ and (z, u) is a system of regular parameters of R which is strictly admissible for J. The conditions (b) and (c) of Proposition 13.3 are satisfied for (z, u) in place of (y, u).*

(2) $\mathbf{w}^+(f, z, u) = \mathbf{w}^+(f, y, u)$, $\mathbf{v}(f, z, u) = \mathbf{v}(f, y, u)$,
$\mathbf{v}^O(f, z, u) = \mathbf{v}^O(f, y, u)$, $\delta(f, z, u) = \delta(f, y, u)$.

(3) $\alpha(f', y', (u_1, \phi')) = \alpha(f', z', (u_1, \phi'))$,
$\beta(f', y', u_1, \phi') < \beta(f', z', u_1, \phi') \leq \gamma^+(f, y, u) = \gamma^+(f, z, u)$.

By Lemma 13.7, if (13.17) holds for (f, y, u), we may replace (f, y, u) by (f, z, u) to show Proposition 13.3. If (f, z, u) is not solvable at $\mathbf{v}(f, z, u)$, then we are done by Corollary 13.6. If (f, z, u) is solvable at $\mathbf{v}(f, z, u)$ and (13.17) holds for (f, z, u), then we apply the same procedure to (z, u) to get a new system of regular parameters of R. This process must stop after finitely many steps, by the last inequality in Lemma 13.7 (3). Thus Proposition 13.3 follows from Corollary 13.6, Lemma 13.7 and the following Lemma 13.8.

Lemma 13.8 *Assume that* (13.17) *does not hold for (f, y, u) and that $(f', y', (u_1, \phi'))$ is solvable at $v' = \mathbf{v}(f', y', (u_1, \phi'))$. Dissolve v' as in* (13.14) *and let z'_j be as in* (13.16). *Then $(f', z', (u_1, \phi'))$ is $\mathbf{v}$-prepared and we have*

$$\beta^O(f', z', (u_1, \phi')) < \beta^O(f, y, u), \quad \alpha(f', z', (u_1, \phi')) = \alpha(f', y', (u_1, \phi')) = \delta - 1.$$

Proof of Lemma 13.7. Conditions (13.17) and (C1) imply

$$(13.18) \qquad \delta - (d\beta' + \alpha_j) > \delta - d(\beta' + 1) \geq \delta - \gamma^+ \geq 0 \quad \text{for } j = 1, \ldots, r,$$

where the last inequality holds in general (cf. the picture below Definition 10.1). This implies $z_j \in R$, and the second assertion of (1) is obvious from (13.15). Let v_{u_2} be the valuation on R with respect to $\langle u_2 \rangle$. By (13.17) and (C1) we have

$$v_{u_2}(\phi^{\beta'} a_j) = d\beta' + \alpha_j < d\beta' + d \leq \gamma^+ .$$

This together with (13.18) implies that the coordinate transformation (13.15) affects only those vertices of $\Delta(f, y, u)$ lying in $\{(a_1, a_2) \in \mathbb{R}^2 | a_1 > \delta - \gamma^+, a_2 < \gamma^+\}$. This shows (2), and that condition (b) of Proposition 13.3 holds for (f, z, u).

The first assertion of (3) follows from (13.10) and the equality $\delta(f, y, u) = \delta(f, z, u)$ implied by (2). The first inequality in the second assertion of (3) is a consequence of the first assertion since $\mathbf{v}(f', y', (u_1, \phi')) \notin \Delta(f', z', (u_1, \phi'))$. Lemma 13.5 implies $\gamma^+(f, z, u) \geq \beta(f', z', (u_1, \phi'))$ and (2) implies $\gamma^+(f, z, u) = \gamma^+(f, y, u)$, which completes the proof of (3).

It remains to show that condition (c) of Proposition 13.3 holds for (f, z, u). Introduce $\rho = (\rho_1, \ldots, \rho_r)$, a tuple of independent variables over k. For $i = 1, \ldots, N$, write

$$in_\delta(f_i)_{(y,u)} = \sum_B Y^B T_{i,B}(U) \quad \text{with } T_{i,B}(U) \in k[U]$$

and substitute $Y_i = Z_i - \rho_i$ in $in_\delta(f_i)_{(y,u)} \in k[Y, U]$ to get

$$G_i(Z, U, \rho) = \sum_B (Z - \rho)^B T_{i,B}(U) \in k[Z, U, \rho].$$

By (13.3) and (13.15) we have $in_\delta(f_i)_{(z,u)} = G_i(Z, U, \rho)_{|\rho = s}$, where

$$s = (s_1, \ldots, s_r) \quad \text{with } s_i = U_1^{\delta - (d\beta' + \alpha_j)} \Phi^{\beta'} A_j(U_1, U_2) \in k[U_1, U_2].$$

Write

$$G_i(Z, U, \rho) = \sum_C Z^C S_{i,C}(U, \rho) \quad (S_{i,C}(U, \rho) \in k[U, \rho]).$$

By condition 13.3 (c) for (f, y, u), i.e., the normalizedness of $in_\delta(f_i)_{(y,u)}$, we have $T_{i,B}(U) \equiv 0$ if $B \in E^r(F_1(Y), \ldots, F_{i-1}(Y))$. By Lemma 13.9 below, this implies $S_{i,C}(U, \rho) \equiv 0$ for $C \in E^r(F_1(Z), \ldots, F_{i-1}(Z))$, which is the normalizedness of $in_\delta(f)_{(z,u)}$, i.e., 13.3 (c) for (f, z, u). $\square$

Lemma 13.9 *Assume given*

$$G(Y) = \sum_A C_A \cdot Y^A \in k[Y] = k[Y_1, \ldots, Y_r],$$

and a subset $E \subset \mathbb{Z}_{\geq 0}^r$ such that $E + \mathbb{Z}_{\geq 0}^r \subset E$ and that $C_A = 0$ if $A \in E$. Let $\rho = (\rho_1, \ldots, \rho_r)$ be a tuple of independent variables over k and write

$$G(Y + \rho) = \sum_K S_K \cdot Y^K \quad \text{with} \quad S_K \in k[\rho].$$

Then $S_K \equiv 0$ if $K \in E$.

Proof We have

$$
\begin{aligned}
G(Y + \rho) &= \sum_A C_A \left[\prod_{i=1}^r (Y_i + \rho_i)^{A_i} \right] \\
&= \sum_A C_A \prod_{i=1}^r \prod_{k_i}^{A_i} \binom{A_i}{k_i} \rho_i^{A_i - k_i} \cdot Y^{k_i} \\
&= \sum_{K=(k_1, \ldots, k_r)} \left[\sum_{A \in K + \mathbb{Z}_{\geq 0}} \left(C_A \prod_{i=1}^m \binom{A_i}{k_i} \right) \rho^{A-K} \right] Y^K.
\end{aligned}
$$

Thus

$$S_K = \sum_{A \in K + \mathbb{Z}_{\leq 0}} \left(C_A \prod_{i=1}^{m} \binom{A_i}{k_i} \right) \cdot \rho^{A-K}.$$

Now, if $S_K \not\equiv 0$ for $K \in E$, then there is an $A \in K + \mathbb{Z}_{\geq 0}^r \subset E$ with $C_A \neq 0$. This completes the proof of the lemma. $\square$

Proof of Lemma 13.8. Let $\tilde{F}_i(Y) \in R[Y]$ be a lift of $F_i(Y) \in k[Y]$. For a tuple of independent variables $\rho = (\rho_1, \ldots, \rho_r)$ over R, write

(13.19) $\qquad \tilde{F}_i(Y + \rho) = \tilde{F}_i(Y) + \sum_{\substack{|B| < n_i \\ |B+D| = n_i}} K_{i,B,D} \cdot Y^B \rho^D \quad$ with $K_{i,B,D} \in R$,

By (13.16) we have $z' = y' + \mu$, where

(13.20) $\qquad \mu = (\mu_1, \ldots, \mu_r) \quad$ with $\mu_i = u_1^{\delta-1}(\phi')^{\beta'} \cdot a_i'$.

Hence (13.19) implies:

$$\tilde{F}_i(z') = \tilde{F}_i(y') + \sum_{\substack{|B| < n_i \\ |B+D| = n_i}} K_{i,B,D} \cdot y'^B \mu^D \quad \text{with } K_{i,B,D} \in R,$$

By (13.8) this implies

(13.21) $\qquad f_i' = \tilde{F}_i(z') + \sum_{|B| < n_i} (z' - \mu)^B \theta_{i,B} + g' \quad$ with $v_\Lambda(g')_{(y',(u_1,\phi'))} > n_i$,

where

$$\begin{aligned}
\theta_{i,B} &= (\phi')^{e_i(B)} q_{i,B}' u_1^{(\delta-1)(n_i - |B|)} - \sum_{|D| = n_i - |B|} K_{i,B,D} \cdot \mu^D \\
&= (\phi'^{\beta'} u_1^{\delta-1})^{n_i - |B|} \omega_{i,B} \ .
\end{aligned}$$

Here we set

(13.22) $\qquad \omega_{i,B} = (\phi')^{b_i(B)} q_{i,B}' - \sum_{|D| = n_i - |B|} K_{i,B,D} \cdot a'^D \quad (a' = (a_1', \ldots, a_r')),$

$$b_i(B) = e_i(B) - (n_i - |B|)\beta'.$$

By (13.10) we have $b_i(B) \geq 0$. For each B write (in $k[U_2]$)

(13.23) $\quad \Phi(1, U_2)^{b_i(B)} Q_{i,B}(1, U_2) - \sum_{|D| = n_i - |B|} \overline{K}_{i,B,D} A(1, U_2)^D = \Phi(1, U_2)^{c_i(B)} \cdot R_{i,B}(1, U_2),$

$\qquad$ with $\quad A(1, U_2) = \big(A_1(1, U_2), \ldots, A_r(1, U_2) \big), \quad \overline{K}_{i,B,D} = K_{i,B,D} \mod \mathfrak{m} \in k,$

where $c_i(B) \in \mathbb{Z}_{\geq 0}$ and $R_{i,B}(U_1, U_2) \in k[U_1, U_2]$ is either 0 or homogeneous and not divisible by Φ nor by U_1. Choose a lift $\tilde{R}_{i,B}(U_1, U_2) \in R[U_1, U_2]$ of $R_{i,B}(U_1, U_2)) \in k[U_1, U_2]$ and set

$$r_{i,B} = \tilde{R}_{i,B}(u_1, u_2) \in R, \quad r_{i,B}' = r_{i,B}/u_1^{\deg R_{i,B}} \in R'.$$

Then (13.22) implies

$$\omega_{i,B} - (\phi')^{c_i(B)} r'_{i,B} \in \mathfrak{m} R' = \langle u_1 \rangle \subset R'$$

so that (13.21) gives

$$(13.24) \qquad f'_i = \tilde{F}_i(z') + \sum_{|B|<n_i} (z'-\mu)^B (\phi')^{(n_i-|B|)\beta'+c_i(B)} u_1^{(n_i-|B|)(\delta-1)} \cdot r'_{i,B} + h',$$

where $v_\Lambda(h')_{(y',(u_1,\phi'))} > n_i$. By Lemma 6.4(3) this implies $v_\Lambda(h')_{(z',(u_1,\phi'))} > n_i$ by noting $z'_i - y'_i \in \langle u_1^{\delta-1} \rangle$ by (13.16). Now we need the following lemma:

Lemma 13.10 *There exist $i \in \{1, \ldots, N\}$ and B with $|B| < n_i$ such that $R_{i,B}(U) \not\equiv 0$ in $k[U_1, U_2]$ (which is equivalent to $r'_{i,B} \notin \mathfrak{m}'$).*

The proof of Lemma 13.10 will be given later. Using the lemma, we see from (13.24) that there is a vertex of $\Delta(f', z', (u_1, \phi'))$ on the line $\{(a_1, a_2) \in \mathbb{R}^2 | \ a_1 = \delta - 1\}$. Since $\Delta(f', z', (u_1, \phi')) \subset \Delta(f', y', (u_1, \phi'))$ and $\mathbf{v}(f', y', (u_1, \phi')) \notin \Delta(f', z', (u_1, \phi'))$, this implies

$$(13.25) \qquad \begin{aligned} \alpha(f', z', (u_1, \phi')) &= \alpha(f', y', (u_1, \phi')) = \delta - 1, \\ \beta' := \beta(f', y', (u_1, \phi')) &< \beta(f', z', (u_1, \phi')). \end{aligned}$$

Moreover there exist $1 \le i \le N$ and B such that

$$(13.26) \qquad \beta(f', z', (u_1, \phi')) = \beta' + \frac{c_i(B)}{n_i - |B|},$$

where $c_i(B)$ is defined by equation (13.23).

Lemma 13.11 *If condition (13.17) does not hold, $c_i(B) < n_i - |B|$ for all $i = 1, \ldots, N$.*

Proof By the assumption we have $\gamma^+(f, y, u) < d(\beta' + 1)$. Then we claim that for all (i, B) such that $Q_{i,B} \not\equiv 0$,

$$d \cdot b_i(B) + \deg_{U_2} Q_{i,B}(1, U_2) < d \cdot (n_i - |B|).$$

Indeed, recalling $b_i(B) = e_i(B) - (n_i - |B|)\beta'$, the assertion is equivalent to

$$\frac{d \cdot e_i(B) + \deg_{U_2} Q_{i,B}(1, U_2)}{n_i - |B|} < d(\beta' + 1)$$

and this follows from (13.6). On the other hand, in (13.23) we have

$$\deg_{U_2} A(1, U_2)^D \le |D| \max\{\deg_{U_2} A_j(1, U_2) \mid 1 \le j \le r\} < d \cdot (n_i - |B|),$$

because $|D| = n_i - |B|$ and $\deg_{U_2} A_j(1, U_2) = \alpha_j < d$ by (C1). By (13.23), this implies

$$\deg_{U_2} \left(\Phi(1, U_2)^{c_i(B)} R_{i,B}(1, U_2) \right) < d \cdot (n_i - |B|),$$

which implies $c_i(B) < n_i - |B|$ since $d = \deg_{U_2} \Phi(1, U_2)$. $\square$

By Lemma 13.11, (13.26) and (13.25) imply

(13.27) $$\beta' < \beta(f', z', (u_1, \phi')) < \beta' + 1 \,.$$

By (13.13) we have $\beta' \in \mathbb{Z}$ so that $\beta(f', z', (u_1, \phi')) \notin \mathbb{Z}$. Hence $(f', z', (u_1, \phi'))$ is not solvable at $\mathbf{v}(f', z', (u_1, \phi'))$.

We now show $(f', z', (u_1, \phi'))$ is normalized at $\mathbf{v}(f', z', (u_1, \phi'))$. Let

$$v'' = \mathbf{v}(f', z', (u_1, \phi')) \quad \text{and} \quad \beta'' = \beta(f', z', (u_1, \phi')).$$

Setting $Z' = (Z'_1, \ldots, Z'_r)$ with $Z'_i = in_{\mathfrak{m}'}(z'_i)$, (13.24) implies

(13.28)
$$in_{v''}(f'_i)_{(z', (u_1, \phi'))} = F_i(Z') + \sum_B (Z' - \overline{\mu})^B \big(\Phi'^{\beta''} U_1^{\delta-1}\big)^{n_i - |B|} \cdot \overline{r}'_{i,B},$$
$$= F_i(Z') + \sum_C Z'^C S_{i,C} \quad \text{with } S_{i,C} \in k'[U_1, \Phi'],$$

where the first sum ranges over such B that $|B| < n_i$ and $\beta' + \dfrac{c_i(B)}{n_i - |B|} = \beta''$ and

$$\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_r) \quad \text{with } \overline{\mu}_i = U_1^{\delta-1} \Phi^{\beta'} \cdot \overline{a}'_i,$$
$$\overline{r}'_{i,B} = r'_{i,B} \mod \mathfrak{m}' = R_{i,B}(1, U_2) \mod \Phi(1, U_2) \in k' \simeq k[U_2]/\langle \Phi(1, U_2)\rangle.$$

Let $B \in E^r(F_1, \ldots, F_{i-1})$. Proposition 13.3 (c) implies $Q_{i,B}(U) \equiv 0$ (cf. (13.3) and (13.4)). This implies that $(F_1, \ldots, F_N)$ is normalized in the sense of Definition 7.11 (1), and Lemma 13.9 implies that $K_{i,B,D} \in \mathfrak{m}$ for all D in (13.19). Hence $R_{i,B}(1, U_2) \equiv 0$ in (13.23) so that $\overline{r}'_{i,B} = 0$ in (13.28). By Lemma 13.9 this implies that $S_{i,C} \equiv 0$ if $C \in E(F_1, \ldots, F_{i-1})$, which proves the desired assertion.

Finally it remains to show

(13.29) $$\beta^O(f', z', (u_1, \phi')) < \beta^O(f, y, u).$$

The proof is divided into the following two cases:

Case (1) $\delta^O(f, y, u) = \delta(f, y, u)$,

Case (2) $\delta^O(f, y, u) < \delta(f, y, u)$.

Note that we always have $\delta^O(f, y, u) \leq \delta(f, y, u)$ since $\Delta(f, y, u) \subset \Delta^O(f, y, u)$.

Assume we are in Case (1). We have

(13.30) $$\gamma^+(f, y, u) \leq \gamma^{+O}(f, y, u) \leq \beta^O(f, y, u),$$

where the first inequality holds by the assumption and the second holds in general. By (13.11) and (13.10), the assumption implies

$$\alpha^O(f', y', (u_1, \phi')) = \alpha(f', y', (u_1, \phi')) = \delta - 1.$$

Since the coordinate transformation $y' \to z'$ in (13.16) affects only those vertices lying in $\{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 \geq \delta - 1\}$, we have

(13.31) $$\alpha^O(f', z', (u_1, \phi')) \geq \alpha^O(f', y', (u_1, \phi')) = \alpha(f', y', (u_1, \phi')) = \delta - 1.$$

By (13.25) we have

$$\alpha(f', y', (u_1, \phi')) = \alpha(f', z', (u_1, \phi')) \geq \alpha^O(f', z', (u_1, \phi')),$$

where the inequality holds since $\Delta(f', z', (u_1, \phi')) \subset \Delta^O(f', z', (u_1, \phi'))$. Thus we get

$$\begin{aligned} \alpha(f', z', (u_1, \phi')) &= \alpha^O(f', z', (u_1, \phi')), \\ \beta^O(f', z', (u_1, \phi')) &\leq \beta(f', z', (u_1, \phi')). \end{aligned} \tag{13.32}$$

On the other hand, from Lemma 13.5 and (13.27), we get

$$\beta'' := \beta(f', z', (u_1, \phi')) < \frac{\gamma^+(f, y, u)}{d} + 1 \leq \frac{\gamma^+(f, y, u)}{2} + 1.$$

If $\gamma^+ := \gamma^+(f, y, u) \geq 2$, then $\beta'' < \gamma^+$, which shows the desired inequality (13.29) thanks to (13.30) and (13.32). If $\gamma^+ < 2$, then Lemma 13.5 implies $\beta' := \beta(f', y', (u_1, \phi')) \leq \gamma^+/2 < 1$ so that $\beta' = 0$ since $\beta' \in \mathbb{Z}$ as noted in (13.13). Hence (13.32) and (13.27) implies

$$\beta^O(f', z', (u_1, \phi')) \leq \beta(f', z', (u_1, \phi')) < \beta' + 1 = 1.$$

On the other hand we have

$$\beta^O(f, y, u) \geq \delta^O(f, y, u) - \alpha^O(f, y, u) > \delta^O(f, y, u) - 1 = \delta(f, y, u) - 1 \geq 1. \tag{13.33}$$

Here the first inequality holds in general, the second inequality holds since $\alpha^O(f, y, u) < 1$ by Lemma 10.5 and the assumption (a), and the last inequality follows from (13.13). This proves the desired inequality (13.29) in Case (1).

Assume we are in Case (2). Write $\delta^O = \delta^O(f, y, u)$. The assumption implies

$$\gamma^{+O} := \gamma^{+O}(f, y, u) = \gamma^{+O}(y, u). \quad \text{(cf. Definition 10.3)}$$

By (13.10) and (13.11) it also implies

$$\alpha^O(f', y', (u_1, \phi')) = \delta^O - 1 < \alpha(f', y', (u_1, \phi')) = \delta - 1.$$

Since the coordinate transformation $y' \to z'$ in (13.16) affects only those vertices lying in $\{(a_1, a_2) \in \mathbb{R}^2 | a_1 \geq \delta - 1\}$ and $\delta^O < \delta$, this implies

$$\alpha^O(f', z', (u_1, \phi')) = \delta^O - 1 < \alpha(f', z', (u_1, \phi')).$$

and hence

$$\beta^O(f', z', (u_1, \phi')) = \beta^O(z', (u_1, \phi')). \tag{13.34}$$

Recall that the δ-face of $\Delta^O(f, y, u)$ is

$$\Delta^O(f, y, u) \cap \{A \in \mathbb{R}^2 | \Lambda^O(A) = 1\} \quad \text{with } \Lambda^O : \mathbb{R}^2 \to \mathbb{R}; \ (a_1, a_2) \to \frac{a_1 + a_2}{\delta^O}.$$

For $B \in O(x)$, the initial form of l_B along this face is written as:

$$in_\delta(l_B) = L_B(Y) + \Phi(U)^{s_B} \Gamma_B(U), \tag{13.35}$$

where $L_B(Y) \in k[Y]$ is a linear form, $s_B \in \mathbb{Z}_{\geq 0}$, and $\Gamma_B(U) \in k[U]$ is either 0 or homogeneous of degree $\delta^O - d \cdot s_B$ and not divisible by Φ. Note that $\Gamma_B(U) \not\equiv 0$ for some $B \in O(x)$. From this we compute

$$
(13.36) \quad \begin{aligned} \gamma^{+O} = \gamma^{+O}(y, u) &= \sup \left\{ \deg_{U_2}(\Phi(1, U_2)^{s_B} \Gamma_B(1, U_2)) \mid B \in O(x),\ \Gamma_B(U) \not\equiv 0 \right\} \\ &\geq d \cdot \{ s_B \mid B \in O(x),\ \Gamma_B(U) \not\equiv 0 \}. \end{aligned}
$$

Choose lifts $\tilde{L}_B(Y) \in R[Y]$ and $\tilde{\Gamma}_B(U) \in R[U]$ of $L_B(Y) \in k[Y]$ and $\Gamma_B(U) \in k[U]$, respectively, and set

$$
\gamma_B = \Gamma_B(u_1, u_2) \in R, \quad \text{and} \quad \gamma_B' = \gamma_B / u_1^{\deg \Gamma_B}.
$$

Note $\gamma_B' \in \mathfrak{m}'$ if and only if $\Gamma_B(U) \equiv 0$. (13.35) implies

$$
l_B = \tilde{L}_B(y) + \phi^{s_B} \gamma_B + \epsilon \quad \text{with } v_{\Lambda^O}(\epsilon)_{(y,u)} > v_{\Lambda^O}(l_B)_{(y,u)} = 1.
$$

By Lemma 8.2 (1), this implies

$$
l_B' = l_B / u_1 = \tilde{L}_B(y') + u_1^{\delta^O - 1} \phi'^{s_B} \gamma_B' + \epsilon' \quad \text{with } v_{\Lambda'^O}(\epsilon')_{(y',(u_1,\phi'))} > 1,
$$

$$
\Lambda'^O : \mathbb{R}^2 \to \mathbb{R} \ ; \ (a_1, a_2) \to \frac{a_1}{\delta^O - 1}.
$$

Substituting $y' = z' - u_1^{\delta - 1} \phi'^{\beta'} \cdot a'$ (cf. (13.16)), we get

$$
l_B' = \tilde{L}_B(z') + u_1^{\delta^O - 1} \phi'^{s_B} \gamma_B' + u_1^{\delta - 1} \cdot h + \epsilon' \quad (h \in R').
$$

By Lemma 6.4 (3), $v_{\Lambda'^O}(\epsilon')_{(y',(u_1,\phi'))} > 1$ implies $v_{\Lambda'^O}(\epsilon')_{(z',(u_1,\phi'))} > 1$ since we have

$$
v_{\Lambda'^O}(z_i' - y_i')_{(z',(u_1,\phi'))} = \frac{\delta - 1}{\delta^O - 1} > 1 \quad \text{for } i = 1, \ldots, r
$$

by (13.16). Hence we get

$$
\beta^O(f', z', (u_1, \phi')) = \beta^O(z', (u_1, \phi')) = \inf \{ s_B \mid B \in O(x),\ \Gamma_B(U) \not\equiv 0 \}
$$

By (13.36) this implies

$$
\beta'' := \beta^O(f', z', (u_1, \phi')) \leq \frac{\gamma^{+O}}{d} \leq \frac{\gamma^{+O}}{2}.
$$

If $\gamma^{+O} \neq 0$, this implies $\beta'' < \gamma^{+O} \leq \beta^O(f, y, u)$ as desired. If $\gamma^{+O} = 0$, then $\beta'' = 0$. On the other hand, as is seen in (13.33), we have

$$
\beta^O(f, y, u) \geq \delta^O(f, y, u) - \alpha^O(f, y, u) > \delta^O(f, y, u) - 1 > 0,
$$

where the last inequality was noted in (11.1). This proves the desired assertion (13.29) and the proof of Lemma 13.8 is complete, up to the proof of Lemma 13.10. $\square$

Proof of Lemma 13.10 Assume the contrary, i.e., that we have $R_{i,B}(U) \equiv 0$ in $k[U]$ for all (i, B) with $|B| < n_i$. Then, for all (i, B) we have

$$
(13.37) \quad \Phi(1, U_2)^{b_i(B)} Q_{i,B}(1, U_2) = \sum_{|D| = n_i - |B|} \overline{K_{i,B,D}} A(1, U_2)^D \in k[U_2]
$$

146

Write

$$\Gamma_j(U_1, U_2) = U_1^{\delta - (d\beta' + \alpha_j)} A_j(U_1, U_2) \ \in \ k[U_1, U_2][U_1^{-1}]$$

$(\delta - (d\beta' + \alpha_j)$ may be negative). Multiplying (13.37) by $U_1^{(n_i - |B|)(\delta - d\beta')}$, we get

(13.38)
$$P_{i,B}(U_1, U_2) = \sum_{|D| = n_i - |B|} \overline{K_{i,B,D}} \Gamma(U_1, U_2)^D$$

where $\Gamma(U)^D = \prod_{j=1}^{r} \Gamma_j(U)^{D_j}$. In fact, for the left hand side we note that $b_i(B) = e_i(B) - (n_i - |B|)\beta'$ and that $Q_{i,B}(U_1, U_2)\Phi(U_1, U_2)^{e_i(B)} = P_{i,B}(U_1, U_2)$ is either 0 or homogeneous of degree $\delta(n_i - |B|)$. For the right hand side recall that $A_j(U_1, U_2) = U_1^{\alpha_j} A_j(1, U_2)$ and that

$$U_1^{|D|(\delta - d\beta')} = \prod_{j=1}^{r} U_1^{D_j(\delta - d\beta' - \alpha_j)} \cdot U_1^{D_j \cdot \alpha_j} \ .$$

In view of (13.19), equations (13.38) and (13.3) imply

(13.39)
$$\begin{aligned} F_i(Y + \Gamma(U)) =& F_i(Y) + \sum_{|B+D| = n_i} (-1)^{|D|} \overline{K}_{i,B,D} Y^B \Gamma(U)^D \\ =& F_i(Y) + \sum_{|B| < n_i} P_{i,B}(U) = in_\delta(f_i) \end{aligned}$$

Now we claim:

(13.40)
$$\delta \geq d\beta' + \alpha_j \text{ for } j = 1, \ldots, r, \quad \text{i.e.,} \quad \Gamma_j(U) \in k[U] \subset k[U][U_1^{-1}].$$

Admitting this claim, (13.39) implies that one can dissolve all the vertices of $\Delta(f, y, u)$ on the line $\{A \in \mathbb{R}^2 | \ L(A) = 1\}$, which contradicts assumption (b). This completes the proof of Lemma 13.10.

It remains to show claim (13.40). We show that $\Gamma_j(U) \in k[U_1.U_2] \subset k[U_1, U_2, U_1^{-1}]$. Recall from (13.39) that in any case

(13.41)
$$F_i(Y + \Gamma(U)) = in_\delta(f_i) \in k[Y, U] \ .$$

Denote the variables $Y_1, \ldots, Y_r, U_1, U_2$ by $X_1, \ldots, X_n$ (so that $n = r + 2$), and define derivations $D_A : k[X] \to k$ for $A \in \mathbb{Z}_{\geq 0}^n$ by

$$G(X + \rho) = \sum_A D_A G \cdot \rho^A \quad (G(X) \in k[X]),$$

where $\rho = (\rho_1, \ldots, \rho_n)$ are new variables. We now apply [H5] (1.2) and [Gi3] Lemmas 1.7 and 3.3.4 (the assumptions of the lemmas are satisfied by Lemma 13.9 and the normal-izedness of $(F_1, \ldots, F_N)$ implied by Proposition 13.3(c)). According to these results, after possibly changing the ordering of $X_1, \ldots, X_n$, we can find:

- f, an integer with $0 \leq f \leq n$,

- P_j for $1 \leq j \leq f$, homogeneous polynomials in the variables $X_{A,i}$ indexed by $A \in \mathbb{Z}_{\geq 0}^n$ and $1 \leq i \leq N$, with coefficients in k,

- $q_1 \leq q_2 \leq \ldots q_f$, numbers which are powers of the exponential characteristic of k (so that $q_j = 1$ for $j = 1, \ldots, f$ if $\mathrm{char}(k) = 0$),

- $\psi_j = c_{j,j+1} X_{j+1}^{q_j} + \ldots c_{j,r} X_n^{q_j}$ ($c_{j,\nu} \in k$, $1 \leq j \leq f$, $j \leq \nu \leq n$), additive polynomials homogeneous of degree q_j,

such that for $i = 1, \ldots, N$, we have

$$
\begin{aligned}
P_1(D_A F_i) &= X_1^{q_1} + \psi_1(X_2, \ldots, X_n) \\
P_2(D_A F_i) &= X_2^{q_2} + \psi_2(X_3, \ldots, X_n) \\
&\quad\vdots \\
P_f(D_A F_i) &= X_f^{q_f} + \psi_f(X_{f+1}, \ldots, X_n)\,.
\end{aligned}
$$

Moreover the equations on the right hand side define the so-called ridge (faîte in French) $F(C_x(X)) = F(C(R/J))$ of the tangent cone $C_x(X) = C(R/J) = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(R/J))$, i.e., the biggest group subscheme of $T_x(Z) = \mathrm{Spec}(\mathrm{gr}_{\mathfrak{m}}(R)) = \mathrm{Spec}(k[X_1, \ldots, X_n])$ which respects $C(R/J)$ with respect to the additive structure of $T_x(Z)$. Since $\mathrm{Dir}(R/J) \subseteq F(C(R/J))$ and $e(R/J) = \bar{e}(R/J) = 2$ (cf. Definition 1.21) by the assumption, all these schemes have dimension 2. Hence we must have $f \geq n - 2 = r$. Since $F_i(Y) \in k[Y]$, the variables U_1 and U_2 do not occur in the above equations so that $f \leq r = n - 2$. Thus we get $f = r$. Hence, after a permutation of the variables $Y_1, \ldots, Y_r$, the equations become

$$
\begin{aligned}
P_1(D_A F_i(Y)) &= Y_1^{q_1} + \psi_1(Y_2, \ldots, Y_r) \\
P_2(D_A F_i(Y)) &= Y_2^{q_2} + \psi_2(Y_3, \ldots, Y_r) \\
&\quad\vdots \\
P_{r-1}(D_A F_i(Y)) &= Y_{r-1}^{q_{r-1}} + \psi_{r-1}(Y_r) \\
P_r(D_A F_i(Y)) &= Y_r^{q_r}\,,
\end{aligned}
$$

This implies

$$
P_r(D_A F_i(Y + \Gamma(U))) = Y_r^{q_r} + \Gamma_r(U)^{q_r}\,.
$$

By (13.41) this gives $\Gamma_r(U)^{q_r} \in k[U]$ and hence $\Gamma_r(U) \in k[U]$. Noting $\psi_j = c_{j,j+1} Y_{j+1}^{q_j} + \ldots + c_{j,r} Y_r^{q_j}$, we easily conclude inductively from (13.41) that $\Gamma_{r-1}(U), \ldots, \Gamma_1(U) \in k[U]$. This completes the proof of claim (13.40). $\square$

14 Non-existence of maximal contact in dimension 2

Let Z be an excellent regular scheme and let $X \subset Z$ be a closed subscheme.

Definition 14.1 *A closed subscheme $W \subset Z$ is said to have maximal contact with X at $x \in X$ if the following conditions are satisfied:*

(1) $x \in W$.

(2) Take any sequence of permissible blowups (cf. (5.1)):

$$
\begin{array}{ccccccccccc}
Z = & Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \leftarrow & Z_{n-1} & \xleftarrow{\pi_n} & Z_n & \leftarrow \cdots \\
& \cup & & \cup & & \cup & & \cup & & \cup & \\
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{n-1} & \xleftarrow{\pi_n} & X_n & \leftarrow \cdots
\end{array}
$$

where for any $n \geq 0$

$$
\begin{array}{ccccc}
Z_{n+1} & = & B\ell_{D_n}(Z_n) & \xrightarrow{\pi_{n+1}} & Z_n \\
& & \cup & & \cup \\
X_{n+1} & = & B\ell_{D_n}(X_n) & \xrightarrow{\pi_{n+1}} & X_n
\end{array}
$$

and $D_n \subset X_n$ is permissible. Assume that there exists a sequence of points $x_n \in D_n$ $(n = 0, 1, \ldots)$ such that $x_0 = x$ and x_n is near to x_{n-1} for all $n \geq 1$. Then $D_n \subset W_n$ for all $n \geq 0$, where W_n is strict transform W in Z_n.

Remark 14.2 *The above definition is much weaker than Hironaka's original definition (see [AHV]).*

In this section we prove the following:

Theorem 14.3 *Let k be a field of characteristic $p > 0$ and let y, u_1, u_2 be three variables over k. Consider $Z = \mathbb{A}_k^3 = \operatorname{Spec}(k[y, u_1, u_2])$ and let $X \subset Z$ be the hypersurface defined by the equation:*

$$(14.1) \qquad f = y^p + y u_1^N u_2^N + u_1^a u_2^b (u_1 + u_2)^{pA}$$

where A, a, b, N are integers satisfying the condition:

$$(14.2) \qquad 0 < a, b < p, \; a + b = p, \; A > p, \; p \nmid A, \; N \geq p^2 A.$$

Let x be the origin of $Z = \mathbb{A}_k^3$ and let $U \subset Z$ be an open neighborhood of x. Then there is no smooth hypersurface $W \subset U$ which has maximal contact with $X \times_Z U$ at x.

We observe the following consequence of maximal contact, from which we will derive a contradiction. Let $x \in X \subset Z$ be as in Definition 14.1. Let $R = \mathcal{O}_{Z,x}$ with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let $J \subset R$ be the ideal defining $X \times_Z \operatorname{Spec}(R)$.

Lemma 14.4 *Assume $Z = \operatorname{Spec}(R)$. Set $e = e_x(X)$ and $r = \dim(R) - e$, and assume that $e \geq 1$. Let $t_1, \ldots, t_r$ be a part of a system of regular parameters of R and assume $W := \operatorname{Spec}(R/\langle t_1, \ldots, t_r \rangle)$ has maximal contact with X at x. Let (f, y, u) be a δ-prepared label of (X, Z) at x (see Definition 9.1). Assume $\delta(f, y, u) > 2$, and let $m \geq 2$ be the integer such that*

$$(14.3) \qquad m < \delta(f, y, u) \leq m + 1.$$

Then the following holds.

(1) $\langle t_1 \ldots, t_r \rangle = \langle y_1 \ldots, y_r \rangle$ in $R/\mathfrak{m}^2$. In particular $(t, u) = (t_1, \ldots, t_r, u_1, \ldots, u_e)$ is a system of regular parameters of R.

(2) $m \leq \delta(f, t, u) \leq m + 1$ so that $|\delta(f, y, u) - \delta(f, t, u)| \leq 1$.

Proof Consider the fundamental sequence of permissible blowups as in Definition 5.34 with $\mathcal{B} = \emptyset$:

$$
\begin{array}{ccccccccccc}
Z = & Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \leftarrow & Z_{m-1} & \xleftarrow{\pi_m} & Z_m \\
& \cup & & \cup & & \cup & & \cup & & \cup \\
X = & X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \leftarrow & X_{m-1} & \xleftarrow{\pi_m} & X_m \\
& \uparrow & & \cup & & \cup & & \cup & & \cup \\
& x & \leftarrow & C_1 & \xleftarrow{\sim} & C_2 & \xleftarrow{\sim} \ldots \xleftarrow{\sim} & C_{m-1} & \xleftarrow{\sim} & C_m
\end{array}
$$

By Corollary 9.6 the integer m in (14.3) coincides with the length of the above sequence. Let W_q is the strict transform of W in Z_q. By Definition 14.1 we have

(14.4) $\qquad C_q \subset W_q$ so that $W_{q+1} \simeq B\ell_{C_q}(W_q)$ for all $q = 1, \ldots, m - 1$.

For $1 \leq q \leq m$, let η_q be the generic point of C_q and let $R_{\eta_q} = \mathcal{O}_{Z_q, \eta_q}$ with maximal ideal $\mathfrak{m}_{\eta_q}$. Let $J_{\eta_q} \subset R_{\eta_q}$ be the ideal defining $X_q \times_{Z_q} \mathrm{Spec}(R_{\eta_q})$. Write

$$
f_i^{(q)} = f_i/u_1^{qn_i}, \ y_i^{(q)} = y_i/u_1^q, \ u_i' = u_i/u_1 \ (2 \leq i \leq e), \ t_i^{(q)} = t_i/u_1^q, \ .
$$

By Claim 9.5, we know

(14.5) $\qquad (f^{(q)}, y^{(q)}, u_1)$ is a δ-prepared label of of (X_q, Z_q) at η_q for $q \leq m - 1$.

Claim 14.5 $(t^{(q)}, u_1) = (t_1^{(q)}, \ldots, t_r^{(q)}, u_1)$ is a system of regular parameters of R_{η_q}.

Proof Condition (14.4) implies (by induction on q) that the strict transform of W in $\mathrm{Spec}(R_{\eta_q})$ is defined by $\langle t_1^{(q)}, \ldots, t_r^{(q)} \rangle$. It also implies that W_q is transversal with the exception divisor of $Z_q \to Z_{q-1}$ which is defined by $\langle u_1 \rangle$ in $\mathrm{Spec}(R_{\eta_q})$. Thus the claim follows. $\square$

By claim (14.5) we have $\langle y^{(q)}, u_1 \rangle = \mathfrak{m}_{\eta_q} = \langle t^{(q)}, u_1 \rangle \subset R_{\eta_q}$. Lemma 14.4 (1) follows easily using this fact for $q = 1$. By Definition 9.1 (2), (14.5) implies $\delta(f^{(q)}, y^{(q)}, u_1) > 1$ so that

$$
f_j^{(q)} \in \mathfrak{m}_{\eta_q}^{n_j} = \langle t^{(q)}, u_1 \rangle^{n_j} \subset R_{\eta_q} \quad (j = 1, \ldots, N)
$$

By Definition 7.2 (3) this implies

(14.6) $\qquad \delta(f^{(q)}, t^{(q)}, u_1) \geq 1 \quad$ if $1 \leq q \leq m - 1$.

On the other hand, we have

(14.7) $\qquad \delta(f^{(q)}, t^{(q)}, u_1) = \delta(f, t, u) - q$.

Indeed, noting that (t, u) is a system of regular parameters of R by Lemma 14.4 (1), we can write, as in (6.3):

$$
f_i = \sum_{(A,B)} C_{i,A,B} \, t^B u^A \quad \text{with} \quad C_{i,A,B} \in R^\times \cup \{0\}
$$

150

We compute
$$f_i^{(q)} = f_i/u_1^{n_j} = \sum_{(A,B)} \left(C_{i,A,B} u'^A\right) (t^{(q)})^B u_1^{|A|+q(|B|-n_i)},$$

$$u'^A = u_2'^{a_2} \cdots u_e'^{a_e} \quad \text{for } A = (a_1, a_2, \ldots, a_e).$$

Noting that $k(\eta_q) = R_{\eta_q}/\mathfrak{m}_{\eta_q} \simeq k(u') = k(u_1', \ldots, u_e')$ and that $R \to R_{\eta_q} \to k(\eta_q)$ factors through $R \to k$ (see Claim 9.5), we see

$$\delta(f^{(q)}, t^{(q)}, u_1) = \min\{\frac{|A| + q(|B| - n_i)}{n_i - |B|} \mid |B| < n_i, \ C_{i,A,B} \neq 0\}$$
$$= \delta(f, y, u) - q.$$

Now (14.6) and (14.7) give us $\delta(f, t, u) \geq m$. As (f, y, u) is δ-prepared, Theorem 7.16 implies $\delta(f, y, u) \geq \delta(f, t, u)$. Thus we get $m \leq \delta(f, t, u) \leq \delta(f, y, u) \leq m+1$ and Lemma 14.4 (2) is shown. $\square$

Now we start the proof of Theorem 14.3. Let x be the origin of $Z = \mathbb{A}_k^3$ and let $R = \mathcal{O}_{Z,x}$ with the maximal ideal $\mathfrak{m}$. Write $u = (u_1, u_2)$.

Claim 14.6 (f, y, u) is a δ-prepared label of (X, Z) at x and $\delta(f, y, u) = A + 1$.

Proof From (14.1) and (14.2) one easily checks that $I\mathrm{Dir}_x(X) = \langle Y \rangle$ with $Y = in_{\mathfrak{m}}(y)$ and that $\Delta(f, y, u) \subset \mathbb{R}^2$ is the polyhedron with the two vertices

$$(A + \frac{a}{p}, \frac{b}{p}), \quad (\frac{a}{p}, A + \frac{b}{p})$$

Since these vertices are not integral points, the polyhedron is δ-prepared. We have $\delta(f, y, u) = A + 1 > 1$ as $a + b = p$ and the claim follows from Definition 9.1 (2). $\square$

We will use three sequences of permissible blowups:

Sequence I: First consider $\pi : Z' = Bl_x(Z) \to Z$ and the strict transform $X' \subset Z'$ of X. Look at the point $x' \in Z'$ of parameters

$$(y', u_1, v) = (\frac{y}{u_1}, u_1, 1 + \frac{u_2}{u_1}).$$

Let $R' = \mathcal{O}_{Z',x'}$. Then $X' \times_{Z'} \mathrm{Spec}(R')$ is defined by the equation:

$$f' := \frac{f}{u_1^p} = y'^p + y' u_1^{2N-p+1}(v-1)^N + u_1^{pA} v^{pA}(v-1)^b.$$

Using (14.2), one can check that $\Delta(f', y', (u_1, v))$ has the unique vertex (A, A) so that $\delta(f', y', (u_1, v)) = 2A > 1$. Hence x' is very near to x by Theorem 8.6. By Theorems 8.1 and 8.3, f is a (u_1, v)-standard base and (u_1, v) is admissible for $J = \langle f \rangle \subset R$. On the other hand, $(f', y', (u_1, v))$ is not prepared at the vertex and we dissolve it by the coordinate change

$$z := y' + \epsilon u_1^A v^A. \quad (\epsilon^p = (-1)^b)$$

151

Setting $\lambda = (v-1)^N \in R'^\times$ and $\mu = v^{-1}\big((v-1)^b - \epsilon^p\big) \in R'^\times$, we compute

(14.8)
$$f' = z^p + \lambda z u_1^{2N-p+1} + \mu u_1^{pA} v^{pA+1} - \lambda \epsilon u_1^{2N-p+1+A} v^A.$$

Using (14.2), we see that $\Delta(f', z, (u_1, v))$ is the polyhedron with three vertices

$$(A, A + \frac{1}{p}), \quad (\frac{2N - p + 1 + A}{p}, \frac{A}{p}), \quad (\frac{2N}{p-1} - 1, 0),$$

and that the δ-face of $\Delta(f', z, (u_1, v))$ is the first vertex. In fact, we have

$$\frac{2N - p + 1 + A}{p} + \frac{A}{p} > A + A + \frac{1}{p} \quad \text{and} \quad \frac{2N}{p-1} - 1 > A + A + \frac{1}{p}.$$

Since this first vertex is not an integral point, the polyhedron is δ-prepared. We have $\delta(f', z, (u_1, v)) = 2A + \frac{1}{p} > 1$. Hence $(f', z, (u_1, v))$ is a δ-prepared label of (X', Z') at x' by Definition 9.1 (2).

We now extend $\pi : Z' \to Z$ to the following sequence of permissible blowups:

$$
\begin{array}{ccccccccc}
Z & \xleftarrow{\pi} & Z' & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \xleftarrow{\pi_p} & Z_p \\
\cup & & \cup & & \cup & & \cup & & \cup \\
X & \xleftarrow{\pi} & X' & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \xleftarrow{\pi_p} & X_p \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
x & \leftarrow & x' & \leftarrow & x_1 & \leftarrow & x_2 & \leftarrow \ldots \leftarrow & x_p
\end{array}
$$

Here $Z_1 = B\ell_{x'}(Z')$ and $Z_{q+1} = B\ell_{x_q}(Z_q)$ $(q = 1, \ldots, p-1)$ where $x_q \in Z_q$ is the unique point lying on the strict transform of $\{z = v = 0\} \subset \operatorname{Spec}(R')$, and $X_q \subset Z_q$ is the strict transform of X. For $q = 1, \ldots, p$, write $R_q = \mathcal{O}_{Z_q, x_q}$ and

$$f^{(q)} = f'/u_1^{pq} = f/u_1^{p(q+1)}, \quad z^{(q)} = z/u_1^q, \quad v_i^{(q)} = v/u_1^q.$$

By convention we write $x_0 = x'$.

Claim 14.7 *For $q = 1, \ldots, p$, x_q is very near to x_{q-1}, $(f^{(q)}, z^{(q)}, (u_1, v^{(q)}))$ is a δ-prepared label of (X_q, Z_q) at x_q, and the δ-face of $\Delta(f^{(q)}, z^{(q)}, (u_1, v^{(q)}))$ is the vertex*

$$(A + q(A - 1 + \frac{1}{p}), A + \frac{1}{p}).$$

Proof By induction on q, one easily shows that $(z^{(q)}, u_1, v^{(q)})$ is a system of regular parameters of R_q. From (14.8), one computes

(14.9) $\quad f^{(q)} = (z^{(q)})^p + \lambda z^{(q)} u_1^{2N-p+1+q(1-p)} + \mu u_1^{pA+q(pA+1-p)} (v^{(q)})^{pA+1}$
$$- \lambda \epsilon u_1^{2N-p+1+A+q(A-p)} (v^{(q)})^A.$$

This implies $f^{(q)} \in R_q$ and $X_q \times_{Z_q} \operatorname{Spec}(R_q) = \operatorname{Spec}(R_q/\langle f^{(q)}\rangle)$. It also implies the last assertion of the claim by noting (14.2). In fact, for the vertex coming from the second term we have

$$\frac{2N}{p-1} - 1 - q > A + q(A - 1 + \frac{1}{p}) + A + \frac{1}{p} \quad \text{for} \quad 0 \le q \le p$$

(see the worst case $q = p$), and the case of the last term is even simpler. Since the vertex in the claim is not an integral point, $(f^{(q)}, z^{(q)}, (u_1, v^{(q)}))$ is δ-prepared. Noting $A > p$, we get

$$\delta(f^{(q)}, y^{(q)}, (u_1^{(q)}, u_2)) = 1 + A + q(\frac{a}{p} - 1) > 1 + a.$$

Now the claim follows from Theorems 8.6, 8.1, 8.3 and Corollary 7.17 as before. $\square$

Sequence II: We consider the following sequence of permissible blowups:

$$
\begin{array}{ccccccccc}
Z & = Z_0 & \xleftarrow{\pi_1} & Z_1 & \xleftarrow{\pi_2} & Z_2 & \leftarrow \ldots \xleftarrow{\pi_p} & Z_p \\
& \cup & & \cup & & \cup & & \cup \\
X & = X_0 & \xleftarrow{\pi} & X_1 & \xleftarrow{\pi_2} & X_2 & \leftarrow \ldots \xleftarrow{\pi_p} & X_p \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
x & = x_0 & \leftarrow & x_1 & \leftarrow & x_2 & \leftarrow \ldots \leftarrow & x_p
\end{array}
$$

Here $Z_{q+1} = B\ell_{x_q}(Z_q)$ $(q = 0, \ldots, p-1)$ where $x_q \in Z_q$ is the unique point lying on the strict transform of $\{y = u_2 = 0\} \subset \text{Spec}(R)$, and $X_q \subset Z_q$ is the strict transform of X. For $q = 0, \ldots, p$, write $R_q = \mathcal{O}_{Z_q, x_q}$ and

$$f^{(q)} = f/u_2^{pq}, \ y^{(q)} = y/u_2^q, \ u_1^{(q)} = u_1/u_2^q.$$

Claim 14.8 *For $q = 1, \ldots, p$, x_q is very near to x_{q-1}, and $(f^{(q)}, y^{(q)}, (u_1^{(q)}, u_2))$ is a δ-prepared label of (X_q, Z_q) at x_q, and $\Delta(f^{(q)}, y^{(q)}, (u_1^{(q)}, u_2))$ is the polyhedron with the unique vertex*

$$(\frac{a}{p}, \frac{b}{p} + A + q(\frac{a}{p} - 1)).$$

Proof By induction on q, one easily shows that $(y^{(q)}, (u_1^{(q)}, u_2))$ is a system of regular parameters of R_q. From (14.1), one computes

$$f^{(q)} = (y^{(q)})^p + y^{(q)}(u_1^{(q)})^N u_2^{N+q(N+1-p)} + (u_1^{(q)})^a u_2^{b+pA+q(a-p)} (u_1^{(q)} u_2^{q-1} + 1)^{pA}.$$

This shows $f^{(q)} \in R_q$ and it is an equation for $X_q \times_{Z_q} \text{Spec}(R_q)$. The last assertion easily follows by noting (14.2). As $p \nmid a$, the polyhedron is δ-prepared. Then we get

$$\delta(f^{(q)}, y^{(q)}, (u_1^{(q)}, u_2)) = 1 + A + q(\frac{a}{p} - 1) > 1 + a.$$

by noting $A > p \geq q$ and $p = a + b$. Now the claim follows from Theorems 8.6, 8.1, 8.3 and Corollary 7.17 as before. $\square$

Sequence III: This is the sequence of permissible blowups, which looks the same as the sequence II, except that now $x_q \in Z_q$ is the unique point lying on the strict transform of $\{y = u_1 = 0\} \subset \text{Spec}(R)$.

Now assume that there exists $s \in \mathfrak{m} - \mathfrak{m}^2$ such that

(∗) $W := \text{Spec}(R/\langle s \rangle) \subset \text{Spec}(R)$ has maximal contact with $X \times_Z \text{Spec}(R)$ at x.

Then we want to deduce a contradiction. Let $\hat{R} = k[[y, u_1, u_2]]$ be the completion of R. It is easy to see that the assumption (∗) and Claims 14.6, 14.7 and 14.8 hold even after replacing R with $\hat{R}$. Thus we may work with $\hat{X} := \text{Spec}(\hat{R}/\langle f \rangle)$ and $\hat{W} = \text{Spec}(\hat{R}/\langle s \rangle) \subset \hat{Z} = \text{Spec}(\hat{R})$ which has maximal contact with $\hat{X}$ at the closed point $x \in \text{Spec}(\hat{R})$.

Claim 14.9 *There exists $t \in \hat{R}$ such that $\langle s \rangle = \langle t \rangle \subset \hat{R}$ and*

$$t = y + \gamma \ \in \widehat{R} \quad \text{where } \gamma \in \mathfrak{n}^{A+1}.$$

Here $\mathfrak{n} = \langle u_1, u_2 \rangle \subset k[[u]] = k[[u_1, u_2]] \subset \hat{R}$.

Proof By Lemma 14.4 (1) and Claim 14.6, we can write $\epsilon s = y + a$ with $\epsilon \in k^\times$ and $a \in \mathfrak{m}^2$. Noting $\hat{R} = k[[y, u_1, u_2]] = k[[t, u_1, u_2]]$, we can write

$$a = \gamma + b \quad \text{with } \gamma \in \mathfrak{n}^2, \ b \in s\hat{R} \cap \mathfrak{m}^2 \hat{R}.$$

Setting $t = \epsilon s - b$, this implies $y = t + \gamma$ and $\langle s \rangle = \langle t \rangle \in \hat{R}$. From (14.1) we get

(14.10) $$f = t^p + t u_1^N u_2^N + u_1^a u_2^b (u_1 + u_2)^{pA} - \gamma^p - \gamma u_1^N u_2^N.$$

From Definition 7.2 (3), this implies

$$\delta(f, t, u) \le c = \max\{i| \ \gamma \in \mathfrak{n}^i\}.$$

Since $\delta(f, y, u) = A + 1$ by Claim 14.6, Lemma 14.4 (2) implies $A < \delta(f, t, u)$ and hence $c \ge A + 1$. This completes the proof of the claim. $\square$

To ease the notation, we write R, X, Z for $\hat{R}, \hat{X}, \hat{Z}$ in what follows. Write

$$\gamma = \Gamma + \theta \quad \text{with } \theta \in \mathfrak{n}^{A+2} \text{ and } \Gamma = 0 \text{ or } \Gamma \in k[u_1, u_2] \text{ homogeneous of degree } A + 1,$$

and let $C = \max\{m \in \mathbb{N} \cup \{\infty\} \ | \ (u_1 + u_2)^m | \Gamma\}$. There are two cases:

Case $C \ne A$: In this case (14.10) implies

$$f = t^p + t u_1^N u_2^N + (u_1 + u_2)^{pB} H(u_1, u_2) + \phi,$$

where $B = \min\{C, A\}$, and $H(u_1, u_2) \in k[u_1, u_2]$ is homogeneous of degree $p(A + 1 - B)$ which is not divisible by $u_1 + u_2$, and $\phi \in \mathfrak{n}^{p(A+2)}$.

Now we consider the sequence I. Let $t' = t/u_1$ and $t^{(q)} = t'/u_1^q$. By the same argument as in the proof of Claim 14.5, we can show that (t', u_1, v) (resp. $(t^{(q)}, u_1, v^{(q)})$) is a system of regular parameters for $R' = \mathcal{O}_{Z', x'}$ (resp. $R_q = \mathcal{O}_{Z_q, x_q}$). We compute
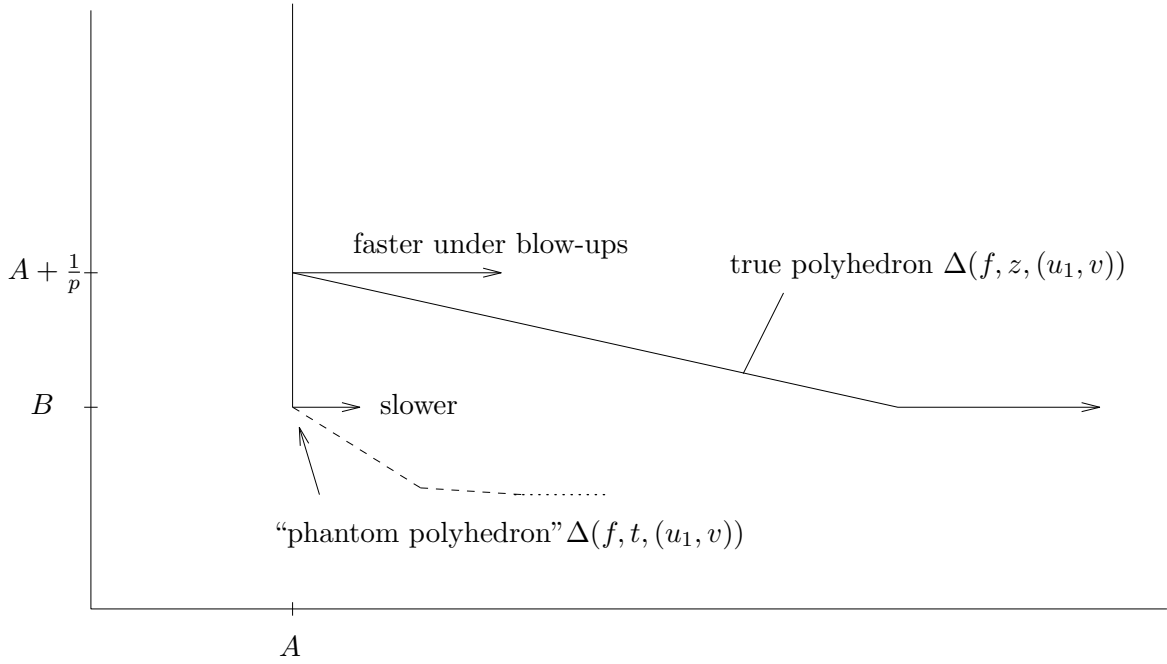
$$f' = t'^p + \lambda t' u_1^{2N-p+1} + \mu' u_1^{pA} v^{pB} + u_1^{p(A+1)} \phi',$$

where $\lambda = (v - 1)^N \in R'^\times$, and $\mu' = H(1, u_2/u_1) \in k[v] \subset R'$, and $\phi' = \phi/u_1^{p(A+2)} \in k[[u_1, u_2]][v] \subset \hat{R}'$. Note $\mu' \in R'^\times$ since $H(u_1, u_2)$ is not divisible by $u_1 + u_2$. From this we compute

$$f^{(q)} = (t^{(q)})^p + \lambda t^{(q)} u_1^{2N-p+1+q(1-p)} + \mu' u_1^{p(A+q(B-1))} (v^{(q)})^{pB} + u_1^{p(A+1-q)} \phi'.$$

By Definition 7.2 (3), this implies

$$\delta(f^{(q)}, t^{(q)}, (u_1, v^{(q)})) \le A + q(B - 1) + B.$$

Setting $\sigma_q = \delta(f^{(q)}, z^{(q)}, (u_1, v^{(q)})) - \delta(f^{(q)}, t^{(q)}, (u_1, v^{(q)}))$, and noting $A \geq B$, we get

$$\sigma_q \geq \left(A + q(A - 1 + \tfrac{1}{p}) + A + \tfrac{1}{p}\right) - \left(A + q(B - 1) + B\right) \geq \frac{q+1}{p},$$

from Claim 14.7. Hence $\sigma_q > 1$ for $q = p$, which contradicts Lemma 14.4(2) since $(f^{(q)}, z^{(q)}, (u_1, v^{(q)}))$ is a δ-prepared label of (X_q, Z_q) at x_q and the strict transform $W_q \subset \mathrm{Spec}(R_q)$ of W has maximal contact with $X_q \times_{Z_q} \mathrm{Spec}(R_q)$ at x_q by Definition 14.1.

Case $C = A$: There exist $c_1, c_2 \in k$ with $c_1 \neq c_2$ and $(c_1, c_2) \neq (0, 0)$ such that

$$\gamma = (u_1 + u_2)^A(c_1 u_1 + c_1 u_2) + \theta \quad \text{with } \theta \in \mathfrak{n}^{A+2} \subset k[[u_1, u_2]].$$

From (14.10) we get

$$f = t^p + t u_1^N u_2^N + (u_1 + u_2)^{pA}\left(u_1^a u_2^b - (c_1 u_1 + c_2 u_2)^p\right) + \phi,$$

where $\phi = -\theta^p - \gamma u_1^N u_2^N \in \mathfrak{n}^{p(A+2)}$.

Assume $c_2 \neq 0$ and consider the sequence II. Let $t^{(q)} = t/u_2^q$. By the same argument as in the proof of Claim 14.5, we can show that $(t^{(q)}, u_1^{(q)}, u_2)$ is a system of regular parameters of $R_q = \mathcal{O}_{Z_q, x_q}$. We compute

$$f^{(q)} = (t^{(q)})^p + t^{(q)}(u_1^{(q)})^N u_2^{N+q(N+1-p)} + \xi u_2^{p(A+1-q)} + u_1^{p(A+2-q)}\phi',$$

where $\phi' = \phi/u_2^{p(A+2)} \in R_q$ and

$$\xi = (u_1^{(q)} u_2^{q-1} + 1)^{pA}\left((u_1^{(q)})^a u_2^{a(q-1)} - (c_1 u_1^{(q)} u_2^{q-1} + c_2)^p\right) \in R_q^\times.$$

Here we have used $a + b = p$. By Definition 7.2 (3), this implies

$$\delta(f^{(q)}, t^{(q)}, (u_1^{(q)}, u_2)) \leq A + 1 - q.$$

Set $\sigma_q = \delta(f^{(q)}, z^{(q)}, (u_1, v^{(q)})) - \delta(f^{(q)}, t^{(q)}, (u_1, v^{(q)}))$. Then Claim 14.8 implies

$$\sigma_q \geq \left(1 + A + q\left(\frac{a}{p} - 1\right)\right) - \left(A + 1 - q\right) \geq \frac{qa}{p}.$$

Hence $\sigma_q > 1$ for $q = p$, which contradicts Lemma 14.4 (2) by the same reason as in the previous case.

It remains to treat the case $c_1 \neq 0$. By symmetry, the proof is given by the same argument using the sequence III instead of II. This completes the proof of Theorem 14.3.

15 An alternative proof of Theorem 5.17

We give another proof of Theorem 5.17 which uses more classical tools in algebraic geometry.

Recall that the Hilbert polynomial $P = P(A)$ of a graded k-algebra A (for a field k) is the unique polynomial in $\mathbb{Q}[T]$ with

$$P(n) = H(A)(n) \qquad \text{for} \qquad n \gg 0\,.$$

It is known that the degree of $P(A)$ is $\dim(A) - 1$ where, by definition, the zero polynomial has degree -1. The following property is well-known (see [AP] (Corollary 3.2 and following remarks for a proof).

Lemma 15.1 *A polynomial $P(T) \in \mathbb{Q}[T]$ is the Hilbert polynomial of a standard graded k-algebra A if and only if*
(15.1)
$$P(T) = \binom{T + a_1}{a_1} + \binom{T + a_2 - 1}{a_2} + \ldots + \binom{T + a_i - i + 1}{a_i} + \ldots + \binom{T + a_s - s + 1}{a_s}$$

for certain integers $a_1 \geq a_2 \geq \ldots \geq a_s \geq 0$ with $s \geq 0$. Moreover, one has $\deg(P) = a_1$ for $s \neq 0$, and the family $a(P) := (a_1, \ldots, a_s)$ is uniquely determined by P.

The equality $a_1 = \dim(A) - 1$ follows from the fact that

$$\binom{T + a_i - i + 1}{a_i} = \frac{(T + a_i - i + 1)(T + a_i - i) \cdots (T - i + 2)}{a_i!}$$

has degree a_i. The case $P(T) = 0$ is obtained as the case where $s = 0$ and $a(P)$ is the empty family. The lemma also shows that the set HP of all Hilbert polynomials of standard graded k-algebras does not depend on k.

Recall that HP is totally ordered with respect to the ordering

(15.2) $$P(T) \geq P'(T) \quad \Longleftrightarrow \quad P(n) \geq P'(n) \qquad \text{for} \qquad n \gg 0\,.$$

We shall need the following description of this ordering.

Lemma 15.2 *For two Hilbert polynomials $P, P' \in HP$ one has $P \geq P'$ if and only if $a(P) \geq a(P')$ in the lexicographic ordering (where we formally fill up the shorter family with entries $-\infty$ at the right until it has the same length as the longer family).*

Proof Since $P = P' \Longleftrightarrow a(P) = a(P')$, and since both orderings are total orderings, it suffices to show that $P > P'$ implies $a(P) > a(P')$. Let $a(P) = (a_1, \ldots, a_s)$ and $a(P') = (b_1, \ldots, b_t)$, and assume $P > P'$. We proceed by induction on $\min(s, t)$. If $\min(s, t) = 0$, then we must have $t = 0$ and $P \neq 0$, so that $a(P) > a(P')$. If $s, t \geq 1$ and $\deg(P) > \deg(P')$, then $a_1 > b_1$ and hence $a(P) > a(P')$. If $a_1 = b_1$, then for

$$P_1(T) = P(T) - \binom{T + a_1}{a_1} \quad , \quad P_1'(T) = P'(T) - \binom{T + a_1}{a_1}$$

the polynomials $Q(T) = P_1(T + 1)$ and $Q'(T) = P_1'(T + 1)$ are again Hilbert polynomials by Lemma 15.1, with associated invariants $a(Q) = (a_2, \ldots, a_s)$ and $a(Q') = (b_2, \ldots, b_t)$, respectively, and $P > P'$ implies $Q > Q'$, so that $a(Q) > a(Q')$ in the lexicographic ordering by induction. Together with $a_1 = b_1$ this implies the claim for P and P'.

Lemma 15.2 immediately implies:

Theorem 15.3 *In the ordered set HP, every strictly descending sequence $P_1 > P_2 > P_3 > \ldots$ is finite.*

Now let $HF \subset \mathbb{N}^{\mathbb{N}}$ be the set of all Hilbert functions of standard graded k-algebras. For $\nu \in HF$, write P_ν for the associated Hilbert polynomial. We trivially have

$$(15.3) \qquad\qquad \nu \geq \nu' \quad \Longrightarrow \quad P_\nu \geq P_{\nu'} .$$

Let X be an excellent scheme. For $x \in X$, let $P_X(x)$ be the Hilbert polynomial associated to $H_X(x)$ and let

$$\Sigma_X^{P\text{-}max} = \{\nu \in \Sigma_X | \ P_\nu = P_X^{\max}\} \quad \text{and} \quad X_{P\text{-}max} = \bigcup_{\nu \in \Sigma_X^{P\text{-}max}} X(\nu),$$

where $P_X^{\max} = \max\{P_X(x) \mid x \in X\}$. Theorem 1.33 implies:

Lemma 15.4 *$X_{P\text{-}max}$ is closed in X.*

In fact, if $\nu \in \Sigma_X^{P\text{-}max}$ and $\mu \geq \nu$, then $\mu \in \Sigma_X^{P\text{-}max}$ by (15.3), so that $X_{P\text{-}max}$ is the union of the finitely many closed sets $X(\geq \nu)$ for $\nu \in \Sigma_X^{P\text{-}max}$. Theorem 2.10 (1) and (15.3) imply

Lemma 15.5 *If $\pi : X' \longrightarrow X$ is a permissible blow-up, then $P_{X'}^{\max} \leq P_X^{\max}$.*

Another proof of Theorem 5.17 Suppose there exists an infinite sequence $X = X_0 \leftarrow X_1 \leftarrow \ldots$ of $\Sigma^{\max}$-eliminations such that no X_n is locally equisingular. For each $n \geq 0$ let $X_n^0 \subseteq X_n$ be the union of those connected components of X_n which are not equisingular, and let $Y_n = (X_n^0)_{P\text{-}max}$. Then $X_{n+1}^0 \subseteq \pi_n^{-1}(X_n^0)$, so by Remark 5.13 (a) and Lemma 15.5 we have $P_{X_{n+1}^0}^{max} \leq P_{X_n^0}^{max}$. By Theorem 15.3 we may assume that $P_{X_n^0}^{\max} = P_{X_{n+1}^0}^{\max}$ for all $n \geq 0$. By Theorem 2.10 this implies $\pi_n(Y_{n+1}) \subset Y_n$, and we get an infinite sequence of proper morphisms $Y = Y_0 \leftarrow Y_1 \leftarrow \cdots$. By Lemma 15.7 below one can choose a sequence of points $x_n \in Y_n$ for $n = 0, 1, \ldots$ such that $x_n = \pi_n(x_{n+1})$. By [H2] Theorem (1.B) and Theorem 2.10 (3) we may assume $H_{X_n}(x_n) = H_{X_{n+1}}(x_{n+1})$ for all $n \geq 0$, so that there exists a $\nu_0 \in HF$ such that $\nu_0 \in \Sigma_{X_n^0}$ for all $n \geq 0$. We claim that $\nu_0 \in \Sigma_{X_n^0}^{max}$ for some n. Then 5.15 (ME2) contradicts $\nu_0 \in \Sigma_{X_{n+1}^0}$. Let $S_n = \{\nu \in \Sigma_{X_n^0}^m ax \mid \nu > \nu_0\}$. We want to show $S_n = \emptyset$ for some n. Let $\nu_1, \ldots, \nu_r$ be the elements of S_0 and put

$$\Lambda = \{\mu \in HF \mid \nu_0 \leq \mu < \nu_i \text{ for some } i \in [1, r]\}.$$

Thus the claim follows from the following:

Lemma 15.6 *Λ is finite, $S_n \subset \Lambda$, and $S_n \cap S_m = \emptyset$ if $n \neq m \geq 0$.*

Proof The first claim follows from the assumption that $P_{\nu_i} = P_\nu$ so that there exists $N > 0$ such that $\nu_0(n) = \nu_i(n)$ for all $n \geq N$ and all $i = 1, \ldots, r$. The second and third claim follow from 5.15 (ME3) and (ME2), respectively. $\square$

Lemma 15.7 *Let* $Z_0 \xleftarrow{\pi_0} Z_1 \xleftarrow{\pi_1} Z_2 \xleftarrow{\pi_2} \cdots$ *be an infinite sequence of proper morphisms of non-empty noetherian schemes. Then there exists a sequence of points* $x_n \in Z_n$ *(*$n = 0, 1, 2, \dots$*) such that* $x_n = \pi_n(x_{n+1})$.

Proof For $m > n \geq 0$, let $\pi_{n,m} : Z_m \to Z_n$ be the composite of π_i for $i = n, \dots, m-1$ and put $Z_{n,m} = \pi_{n,m}(Z_m)$. By the properness, $Z_{n,m}$ is non-empty and closed in Z_n. Clearly we have $Z_n \supset Z_{n,n+1} \supset Z_{n,n+2} \supset \cdots$ and

$$(15.4) \qquad\qquad Z_{n,l} = \pi_{n,m}(Z_{m,l}) \quad \text{for } l > m > n.$$

Put $Z_{n,\infty} = \underset{m>n}{\cap}\, Z_{n,m}$. By the Noetherian condition, there exists $N(n) > n$ such that $Z_{n,\infty} = Z_{n,N(n)}$ so that $Z_{n,\infty} \neq \emptyset$. Then (15.4) implies $Z_{n,\infty} = \pi_{n,m}(Z_{m,\infty})$ for $m > n$. Thus we get an infinite sequence of proper surjective morphisms $Z_{0,\infty} \leftarrow Z_{1,\infty} \leftarrow Z_{2,\infty} \leftarrow \cdots$, which inductively gives the desired claim. $\square$

16 Functoriality, and algebraic spaces and stacks

In this section we reformulate the obtained functoriality for our resolution and apply this to obtain resolution of singularities for algebraic spaces and algebraic stacks.

Definition 16.1 *Let X be an excellent scheme of dimension at most two. The resolution morphism*

$$r = r(X) : X' = R(X) \longrightarrow X$$

is defined as the composition of the morphisms $X' = X_n \to \dots X_1 \to X_0 = X$ in the canonical resolution sequence of Theorem 0.1 (or, what amounts to the same, of the morphisms in the canonical resolution sequence $S_0(X, \emptyset)$ of Theorem 5.6 (b)).

Then we can state the functoriality in the mentioned theorems as follows.

Theorem 16.2 *(a) If $\alpha : U \to X$ is an étale morphism, then there is a unique morphism $\alpha' = R(\alpha) : R(U) \to R(X)$ making the diagram*

$$
\begin{array}{ccc}
R(U) & \xrightarrow{R(\alpha)} & R(X) \\
{\scriptstyle r(U)}\downarrow & & \downarrow{\scriptstyle r(X)} \\
U & \xrightarrow{\ \alpha\ } & X
\end{array}
$$

commutative. Moreover, the diagram is cartesian.
(b) The same functoriality holds for a boundary $\mathcal{B}$ on X and the analogous resolution morphism $r(X, \mathcal{B}) : X'' = R(X, \mathcal{B}) \to X$ obtained from Theorem 5.6 by composing the morphisms in $S(X, \mathcal{B})$, and for an embedded situation $(X, Z, \mathcal{B})$ as in the Theorem 5.9 and the resolution morphism $r(X, Z, \mathcal{B}) : (X''', Z''', \mathcal{B}''') \to (X, Z, \mathcal{B})$ obtained by composing the morphisms in the sequence $S(X, Z, \mathcal{B})$ of that theorem.
(c) More generally, let X be of arbitrary dimension, let $\mathcal{B}$ be a boundary on X, let $\alpha : X^* \to X$ be a flat morphism with geometrically regular fibers (e.g., a smooth morphism), and let $\mathcal{B}^*$ be the pull-back of $\mathcal{B}$ to X^*. Assume that the canonical resolution sequence constructed by Remark 5.29 and Corollary 5.27 for $(X, \mathcal{B})$ is finite. Then the corresponding sequence for $(X^*, \mathcal{B}^*)$ is finite as well, and there is a unique morphism $\alpha' = R(\alpha) : R(X^*, \mathcal{B}^*) \to R(X, \mathcal{B})$ making the diagram

$$
\begin{array}{ccc}
(X^*)' = R(X^*, \mathcal{B}^*) & \xrightarrow{\alpha' = R(\alpha)} & R(X, \mathcal{B}) = X' \\
{\scriptstyle r^* := r(X^*, \mathcal{B}^*)}\downarrow & & \downarrow{\scriptstyle r := r(X, \mathcal{B})} \\
X^* & \xrightarrow{\ \alpha\ } & X
\end{array}
$$

commutative, where the vertical morphisms are the compositions of the morphisms in the respective resolution sequences. Moreover, the diagrams are cartesian, and the boundary $(\mathcal{B}^*)'$ on $(X^*)'$ is the pull-back of the boundary $\mathcal{B}'$ on X'.

Proof The finiteness of the resolution sequence for $(X^*, \mathcal{B}^*)$, the existence of α' and the fact that the diagram is cartesian follow from Proposition 5.31 (ii). Since $r(X, \mathcal{B})$ is the composition of a sequence $X' = X_n \to X_{n-1} \to \dots \to X_1 \to X_0 = X$ of blow-ups, the uniqueness follows from the following Lemma.

Lemma 16.3 *Consider a cartesian diagram of schemes*

$$
\begin{array}{ccc}
(Y)' & \xrightarrow{\ \alpha'\ } & X' \\
\rho \downarrow & & \downarrow r \\
Y & \xrightarrow{\ \alpha\ } & X \ ,
\end{array}
$$

where α is flat and r is a composition of blow-ups. Then α' is the unique morphism making the diagram commutative.

Proof Let β be another morphism making the diagram commutative. Write π as the composition of blow-ups $X' = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = X$. Then we get a commutative diagram with cartesian squares

$$
\begin{array}{ccccc}
Y' = & Y_n & \overset{\alpha_n = \alpha'}{\longrightarrow} & X_n & = X' \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & \\
 & Y_{m+1} & \overset{\alpha_{m+1}}{\longrightarrow} & X_{m+1} & \\
 & \downarrow & & \downarrow \pi_m & \\
 & Y_m & \overset{\alpha_m}{\longrightarrow} & X_m & \\
 & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
Y = & Y_0 & \overset{\alpha_0 = \alpha}{\longrightarrow} & X_0 & = X' \ .
\end{array}
$$

Since α is flat, all α_m are flat as well. Hence, if $\pi_m : X_{m+1} \to X_m$ is the blow-up in D_n, the morphism $Y_{m+1} \to Y_m$ identifies with the blow-up in $Y_m \times_{X_m} D_m$. Write $r_m : X_n \to X_m$ for the composition in the right column and $\rho_m : Y_n \to Y_m$ for the composition in the left column. We show inductively that

$$(16.1) \qquad\qquad r_m \beta = r_m \alpha' \ ,$$

which gives the claim for $m = n$, where $r_n = id$. By assumption (16.1) holds for $m = 0$, since $r_0 \beta = r \beta = \alpha \rho = r \alpha' = r_0 \alpha'$. If equation (16.1) holds for m, then we show it for $m + 1$ as follows. If J_m is the ideal sheaf of D_m in X_m, then $L_1 = \pi_m^{-1} J_m \mathcal{O}_{X_{m+1}}$ is an invertible sheaf on X_{m+1} by the universal property of blow-ups. Since α_{m+1} is flat, $L_2 = \alpha_{m+1}^{-1} L_1 \mathcal{O}_{Y_{m+1}}$ is invertible on Y_{m+1}. Then $L_3 = \rho_m^{-1} L_2 \mathcal{O}_{Y_n}$ is invertible on Y_n. In fact, for any blow-up morphism $f : Z' \to Z$ it is obvious that $f^{-1} M \mathcal{O}_{Z'}$ is invertible for any invertible ideal sheaf M on Z. Consider the morphism

$$\varphi = r_m \beta = r_m \alpha' = \alpha_m \rho_m = \pi_m \alpha_{m+1} \rho_{m+1} : Y_n \longrightarrow X_m \ .$$

Since we have shown that $\varphi^{-1} J_m \mathcal{O}_{Y_n}$ is invertible, it follows by the universal property of the blow-up $\pi_m : X_{m+1} \to X_m$ that there is a unique morphism $\psi : Y_n \to X_{m+1}$ with $\pi_m \psi = \varphi$. Since

$$\pi_m r_{m+1} \beta = r_m \beta = \varphi = r_m \alpha' = \pi_m r_{m+1} \alpha' \ ,$$

we get $r_{m+1} \beta = r_{m+1} \alpha'$ as wanted.

Now we apply this to get resolution for algebraic stacks of dimension ≤ 2.

Theorem 16.4 *Let $\mathcal{X}$ be an excellent algebraic stack for a Grothendieck topology whose coverings are flat morphisms with geometrically regular fibers (e.g., a Deligne-Mumford stack or an Artin stack). Assume that there is a representable covering morphism $X \to \mathcal{X}$ in the chosen topology such that X is an excellent scheme with $\dim(X) \leq 2$. Then there exists a proper surjective morphism $\mathcal{X}' \to \mathcal{X}$ of stacks such that $\mathcal{X}'$ is regular.*

Proof This follows from the functoriality in the same way as in [Te] section 5.2. By the assumption one may represent $\mathcal{X}$ by a groupoid scheme

$$R \underset{t}{\overset{s}{\rightrightarrows}} X$$

(with the usual further structural morphisms m, e and i) such that R and X are excellent schemes and the morphisms s and t are flat with regular fibers. By the functorial resolution for X (which then gives finiteness for the canonical resolution strategy for R as well, by 16.2 (c) and using s, say) one defines the structural morphism s' by the cartesian diagram

$$
\begin{array}{ccc}
R' & \xrightarrow{\ s'\ } & X \\
{\scriptstyle r(R)}\downarrow & & \downarrow{\scriptstyle r(X)} \\
R & \xrightarrow{\ s\ } & X
\end{array}
$$

with the vertical resolution morphisms, in the same way one defines t' from t, and similarly one defines the 'composition' structural morphism m' by the cartesian diagram

$$
\begin{array}{ccc}
R' \times_{t'X's'} R' = (R \times_{tXs} R)' & \xrightarrow{\ m'\ } & R' \\
{\scriptstyle r(R\times_{tXs}R)}\downarrow & & \downarrow{\scriptstyle r(R)} \\
R \times_{tXs} R & \xrightarrow{\ m\ } & R
\end{array}
$$

with vertical resolution morphisms, since m is flat with regular fibers as well. The further structural morphisms e, i are obtained via base change, and the rules for s', t', m', e' and i' follow by the functoriality from those for s, t, m, e and i, because the equalities are checked on flat morphisms with regular fibers.

In the special case of an algebraic space one sees that one gets a resolution by an algebraic space again.

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