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Algebraic Geometry I

Winter term 2008/2009

Exercise sheet 3

28th October 2008

In the following let k be an algebraically closed field and $\mathbb{A}^n(k) = k^n$ the affine space endowed with the Zariski-topology.

Exercise 1. Show:

a) If $Z \subseteq \mathbb{A}^n(k)$ and $Z' \subseteq \mathbb{A}^m(k)$ are two affine k-varieties, then $Z \times Z' \subseteq \mathbb{A}^{m+n}(k)$ is an affine k-variety. k-variety.

b) The subspace topology of $Z \times Z' \subseteq \mathbb{A}^{m+n}(k)$ is finer than the product topology.

c) If $U \subseteq \mathbb{A}^n(k)$ and $U' \subseteq \mathbb{A}^m(k)$ are quasi-affine k-varieties, then $U \times U' \subseteq \mathbb{A}^{m+n}(k)$ is an quasi-affine k-variety.

(4 points)

Exercise 2. Prove:

a) A non-empty subset $U \subseteq \mathbb{A}^1(k)$ is open if and only if the complement of U is finite.

b) The Zariski topology on $\mathbb{A}^2(k) = \mathbb{A}^1(k) \times \mathbb{A}^1(k)$ is strictly finer than the product topology with respect to the two factors.

(4 points)

Exercise 3. Let $\phi : Z \to Z'$ be a regular map of affine varieties and let

$$\Gamma(\phi) = \{(a, b) \in Z \times Z' \mid b = \phi(a)\}$$

be the graph of ϕ. Show:

a) $\Gamma(\phi)$ is a affine variety.

b) The map

$$\begin{array}{rcccc} \psi & : & Z & \to & \Gamma(\phi) \\ & & a & \mapsto & (a, \phi(a)) \end{array}$$

is an isomorphism of affine varieties. Is this also true in the case of quasi-affine varieties?

(4 points)

Exercise 4. Show that Neil's parabola

$$V = \{(x, y) \in \mathbb{A}^2(k) \mid y^2 = x^3\}$$

is not isomorphic to $\mathbb{A}^1(k)$.
Hint: Show, e.g., that $k[X, Y]/ < Y^2 - X^3 >$ is not a principal ideal domain.

(4 points)