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Algebraic Geometry I

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Exercise sheet 6

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Exercise 1. Let X be a topological space, $Y \subseteq X$ a subset and A a non-trivial abelian group.

a) Let U run through all open subsets of X and define $\mathcal{F}$ by

$$\mathcal{F}(U) = \begin{cases} A, & U \cap Y \neq \emptyset, \\ 0, & U \cap Y = \emptyset. \end{cases}$$

Show that $\mathcal{F}$ endowed with the natural restriction maps (i.e. identity and zero maps, respectively) is a pre-sheaf on X.

b) Show that $\mathcal{F}$ is a sheaf if and only if $Y \subseteq X$ is irreducible.

c) Define the *modified constant pre-sheaf* by

$$A^p(U) = \begin{cases} A, & U \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

and the natural restriction maps (identity and zero maps). When is it a sheaf?

(4 points)

In all the following let k be an algebraically closed field.

Exercise 2. Consider the affine space $\mathbb{A}^4(k)$ with the coordinates X, Y, Z, W, the affine variety $V = Z(X \cdot W - Y \cdot Z) \subseteq \mathbb{A}^4(k)$ and the quasi-affine variety $U = (D(Y) \cup D(W)) \cap V$.
Define $h : U \to \mathbb{A}^1(k)$ by

$$h(w, x, y, z) = \begin{cases} x/y, & \text{für } y \neq 0, \\ z/w, & \text{für } w \neq 0. \end{cases}$$

Show: h is a well-defined regular function on U, but is not a global quotient of regular functions of V.

(4 points)

Exercise 3. Consider the affine variety $V \subseteq \mathbb{A}^n(k)$ and a fixed standard embedding $\mathbb{A}^n(k) \subseteq \mathbb{P}^n(k)$. Show: The topological closure $\bar{V} \subseteq \mathbb{P}^n(k)$ of V equals $Z_+(I^*)$, where I^* denotes the ideal which is generated by all homogenisations f^* of the elements $f \in I$, the vanishing ideal of V.

(4 points)

Exercise 4. Let $p, q > 0$ be fixed integral numbers, and $V = \{(t^p, t^q) \mid t \in k\} \subseteq \mathbb{A}^2(k)$.
Show: V is an affine variety and determine the vanishing ideal $I(V) \subseteq k[X, Y]$.

(4 points)

Supplementary question: When is V irreducible? *Hint for this*: (Irreducibility criterion) Let K be a field, $a \in K^\times$. The polynomial $X^n - a$ is irreducible in $K[X]$ if and only if the following conditions are true:

- $a \notin (K^\times)^p$ for all primes $p | n$,
- $a \notin -4(K^\times)^4$, if $4 | n$.