University of Regensburg
Faculty of Sciences I – Mathematics

Prof. Dr. Uwe Jannsen
Patrick Forré

Algebraic Geometry I

Winter term 2008/2009

Exercise sheet 7

25th November 2008

In all of the following let k be an algebraically closed field.

Exercise 1. Determine the vanishing ideal of the irreducible components of the following varieties in $\mathbb{A}^3(k)$ and give a geometrical description:

a) $U = Z(X^2 - YZ,\, X - XZ)$,

b) $V = Z(X^2 - YZ,\, X^3 - Y^3)$,

c) $W = Z(X^2 + Y^2 + Z^2,\, X^2 - Y^2 - Z^2 + 1)$.

(4 points)

Exercise 2. Let $V \subseteq \mathbb{A}^m(k)$ and $W \subseteq \mathbb{A}^n(k)$ be two irreducible varieties. Show: The product $V \times W \subseteq \mathbb{A}^{m+n}(k)$ is irreducible, too.
Hint: Suppose given a decomposition $V \times W = X_1 \cup X_2$ in subvarieties X_i and consider $V_i := \{v \in V \,|\, \{v\} \times W \subseteq X_i \}$ for $i = 1, 2$.

(4 points)

Exercise 3. Determine $\mathrm{Spec}(\mathbb{Z}[X])$ in a similar way as in exercise 3 on the 5th exercise sheet.

(4 points)

Exercise 4. Let X be a topological space. Prove the following statements:

a) If $Y \subseteq X$ is a subset, then $\dim Y \leq \dim X$.

b) If $\{U_i \,|\, i \in I\}$ is an open covering of X, then $\dim X = \sup_{i \in I} \dim U_i$.

c) If X is irreducible and finite-dimensional and $Y \subseteq X$ is a closed subspace with $\dim Y = \dim X$, then $Y = X$.

d) Give an example of a topological space X and an open and dense subspace $U \subseteq X$ with $\dim U < \dim X$.
Hint: Look at $\mathrm{Spec}(\mathbb{Z}_{<p>})$.

(4 points)

Submission: 2nd December 2008