

A new p -adic cohomology theory in characteristic p

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Principal aims

- Development of a cohomology theory in characteristic p which refines the crystalline cohomology
–and works well for torsion.
- Development of a sheaf theory which generalizes the Dieudonné theory
–and works well for torsion.

Gauges and φ -gauges over a perfect field

k perfect field of char. p ; $W_n, W_\infty = W$ Witt vectors of k .

Let $D_n = W_n[f, v]/(fv - p)$ as a graded commutative ring with $\deg f = 1$ and $\deg v = -1$.

Definition: A W_n -gauge is a graded D_n -module.

explicitly: $\dots \rightleftarrows M^r \begin{matrix} \xrightarrow{f} \\ \xleftarrow{v} \end{matrix} M^{r+1} \rightleftarrows \dots$ where $fv = p = vf$.

Definition: A φ - W_n -gauge is a gauge with a σ -linear isomorphism

$$\varphi : M^\infty := \varinjlim M^r \xrightarrow{\cong} M^{-\infty} := \varprojlim M^r.$$

Morphisms are morphisms of graded D_n -modules respecting φ .

Finite type gauges

Call a W_n -gauge of finite type, if it is finitely generated as D_n -module. For integers $a \leq b$ call a finite type gauge concentrated in the interval $[a, b]$, if f is an isomorphism for $n \geq b$ and v is an isomorphism for $n \leq a$.

This category is equivalent to the category of finite diagrams

$$M^a \underset{v}{\overset{f}{\rightleftarrows}} M^{a+1} \dots M^{b-1} \underset{v}{\overset{f}{\rightleftarrows}} M^b.$$

Here $M^\infty = M^b$ and $M^{-\infty} = M^a$.

Any finite type gauge is concentrated in some interval. Call a gauge effective if it is concentrated in $[0, \infty]$.

Remark: The category of finite type φ - W -gauges concentrated in $[0, 1]$ is equivalent to the category of Dieudonné modules:

$$M^0 \underset{v}{\overset{f}{\rightleftarrows}} M^1 \xrightarrow{\varphi} M^0_\sigma \quad \rightsquigarrow \quad M^0 \text{ mit } F = \varphi f \text{ und } V = v\varphi^{-1}.$$

The prototype

Let M be an F-crystal, i.e., a finitely generated free W -module with an injective σ -linear endomorphism $\varphi : M \longrightarrow M$.

We get an associated φ - W -gauge as follows:

- $M^r = \{ m \in M \mid \varphi(m) \in p^r M \}$,
- $f : M^r \rightarrow M^{r+1}$ the multiplication by p ,
- $v : M^{r+1} \rightarrow M^r$ the inclusion.

There is an $N \geq 0$ with $p^N M \subset \varphi(M)$. Then $\varphi(M^N) = M$, and

$$M = M^0 \xrightarrow{\sim} M^{-\infty} \quad \text{and} \quad M^N \xrightarrow{\sim} M^{\infty}.$$

Define a σ -linear isomorphism $\varphi : M^{\infty} \longrightarrow M^{-\infty}$ by the maps

$$\varphi_r : M^r \longrightarrow M, \quad \varphi_r(m) = p^{-r} \varphi(m).$$

F -crystals and φ -gauges

This gives a full embedding of categories

F -crystals \hookrightarrow finite type free effective φ - W -gauges .

More generally we have a full embedding of categories

virtual F crystals \hookrightarrow finite type free φ - W -gauges .

Here a virtual F -crystal is an F -isocrystal D (i.e., a finite-dimensional vector space over $K_0 = \text{Frac}(W)$ with a σ -linear isomorphism $\varphi : D \rightarrow D$) together with a W -lattice $M \subset D$. The associated φ - W -gauge is literally defined as above. The above embeddings are not surjective on objects. Thus the category of all φ - W -gauges is much bigger than the category of F -crystals.

Some constructions and definitions for φ -gauges

Reduction: If $m < n \leq \infty$, and M is a φ - W_n -gauge, then we get an induced φ - W_m -gauge $M \otimes_{W_n} W_m = M/p^m$.

For $r \in \mathbb{Z}$ define the Tate φ - W_n -gauge $W_n(r)$ as the gauge which is W_n concentrated in the place $-r$ (i.e., the interval $[-r, -r]$):

$$\dots \begin{array}{c} p \\ \leftarrow \\ W_n \\ \rightarrow \\ id \end{array} \begin{array}{c} p \\ \leftarrow \\ W_n \\ \rightarrow \\ id \end{array} \begin{array}{c} id \\ \leftarrow \\ W_n \\ \rightarrow \\ p \end{array} \begin{array}{c} id \\ \leftarrow \\ W_n \\ \rightarrow \\ p \end{array} \dots$$

with the middle W_n place in degree $-r$.

Definition Call a W_n -gauge M

(a) strict if $(f_r, v_r) : M^r \rightarrow M^\infty \oplus M^{-\infty}$ is injective for all r ,

(b) quasi-rigid if $M^r \xrightarrow{f} M^{r+1} \xrightarrow{v} M^r \xrightarrow{f} M^{r+1}$ is exact for all r ,

(c) rigid if it is strict and quasi-rigid.

Reductions of virtual F -crystals and free φ - W -gauges

Theorem

1. For a finite type φ - W_1 -gauge M the following are equivalent:
 - (a) M is the reduction of a virtual F -crystal.
 - (b) M is strict.
 - (c) As a gauge, M is the direct sum of gauges $W_1(r_i)$.
 - (d) M corresponds to an F -zip (see below).
2. If M is the reduction of a free φ - W -gauge, then M is quasi-rigid.

Hence not all φ - W_1 -gauges come via reduction from virtual F -crystals or free φ - W -gauges.

Algebraic properties

One has tensor products of W_n -gauges and φ - W_n -gauges: The tensor product of graded modules M and N over a graded ring A is given by

$$(M \otimes_A N)^n = \left(\bigoplus_{i+j=n} M^i \otimes_{A^0} N^j \right) / U^n,$$

where U^n is the sub- A^0 -module generated by the elements $\lambda x \otimes y - x \otimes \lambda y$ for $x \in M^i, y \in N^j$ and $\lambda \in A^k$ with $i + j + k = n$. One easily checks that for W_n -gauges M and N one has canonically

$$(M \otimes N)^\infty \cong M^\infty \otimes_{W_n} N^\infty \quad \text{and} \quad (M \otimes N)^{-\infty} \cong M^{-\infty} \otimes_{W_n} N^{-\infty}.$$

So for φ -gauges M and N the gauge $M \otimes N$ becomes a φ -gauge by $\varphi_M \otimes \varphi_N : M^\infty \otimes N^\infty \rightarrow M^{-\infty} \otimes N^{-\infty}$.

Similarly one has internal Homs, and (Tate) twists $M(m)$ by $(M(m))^n = M^{n+m}$. In fact, $M(m) = M \otimes W_n(m)$.

The sheaves \mathcal{O}_n^{cris}

A morphism of schemes $f : X \rightarrow Y$ is called syntomic, if it is flat and locally a complete intersection (factors locally as $X \xrightarrow{i} Z \xrightarrow{p} Y$ where p is smooth and i is a regular immersion). The syntomic topology is the topology generated by these morphisms.

Work of Fontaine and Messing: For $n \geq 1$ let \mathcal{O}_n^{cris} be the presheaf on the site X_{syn} of all syntomic schemes over k defined by the crystalline cohomology:

$$\mathcal{O}_n^{cris}(X) = H_{cris}^0(X/W_n(k), \mathcal{O}_{X/W_n}).$$

Then one has canonical isomorphisms for all $r \geq 0$

$$H^r(X, \mathcal{O}_n^{cris}) := H^r(X_{syn}, \mathcal{O}_n^{cris}) \cong H_{cris}^r(X/W_n, \mathcal{O}_{X/W_n}),$$

i.e., the sheaves \mathcal{O}_n^{cris} compute the crystalline cohomology.

Explicit description of the sheaves $\mathcal{O}_n^{\text{cris}}$

There is a rather explicit description of the sheaves $\mathcal{O}_n^{\text{cris}}$: For an affine syntomic scheme $\text{Spec}(A)$ let $W_n(A)$ be the ring of n -truncated Witt vectors, and let $W_n^{\text{DP}}(A)$ be the divided power envelope of $W(A)$ w.r.t. the ideal $I = \{(a_0, \dots, a_n) \mid a_0^{p^n} = 0\}$, compatible with the standard divided powers on pW_n . Then there is a canonical isomorphism of sheaves

$$\widetilde{W}_n^{\text{DP}} \xrightarrow{\sim} \mathcal{O}_n^{\text{cris}},$$

$\widetilde{W}_n^{\text{DP}}$ = the sheaf associated to the presheaf $A \mapsto W_n^{\text{DP}}(A)$.
From both description it is clear that one has a canonical Frobenius morphism $\varphi : \mathcal{O}_n^{\text{cris}} \rightarrow \mathcal{O}_n^{\text{cris}}$.

The sheaves $I_1^{[r]}$

Lemma/Definition (a) There is an epimorphism of ring sheaves $\mathcal{O}_1^{cris} \rightarrow \mathcal{O}$, given in terms of Witt vectors by $(a_0) \mapsto a_0^p$.

(b) Its kernel, denoted $I_1^{[1]}$, is a *PD*-ideal. Its r -th *DP*-power is denoted $I_1^{[r]}$.

(c) There is a natural monomorphism $\mathcal{O} \hookrightarrow \mathcal{O}_1^{cris}$. The composition $\mathcal{O}_1^{cris} \twoheadrightarrow \mathcal{O} \hookrightarrow \mathcal{O}_1^{cris}$ is φ , the Frobenius on \mathcal{O}_1^{cris} .

(d) The composition $\mathcal{O} \hookrightarrow \mathcal{O}_1^{cris} \twoheadrightarrow \mathcal{O}$ is the Frobenius on \mathcal{O} .

The universal gauge \mathcal{G}^r

Lemma One has a canonical exact sequence for $m, n \geq 0$

$$0 \rightarrow \mathcal{O}_m^{\text{cris}} \rightarrow \mathcal{O}_{m+n}^{\text{cris}} \rightarrow \mathcal{O}_n^{\text{cris}} \rightarrow 0,$$

where the map $\mathcal{O}_{m+n}^{\text{cris}} \rightarrow \mathcal{O}_n^{\text{cris}}$ is the natural projection, and $\mathcal{O}_m^{\text{cris}} \rightarrow \mathcal{O}_{m+n}^{\text{cris}}$ is induced by the multiplication by p^n . Thus we have a torsion free pro-sheaf

$$\mathcal{O}^{\text{cris}} = (\mathcal{O}_n^{\text{cris}}),$$

with the projections as transition maps. We can now apply the construction of the prototype φ -gauge and define

$$\mathcal{G}^r = \ker(\mathcal{O}^{\text{cris}} \xrightarrow{\varphi} \mathcal{O}^{\text{cris}} \rightarrow \mathcal{O}_r^{\text{cris}} = \mathcal{O}^{\text{cris}} / p^r \mathcal{O}^{\text{cris}}).$$

(The sub-pro-sheaf on which the Frobenius is divisible by p^r .)

The universal gauges \mathcal{G}_n^r

By reduction modulo p^n we obtain a gauge (\mathcal{G}_n^r) .

There is also a more direct description of \mathcal{G}_n^r : For any $m \geq n + r$ define

$$\hat{\mathcal{G}}_m^r = \ker(\mathcal{O}_m^{\text{cris}} \xrightarrow{\varphi} \mathcal{O}_m^{\text{cris}} \rightarrow \mathcal{O}_r^{\text{cris}})$$

and then let $\mathcal{G}_n^r = \hat{\mathcal{G}}_m^r / \mathfrak{p}^r$.

One shows that this is independent of m ($\geq n + r$), and that one has exact sequences

$$0 \rightarrow \mathcal{G}_m^r \rightarrow \mathcal{G}_{m+n}^r \rightarrow \mathcal{G}_n^r \rightarrow 0.$$

The gauges \mathcal{G}_n^r are φ -gauges

One can define a σ -linear Frobenius $\varphi : \mathcal{G}^\infty \rightarrow \mathcal{G}^{-\infty} = \mathcal{G}^0$, by letting “ $\varphi_r = p^{-r}\varphi$ ” on \mathcal{G}^r , similarly for (\mathcal{G}_n^r) . It follows from work of Fontaine and Messing that φ is an isomorphism, so that we get φ -gauges:

In fact, this is easily reduced to the case of (\mathcal{G}_1^r) , and here one argues as follows. Let $F_r\mathcal{O}_1^{\text{cris}} = \text{im}(\varphi_r : \mathcal{G}_1^r \rightarrow \mathcal{O}_1^{\text{cris}})$.

Theorem (‘Cartier isomorphism’; Fontaine, Messing) φ_r induces an isomorphism

$$I_1^{[r]} / I_1^{[r+1]} \xrightarrow{\sim} F_r\mathcal{O}_1^{\text{cris}} / F_{r-1}\mathcal{O}_1^{\text{cris}}.$$

This implies the surjectivity of φ , i.e., $\mathcal{O}_1^{\text{cris}} = \bigcup_{n \geq 0} F_n$, and also the injectivity of φ .

Gauge cohomology

Theorem (Fontaine, J.)

For any n , there is a cohomology theory with values in φ - W_n -gauges, $(H^i(-, W_n)^r, f, v)$, so that the following holds for smooth varieties X over k :

- For each i , the φ -gauge $(H^i(X, W_n)^r)$ is concentrated in the interval $[0, i]$ and vanishes for $i > 2d$ where $d = \dim(X)$.
- $H^i(X, W_n)^0 = H_{\text{cris}}^i(X, W_n)$.
- For X connected, smooth and proper of dimension d the gauges are of finite type, and the Poincaré duality for crystalline cohomology extends to a perfect duality of gauges

$$(H^i(X, W_n)^r) \times (H^{2d-i}(X, W_n)^r) \longrightarrow (H^{2d}(X, W_n)^r) \xrightarrow{\sim} W_n(-d), .$$

Comments 1

1. The cohomology theory is simply obtained by defining $H^i(X, W_n)^r = H_{syn}^i(X, \mathcal{G}_n^r)$ and using transport of structure. Then statement b. is clear.
2. The other results are obtained by projecting to the Zariski site and reducing to the case $n = 1$. If

$$\alpha : X_{syn} \longrightarrow X_{Zar}$$

is the morphism of sites, it follows from results of Berthelot on crystalline cohomology that one has

$$R\alpha_* I_1^{[r]} = \Omega_X^{\geq r}$$

(the upper part, starting with Ω_X^r , of the de Rham complex).

Comments 2

Then one uses induction and the following exact sequences:

$$0 \rightarrow F_r \mathcal{O}_1^{cris} \rightarrow \mathcal{G}_1^{r+1} \xrightarrow{v} \mathcal{G}_1^r \rightarrow F_r \mathcal{O}_1^{cris} \rightarrow 0$$

$$0 \rightarrow I_1^{[r]} \rightarrow \mathcal{G}_1^{r-1} \xrightarrow{f} \mathcal{G}_1^r \rightarrow I_1^{[r]} \rightarrow 0$$

$$\mathcal{G}_1^r \xrightarrow{f} \mathcal{G}_1^{r+1} \xrightarrow{v} \mathcal{G}_1^r \xrightarrow{f} \mathcal{G}_1^{r+1}.$$

The last exactness is the quasi-rigidity of (\mathcal{G}_1^r) ; in fact, this gauge is also rigid.

These sequences can also be used to prove that the cohomology of the \mathcal{G}_n^r can be computed by much coarser topologies, e.g., the so-called quasi-étale (quiet) topology or simply the p -th root topology, generated by open immersions and extraction of p -th roots (conjectured by Fontaine, proved by F. Schnellinger).

Examples 1

1. On $H^1(X, W_n)$ one obtains just a Dieudonné module, see the remark above.
2. $(H^2(X, W_n)^r)$ is a φ -gauge in the interval $[0, 2]$:

$$H^{2,0} \rightleftarrows H^{2,1} \rightleftarrows H^{2,2} \xrightarrow{\varphi} H^{2,0}.$$

This is already a finer structure than the ‘generalized Dieudonné module’ $H^{2,0} = H_{\text{cris}}^2(X, W_n)$ with $F (= \varphi \circ f^2)$ and $V (= v^2 \circ \varphi^{-1})$ satisfying $FV = p^2 = VF$.

Examples 2

3. For X smooth of dimension 2, the projection ($G_1^r = R\alpha_* G_1^r$) of the universal gauge to the Zariski site is given by the diagram

$$G_1^0 \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{v} \end{array} G_1^1 \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{v} \end{array} G_1^2$$

of complexes (where C is the Cartier operator and $Z\Omega_\sigma^1 = \ker d$)

$$\begin{array}{ccccc} \Omega^2 & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{id} \end{array} & \Omega^2 & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{C} \end{array} & \Omega_\sigma^2 \\ \uparrow d & & \uparrow dC & & \uparrow d \\ \Omega^1 & \begin{array}{c} \xrightarrow{0} \\ \xleftarrow{C} \end{array} & Z\Omega_\sigma^1 & \begin{array}{c} \xrightarrow{incl} \\ \xleftarrow{0} \end{array} & \Omega_\sigma^1 \\ \uparrow d & & \uparrow d & & \uparrow d \\ \mathcal{O} & \begin{array}{c} \xrightarrow{C^{-1}=F} \\ \xleftarrow{0} \end{array} & \mathcal{O}_\sigma & \begin{array}{c} \xrightarrow{id} \\ \xleftarrow{0} \end{array} & \mathcal{O}_\sigma \end{array}$$

Relation with Fontaine-Laffaille theory 1

We note that, for a W -gauge M , one has

$$M^\infty = M/(f-1)M \quad \text{and} \quad M^{-\infty} = M/(v-1)M.$$

Now consider $W[v] =$ polynomial ring with v in degree -1 , and the category of graded $W[v]$ -modules with σ -linear isomorphism

$$\varphi : M/(v-p)M \xrightarrow{\sim} M/(v-1)M.$$

If M is p -torsion and of finite type, it is concentrated in an interval $[a, b]$ in a similar way as above. The bijectivity of φ shows that v is injective, and so one can identify M^r , for $a \leq r \leq b$, with a submodule of M^a (via v^{r-a}), and one gets semi-linear maps $\varphi_r : M^r \rightarrow M^a$ satisfying $\varphi_r = p\varphi_{r+1}$. Moreover, M^a is the sum of the images of the φ_r .

Relation with Fontaine-Laffaille theory 2

Thus the considered category is the one of filtered φ -modules considered by Fontaine and Laffaille, which is equivalent to the category of 'crystalline' G_{K_0} - \mathbb{Z}_p -representations, $K_0 = \text{Frac}(W)$.

Now we have a morphism of graded rings

$W[v] \rightarrow D = W[v, f]/(fv - p)$ into the ring we considered up to now, and for an object M as above the D -module

$$\tilde{M} = D \otimes_{W[v]} M$$

has a natural grading. Moreover the morphisms

$$M/(v - 1)M \rightarrow \tilde{M}/(v - 1)\tilde{M} \quad \text{and} \quad M/(v - p)M \rightarrow \tilde{M}/(f - 1)\tilde{M}$$

induced by the map $x \mapsto 1 \otimes x$ are isomorphisms (note that $v - p = (f - 1)v^{-1}$). Hence \tilde{M} becomes a φ - W -gauge, which is also of finite type and concentrated in $[a, b]$.

Relation with Fontaine-Laffaille theory 3

Thus we have a ‘special fiber functor’ (of abelian categories) from finite type crystalline \mathbb{Z}_p -representations over K_0 to φ - W -gauges.

To be checked: That, for a smooth and proper scheme X over W the functor transports $H^i(X_{K_0} \otimes_{K_0} \overline{K_0}, \mathbb{Z}_p)$ to the φ - W -gauge $(H^i(X_k, W)^r)$.

Relation with F -zips and displays 1

Note that φ -gauges can be defined over any scheme S of characteristic p , in a more or less obvious way.

Definition (Moonen and Wedhorn) Let S be a scheme over \mathbb{F}_p . An F -zip over S is a tuple $M = (M, C^\bullet, D_\bullet, \varphi_\bullet)$ where

- (a) M is a locally free sheaf of \mathcal{O}_S -modules of finite type.
- (b) C^\bullet is a descending chain in $M^{(\sigma)}$ such that the quotients are locally free.
- (c) D^\bullet is an ascending chain in M with locally free quotients.
- (d) $(\varphi_i)_{i \in \mathbb{Z}}$ is a family of isomorphisms of \mathcal{O}_S -modules

$$\varphi_i : C^i / C^{i+1} \longrightarrow D_i / D_{i-1}$$

such that ... (some obvious commutations and one further assumption).

Relation with F -zips and displays 2

Proposition (Fontaine/F. Schnellinger) There is a fully faithful embedding of the category of F -zips into the category of φ - \mathcal{O}_S -gauges. The essential image consist of rigid locally free gauges of finite type.

It seems also that the category of (generalized) displays, as defined by Langer and Zink, can be embedded into the category of φ - $W(\mathcal{O}_S)$ -gauges. This was proved for perfect rings \mathcal{O}_S (M. Wid). The image should be the gauges which are locally free.

Sheaves and generalized Dieudonné theory

The idea is to define certain syntomic sheaves which should correspond to the smooth \mathbb{Z}_ℓ sheaves for $\ell \neq p$, by using φ -gauges.

Let M be a φ - W -gauge. Then we get an associated syntomic sheaf $\mathcal{F}(M)$ on the big site of $\mathrm{Spec}(k)$ by defining

$$\mathcal{F}(M) = \underline{\mathrm{Hom}}(\mathbf{1}, \mathcal{G}^\bullet \otimes \mathbf{M}).$$

Here the Hom is the sheaf Hom of sheaves of gauges.

Moreover, $\mathbf{1}$ is the trivial constant sheaf of φ -gauges, \mathbf{M} stands for the constant sheaf of gauges associated to M , and the tensor product is the tensor product of sheaves of φ -gauges.

More explicitly one has

$$\mathcal{F}(M) = \ker((\mathcal{G} \otimes \mathbf{M})^0 \xrightarrow{\varphi - v} (\mathcal{G} \otimes \mathbf{M})^{-\infty}).$$

Outlook

It is hoped that this operation gives a full embedding in the setting of the derived categories. But one can show that one gets a full embedding without deriving after restricting to gauges in smaller intervals.

Moreover, there should be a

Projection formula For a φ - W_n -gauge M in the interval $[-\infty, 0]$ and $\pi : X \rightarrow \text{Spec}(k)$ smooth and proper, one has

$$R\pi_*\pi^*\mathcal{F}(M) = R\mathcal{F}_*(\mathbf{M} \otimes R\pi_*\mathcal{G}_{n,X}).$$