Interpolation Theory and Function Spaces

Exercise Sheet 1

E1 Let $X_0, X_1$ be compatible Banach spaces, i.e., $X_0, X_1 \subset Z$ for some Hausdorff topological space $Z$. Prove that $X_0 + X_1$ equipped with the norm

$$\|x\|_{X_0 + X_1} = \inf_{x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1} \|x_0\|_{X_0} + \|x_1\|_{X_1}$$

is a Banach space.

**Hint:** Proof and use that a normed space $Y$ is complete if and only if for every sequence $(y_j)_{j \in \mathbb{N}}$ with $\sum_{j=0}^{\infty} \|y_j\|_Y < \infty$ the infinite sum $\sum_{j=0}^{\infty} y_j$ converges in $Y$.

E2 For $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$ let

$$E_tf(x) := e_t * f(x) = \int_{\mathbb{R}^n} e_t(x - y)f(y) \, dy, \quad t > 0, x \in \mathbb{R}^n,$$

be the heat semi-group, where

$$e_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \quad t > 0, x \in \mathbb{R}^n,$$

is the heat kernel. Prove that there is a constant $C > 0$ such that for every $1 \leq p \leq q \leq \infty$ the estimate

$$\|E_tf\|_{L^q(\mathbb{R}^n)} \leq C t^{-\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\mathbb{R}^n)}$$

uniformly in $t > 0$ and $f \in L^p(\mathbb{R}^n)$.

E3 Let $(X_0, X_1)$ be an admissible couple of Banach spaces and let $F(X_0, X_1)$ be as in the lecture notes. Prove that $F(X_0, X_1)$ is a Banach space.

E4 Read the proof of the Marcinkiewicz-Interpolation Theorem in the appendix of the lecture notes and prove the case $r = \infty$ in detail.