Blockseminar: Surgery
Long program

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1 Introduction

The aim of this seminar is to give a detailed treatment of the smooth surgery exact sequence for manifolds of dimension 5 and higher and then to review extensions of this sequence to the piecewise linear and topological categories and to give a range of applications.

The main reference for the seminar is Wolfgang Lück's lecture notes, *A basic introduction to surgery* [L] but we will use various other sources.

1.1 Prerequisites

The seminar will assume not go into the following important subjects:

1. The general theory of smooth manifolds, submanifolds, normal bundles and tubular neighbourhoods: see e.g. [M-S, Ch. 11 & Ch. 18].

2. Participants are assumed to be familiar with the statement of the h-cobordism theorem and hopefully it's proof: see [M2].

An excellent place to learn the essential ideas for surgery remains [M1], in particular the first four sections.

2 Program

Note that the program naturally splits into various groups of talks. The speakers from these groups should co-operate with one another both in learning the material and in decided how to organise its presentation.

A Talks 1-3: The aims of these talks is to out-line the proof of the s-coboridism theorem assuming the audience is familiar with the proof of the h-cobordism theorem.
Regensburg

B Talks 4-5: Here the key homotopy theoretic aspects of manifolds are identified: Poincaré duality and its surprising consequence: the existence of the Spivak normal fibration.
Freiburg

C Talks 6-8: Working below the middle dimension we see how surgery and bordism are intimately related and how to make normal maps highly connected.
Regensburg

D Talks 9-11: These talks present the heart of surgery: the surgery obstruction map arising out of the challenge of performing desired surgeries on middle-dimensional homotopy classes. Subtle topology and complex algebra and elegantly gathered into a powerful synthesis here.
Edinburgh

E Talks 12-14: Here we step back a little and see how to assemble all of the previous hard work into the succicnt and powerful surgery exact sequence.
Münster 12 & 13 and Prof. Ammann 14

F Talks 15-18: Now we give applications and show how to work with the surgery exact sequence. Starting with exotic spheres we see the spaces fundamental spaces O, PL, TOP and G and their quotients, compute their homotopy groups and sometimes even their homotopy type. As a result we can calculate manifolds homotopy equivalent to S^n, $\mathbb{C}P^n$ and T^n (in appropriate categories).
Prof. Goette 15, Bonn 16 & 18 and Poznan 17

∗ *Note that generic references are to [L].*

2.1 Description of Talks

1. The s-cobordism theorem I: [L, 1.1-1.3]

   Start with a clear statement of Theorem 1.1 where the Whitehead group $\mathrm{Wh}(\pi)$ appears as a certain group depending only on $\pi_1(M_0)$ and which is to be defined later. As motivation, quickly state and prove Theorems 1.2 & 1.3.

   Summarise the notation and main results of Section 1.1: these are Definition 1.6, Lemma 1.7, Lemma 1.12, Notation 1.15 and Lemma 1.16.

   Review Section 1.2 and then state without proof the Normal Form Lemma 1.24. State and prove Lemmas 1.22 and 1.23.

2. The s-cobordism theorem II: [L, 1.3 & 1.4]

   Prove Lemma 1.24 and then go through Section 1.4 in detail give the first definition of $\mathrm{Wh}(\pi)$ and then state Lemma 1.27. Give the proof of Lemma 1.27 in detail.

3. The s-cobordism theorem III: [L, Ch. 2]

   Follow the Introduction and Sections 2.1 and 2.2 closely. State Theorem 2.1 making explicit that (5) is deep. Ensuring that you state and prove Lemma 2.16 and complete the proof of Theorem 1.1 cover as much of 2.1 and 2.2 as time permits.

4. Poincaré complexes: [L, 3.1]

   Cover the whole section paying particular care with the introduction of homology groups with $\mathbb{Z}[\pi]$-coefficients and local coefficients twisted by the orientation character $w\colon \pi_1(M) \to \mathbb{Z}/2$. Give at least one unorientable example.

5. Spherical fibrations and the normal Spivak fibration: [L, 3.2.2 & 3.2.3]

   Cover the two subsections in detail: Theorem 3.38 and Lemma 3.40 are the main results. Focus on defining spherical fibrations and clearly stating the main results, then move on to the proofs.

6. Normal maps and the Pontrjagin-Thom isomorphism: [L, 3.2.1 & 3.3]

   Briefly recall the main results of Section 3.2.1, Theorems 3.26 and 3.28, without proof. The focus on Section 3.3. The main results are Theorems 3.45 and 3.52.

7. Surgery below the middle dimension : [L, 3.4]

   Use 3.4.1 mainly as background motivation but do prove Lemma 3.55. Use 3.4.2 as a black box but give more examples of sets of regular homotopy classes of immersions, in particular $\mathrm{Imm}(S^n, S^{2n})$. State and prove Theorems 3.59 and 3.61 in their full glory.

8. Intersections and self-intersections: [L, 4.1]

   The main results are Lemma 4.3 and 4.7. Draw lots of pictures. Find and present some examples of immersions with interesting self-intersections.

9. Kernels and forms: [L, 4.2]

   This is a long talk: the main result is Theorem 4.27.

10. Even dimensional surgery obstructions: [L, 4.3 & 4.4]

    The main result is Theorem 4.33 and its elaboration, Theorem 4.36, in the simply connected case.

11. Odd dimensional surgery obstructions: [L, 4.5 & 4.6]

    This is a difficult talk. The main results are Theorems 4.44 and 4.46. It is reasonable to follow Lück but some details are left out. The presenter could also read [R, §12].

12. Manifolds with boundary and simple surgery obstructions: [L, 4.7]

    The main results are Theorems 4.47 and 4.61. You may choose to skip some of the algebraic details related to the simple surgery obstruction.

13. The structure set and Wall realisation: [L, 5.1]

    The main points are the definition of the structure set and Theorem 5.5.

    In addition show that the action of $L_{n+1}(\mathbb{Z}[e])$ on the structure set is via connected sum with homotopy spheres and give a detailed describtion of plumbing: [B, II 4.10] and then [B, V 2.1, 2.9 & 2.11]. Note, do not use Browder's definition of the quadratic refinement rather refer to the exercises for the proof of 2.11.

You may also want to attempt to sketch realisation for odd dimensional L-groups.

14. The smooth surgery exact sequence: [L, 5.2, 5.3 & 6.1]

    Everything comes together in this talk! The main result is Theorem 5.12. You should also state and prove that the set of diffeomorphisms classes of manifolds homotopy equivalent to a given manifold M is equal to the orbit space $\mathcal{S}(M)/\mathcal{E}(M)$ where $\mathcal{E}(M)$ is the group of homotopy automorphisms of M.

    As an example, begin with the surgery exact sequence for S^n in Section 6.1.

15. Exotic spheres: [L, 6.1-6.5 & 6.7]

    There is a good deal of beautiful mathematics to cover in this talk. The main result is Theorem 6.11: quickly make your way to its statement and then explain how computing the various groups and maps in this sequence organises the rest of the talk.

    Note that the status of the Kervaire invariant problem has changed since [L] was written: for the current situation see e.g. [MA, §4].

16. The surgery exact sequence for TOP and PL: [L, 5.4, 6.6]

    The central result if Theorem 5.15. Discuss a little the extra challenges of the new categories: PL is not so hard, TOP is very difficult! Assuming Theorem 5.15 use the surgery exact sequence and the Generalised Poincaé Conjecture to compute the homotopy groups $\pi_n(G/TOP)$.

    As an example describe the PL surgery exact sequence for $\mathbb{C}P^n$ and $S^p \times S^q$ given in [W, Theorem 14C.2] and [K-L, §7]: do not go into details as they will come in the next lecture.

    Then state Sullivan's theorems identifying the homotopy type of G/TOP and Kirby-Siebenmann's result on TOP/PL.

    Then take the remainder of the talk to identify the 2-local homotopy type of G/PL: i.e. give the proof of [M-M, Theorem 4.8].

17. PL manifolds homotopy equivalent to $\mathbb{C}P^n$: [W, 14C], [M-M, 8C]

    State the surgery classification of PL-manifolds homotopy equivalent to $\mathbb{C}P^n$ given in [W, Theorem 14C.2]. In particular emphasise the splitting invariants $s_{4k}(f)$ and $s_{4k+2}(f)$.

    Then go through [M-M, 8C] in detail and in particular prove [M-M, Theorem 8.27].

18. Fake tori: [W, 15A]

    The ambitious goal of this talks is to prove [W, Theorem 15A.2].

    Present Walls' identification of $[T^n, G/PL]$ using the fact that G/PL is an infinite loop space. In order to compute $\mathcal{S}^{PL}(T^n)$ you need to compute the surgery obstruction maps $\sigma\colon [T^n, G/PL] \to L_n(\mathbb{Z}[\mathbb{Z}^n])$. Assume Wall's calculation of the the L-groups $L_*(\mathbb{Z}^n)$ and compute σ by using splitting obstructions along the sub-tori $T^m \subset T^n$.

References

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