Seminar: Recent mathematical progress in General Relativity

Winter term 2013/14

Prof. Bernd Ammann

Part I: Preliminaries on the Einstein equations

Solving the Cauchy problem for the vacuum Einstein equation and constraint equations for their initial data

xx Talk no. 1, *Basic concepts in Lorentz geometry,* WILL BE DISCUSSED IN OTHER TALKS.
Assuming a solid knowledge on Riemannian differential geometry, the speaker should rapidly introduce the following objects: time orientation, chronological and causal future and past, chronology and causality condition for Lorentz manifolds, length of causal curves, and their basic properties [Bär, Sec 2.1]. Describe the major results of [Bär, Sec 2.2], sketch its proof without going into technical details and assuming that the audience is familiar with Jacobi field arguments. Introduce and discuss globally hyperbolic Lorentz manifolds [Bär, Sec 2.5] in detail. Then describe shortly the results around [Bär, Sec 2.10.6].

15. and 22.10. Talk no. 2, *Solving the Cauchy problem for the vacuum Einstein equation and constraint equations for their initial data,* JAN-HENDRIK TREUDE.
Derive the constraint equations for the vaccum Einstein equation of Lorentz geometry. Show that after fixing suitable charts (i.e. Fermi coordinates or normal coordinates, choice of the speaker) the system turns into a hyperbolic partial differential equation. Use standard results about hyperbolic partial differential equations to show the local existence of solutions and its uniqueness up to diffeomorphisms. Conclude that for any initial data set there is a unique maximal Cauchy development. Then sketch rapidly Lichnerowicz's method to construct initial data sets.
For the convenience of the speaker (and thus indirectly of the audience), it is helpful to say several details on the literature. One of the first sources is [CY, Sec. 1.1 & Sec. 3.1– 3.3.1], and this reference roughly outlines the most important aspects to be covered. A more modern textbook style presentation is in [Kri2, Sec. 5.4]. However the book by Kriele contains some typos in this section, ask Bernd. Many more details can be found in [FR, Ren], but these sources go too much into the analytical theory for our purpose. Unfortunately most of the literature uses a coordinate dependent formalism, so it would be desirable to present — at least the derivation of the constraint equations — in a modern coordinate free version. Such a coordinate free presentation for the Riemannian analogue can be found e.g. in [AMM, Sec. 2 before Thm. 2.1], see also [BGM] for some background in notation. For the Lichnerowicz method [BI] is a good reference. See also [Ise] as additional literature (for rounding up).

Part II: Preliminaries around spacetime singularities

The goal of this part is to explain the singularity theorems by Hawking and Penrose and its relations to marginally outer trapped surfaces.

22. and 29.10. Talk no. 3, *Hawking's singularity theorem,* Mihaela Pilca.
Explain the results and proof of [Bär, Sec. 2.8]. Several facts from section before should be recalled or shortly introduced.

29.10. and 05.11. Talk no. 4, *Penroses's singularity theorem,* Nicolas Ginoux.
Discuss Penrose's singularity theorem, following [Bär, Sec. 2.9]. If time admits, discuss the advantages and disadvantages of similar theorems with other conditions and conclusions in [Kri1] and [Kri2].

12. and 19.11. Talk no. 5, *Mass and momentum of asymptotically flat manifolds,* Johannes Kleiner.
Introduce mass and momentum of asymptotically flat manifolds. Literature: [Wa2, Chap. 11], [Chr, Chap. 1]

26.11. Supplementary Talk no. 1, *The censorship conjecture,* Bernd Ammann.
The idea of this talk is to try to give an overview which explains some forms of the censorship conjecture, some physical interpretation and some counter examples (in special situations). The talk will need quite a lot of reading of the speaker, and the courage and the right feeling to choose the relevant pieces to give an impression of what should be probably be the good conjecture: [Wa1]=ArXiv: `gr-qc/9710068`, [Wa2, Sec. 12.1], [Kri3], [Pen], [Ear]

Part III: Preliminaries on asymptotically flat manifolds

10.12 Talk no. 6, *Entfällt wegen Krankheit,*

17.12. Talk no. 7, *The Witten proof of the positive mass theorem,* Michael Völkl.
Explain Witten's proof of the positive mass theorem, [Wit81] and [PT].

7.1. Talk no. 8, *The minimal surface proof of the positive mass theorem,* Manuel Streil.
Explain the proof of the positive mass theorem via minimal hypersurfaces, as originally proven by Schoen and Yau, see [SY1] and [SY2]. It might also be good to check whether some secondary literature is better suited for the seminar, e.g. one might consult [Min] which I did not find while preparing.

14.1. und 21.1. Supplementary Talk no. 2, *Proof of the Penrose inequality,* Farid Madani.
The speaker should decide whether he wants to present to Huisken-Illmanen proof of the Penrose inquality [HI] and [Hui], or whether he wants to follow Bray's approach which yields a better estimate for non-connected horizons [Br1].

??? Supplementary Talk no. 3, *Current status and new aspects of the Penrose inequality,* N.N..
[BrC], [Mar1]=ArXiv: `0906.5566`, [CM]=ArXiv: `0911.0883`.

Part IV: Recent developments

Friday, 6.12. Talk no. 9, J. Metzger.
At this place it was planned: "MOTS (marginally outer trapped surfaces) are similar to minimal surfaces, and they are used to define the apparent horizon of a black hole. In this talk it is shown that this apparent horizon is a smooth surface, [AM]=ArXiv: `0708.4252`, see also whether [AMW]=ArXiv: `0704.2889` contains interesting supplementary information for us." It now happens that Jan Metzger will visit us in Regensburg just at the right time for the seminar. So he can explain this to us directly in a special session on Friday, 10-12. And the description above is obviously not his abstract.

?? Talk no. 10, *The spacetime positive mass theorem in dimensions less than eight,* Olaf Müller + ???.
The positive mass theorem is proven without the often used assumption that the spacetime is static (time independent and of product type), [EHLS]=ArXiv: `1110.2087`.

28.1. Talk no. 11, *A generalization of Hawking's black hole topology theorem to higher dimensions,* N.N..
The topology of horizons in arbitrary dimensions (in 3+1 dimensions they are spheres S^2, in general dimensions a manifold admitting positive scalar curvature. Explain the results and maybe the main ideas of proof of [GS]=ArXiv: `gr-qc/0509107`.

4.2 Talk no. 12, *Rigidity of marginally trapped surfaces and the topology of black holes,* N.N..
Explain the results and maybe the main ideas of proof of [Ga1]=ArXiv: `gr-qc/0608118`.

Seminar-Homepage

`http://www.mathematik.uni-r.de/ammann/lehre/2013w__ART`

Literatur

[AMM] B. Ammann, A. Moroianu, S. Moroianu, *The Cauchy Problems for Einstein Metrics and Parallel Spinors*, Commun. Math. Phys. **320**, 173–198 (2013), ArXiv: `1106.2066`

[AMW] L. Andersson, M. Mars, W. Simon, *Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes*, ArXiv: `0704.2889`

[AM] L. Andersson, J. Metzger, *The area of horizons and the trapped region*, ArXiv: 0708.4252

[Bär] C. Bär, *Lorentzgeometrie*, Vorlesungsskript, http://geometrie.math.uni-potsdam.de/documents/baer/skripte/skript-LorGeo.pdf.

[BGM] C. Bär, P. Gauduchon, A. Moroianu, *Generalized Cylinders in Semi-Riemannian and Spin Geometry*, Math. Z. **249**, 545—580 (2005), ArXiv: math/0303095

[BI] R. Bartnik, J. Isenberg, *The Constraint equations*, ArXiv: gr-qc/0405092

[BeC] R. Beig, P. Chrusciel, *Killing vectors in asymptotically flat space-times: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, ArXiv: gr-qc/9510015

[Br1] H. Bray, *Proof of the Riemannian Penrose Inequality Using the Positive Mass Theorem*, J. Diff. Geom. **59** (2001), 177-267. ArXiv: math/9911173

[Br2] H. Bray, On the positive mass, Penrose, and ZAS inequalities in general dimension. (English summary) Surveys in geometric analysis and relativity, 1–27, Adv. Lect. Math. (ALM), 20, Int. Press, Somerville, MA, 2011. ArXiv: 1101.2230

[BrC] H. Bray, P. Chrusciel, The Penrose inequality. (English summary) The Einstein equations and the large scale behavior of gravitational fields, 39–70, Birkhäuser, Basel, 2004., ArXiv: gr-qc/0312047.

[CY] Y. Choquet-Bruhat, J. York, The Cauchy problem. In *General relativity and gravitation, Vol. 1*, pages 99–172. Plenum, New York, 1980.

[Chr] P. Chrusciel, *Lectures on Energy in General Relativity*, Krakow, March–April 2010

[CM] A. Carrasco, M. Mars, *A counter-example to a recent version of the Penrose conjecture*, ArXiv: 0911.0883

[Ear] D. Eardley, *Cosmic Censorship*, Annals of the New York Academy of Sciences, Volume 688, Issue 1, June 1993, *408—417*

[EHLS] M. Eichmair, L.-H. Huang, D. Lee, R.Schoen, *The spacetime positive mass theorem in dimensions less than eight*, ArXiv: 1110.2087

[FR] H. Friedrich, A. D. Rendall, *The Cauchy Problem for the Einstein Equations*, ArXiv: gr-qc/0002074

[Ga1] G. Galloway, *Rigidity of marginally trapped surfaces and the topology of black holes*, ArXiv: gr-qc/0608118.

[Ga2] Gregory J. Galloway, *Spacetime geometry*, http://www.math.miami.edu/~galloway/beijing.pdf.

[GS] G. Galloway, R. Schoen, *A generalization of Hawking's black hole topology theorem to higher dimensions*, ArXiv: gr-qc/0509107.

[Hui] G. Huisken, *Evolution of hypersurfaces by their curvature in Riemannian manifolds.* Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998). Doc. Math. 1998, Extra Vol. II, *349--360*

[HI] G. Huisken, T. Illmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, J. Diff. Geom. 59, 353–437 (2001)

[Ise] J. Isenberg, *Constant mean curvature solutions of the Einstein constraint equations on closed manifolds*, Classical and Quantum Gravity, **12**, 1995, 2249–2274.

[Kri1] M. Kriele, *A Generalization of the Singularity Theorem of Hawking & Penrose to Space-Times with Causality Violations*, Proc. R. Soc. Lond. A 1990 **431**

[Kri2] M. Kriele, *Spacetime, foundations of general relativity and differential geometry*, Lecture Notes in Physics. New Series m: Monographs, vol. 59, Springer-Verlag, Berlin, 1999.

[Kri3] M. Kriele, *A stable class of spacetimes with naked singularities*, Mathematics of gravitation, Part I (Warsaw, 1996), Banach Center Publ., **41**, 169–178, Polish Acad. Sci. Warsaw, 1997.

[Mar1] M. Mars, Present status of the Penrose inequality, ArXiv: 0906.5566

[Min] W. Minicozzi, *The work of Schoen and Yau on manifolds on positive scalar curvature*. Geometry and analysis. No. 1, 185–190, Adv. Lect. Math. (ALM), 17, Int. Press, Somerville, MA, 2011.

[Pen] R. Penrose, Naked Singularities, Ann. New York Acad. Sci. 224 (1973) *125–134*.

[PT] T. Parker, C. Taubes, *On Witten's proof of the positive energy theorem*, Commun. Math. Phys., 1982, **84**, (223–238)

[Ren] A. Rendall, *Theorems on existence and global dynamics for the einstein equations*, Living Rev. Relativity, 5:6., 2002.ArXiv: gr-qc/0203012

[SY1] R. Schoen, S. T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), *45--76*. http://projecteuclid.org/euclid.cmp/1103904790

[SY2] R. Schoen, S. T. Yau, *Proof of the positive mass theorem. II*, Comm. Math. Phys. **79** Number 2 (1981), *231–260*. http://projecteuclid.org/euclid.cmp/1103908964

[Wa1] R. Wald, *Gravitational Collapse and Cosmic Censorship*, ArXiv: gr-qc/9710068

[Wa2] R. Wald, *General relativity*, University of Chicago Press, Chicago, IL, 1984.

[Wit81] E. Witten. A new proof of the positive energy theorem. *Commun. Math. Phys.*, **80** 381–402, 1981.