Seminar: Positive scalar curvature

Winter term 2014/15

Bernd Ammann, Ulrich Bunke, Georgios Raptis

Motivation for the seminar

In the recent months and years, important progress was obtained about the space of metrics of positive scalar curvature on a given compact manifold M of sufficiently high dimension. This question is tightly related to the conjecture (see [2, Conj. A]) that every compact spin manifold (except S^1 and some surface of low genus) carries a metric with non-invertible Dirac operator.
One approach (by Botvinnik, Ebert, and Randal-Williams, see [1]) towards finding non-trivial homotopy groups in the space of metrics with positive scalar curvature builds on previous homotopy-theoretic work by Galatius, Madsen, Randal-Williams, Tillmann and Weiss.

Theorem 1 (Botvinnik, Ebert, Randal-Williams [1]). *Let M be a connected compact spin manifold of dimension $d \geq 7$. Then there is a map*

$$\pi_k(\mathrm{Riem}^+(M), g_0) \to \mathrm{KO}_{d+k+1}(pt)$$

which is non-zero whenever the target is non-zero (i.e. whenever $d+k+1 \equiv 0, 1, 2, 4 \bmod 8$).

Partial results are announced for $d = 6$. It is expected that this work implies that every spin manifold of dimension at least 6 carries a metric with non-invertible Dirac operator.
The goal of the seminar is **not** to discuss the proof of Theorem 1, but to learn the underlying techniques. Namely we want to learn the following theorem which is the first step in this program.

Theorem 2 (Galatius, Madsen, Tillmann, Weiss [6]). *There is a homotopy equivalence*

$$BC_\theta \simeq \Omega^{\infty-1} MT\theta(d).$$

Here C_θ is a suitable category defined using bordisms with some extra structure, and BC_θ is its classifying space. And $\Omega^{\infty-1}MT\theta(d)$ is the infinite loop spectrum associated to a suitable Thom spectrum (shifted in degree by 1).
The second step is the following:

Theorem 3 (Galatius, Randal-Williams [5]). *For $n \geq 3$ there is a homology equivalence*

$$\operatorname*{hocolim}_{g\to\infty} B\mathrm{Diff}_\partial\Big(\underbrace{(S^n\times S^n)\#(S^n\times S^n)\#\cdots(S^n\times S^n)\#}_{g-times} D^{2n}\Big) \to \Omega_0^\infty MT\theta(2n).$$

Overview of the content

The first part of the seminar will discuss Theorem 2. We do not want to follow the original proof, but an essentially improved approach given in [4].
By "discuss" we mean that we should understand very well all involved definitions and structures, and then to go as far as possible into the proofs. The talks should be — as far as possible — understandable without detailed knowledge in homotopy theory. Many constructions (e.g. Section 2 in [4]) require only little knowledge in homotopy theory, but nevertheless we will need several concepts such as e.g. the nerve of a category, the geometric realization of a simplicial set, infinite loop spaces, Thom spectra of inverses of vector bundles, etc.. In order to get efficiently into this material, we begin with a well-adapted introduction into homotopy theoretic tools (Talk no. 1). The speakers of the following talks should plan their talks only for 60 minutes. The remaining time should be reserved for recalling missing conecpts ad hoc, or after or before the proper talk. We also encourage to pose and discuss exercises; a good source of many exercises is e.g. [3] or on this webpage.
The second part of the seminar will be fixed later, depending on how fast we advance and on the audience's interests. Possible topics include the work by Mark Walsh [8] and Theorem 3.

1 The theorem by Galatius, Madsen, Tillmann and Weiss (Thm. 2)

Talk no. 1: Summary of important techniques for the seminar. The talk should summarize the most important homotopy theoretical concepts used in this seminar such as: homotopy colimits; n-connective covers; spectra and its homotopy groups; infinite loop spaces; Thom spectra and its relation to bordism groups; Thom spectra of inverses of vector bundles; concepts from category theory such as the nerve of a category, geometric realisation of a simplicial set, classifying spaces of a category; Serre fibrations. As the above material is probably much more than what can be covered in one session, the speaker has to make a choice, and further parts will be discussed when they appear.

Talk no. 2: The space of submanifolds of $\mathbb{R}^\infty$.
The talk should cover Section 2.1 and Section 2.2 of [4]. Roughly speaking, one has to work in [4] with the "space of all d-dimensional manifolds". For this, it does not suffice to discuss manifolds up to diffeomorphisms. Thus we study the space $\Psi(\mathbb{R}^\infty)$ of all submanifolds in $\mathbb{R}^N$, $N >> 0$, closed as a subset. Similarly one defines $\Psi(U)$ for U open in $\mathbb{R}^N$. On equips $\Psi(U)$ with the compactly supported topology and the inverse limit of the K-topologies, see [4, Sec. 2.1]. Then discuss continuity and openness of the restriction map for those topologies, which finally leads to a characterization of continuous maps into $\Psi(U)$.

Talk no. 3: Tangential structures and smooth maps into $\Psi(U)$.
The talk should cover Section 2.3 and Section 2.4 of [4]. Tangential structures are a common language to describe additional structures on manifolds such as orientations, spin structures, almost complex structures or framings. They are characterised by a map $\theta : X \to BO(d)$. This should be explained in the talk, following [4, Sec. 2.13]. Submanifolds $M \in \Psi(U)$ together with such a tangential structure form the elements of $\Psi_\theta(U)$. One should explain the map

$$\mathrm{Emb}(U, V) \times \psi_\theta(U) \to \Psi_\theta(V)$$

from [4, Thm. 2.13]. Then we should understand smooth maps $f : X \to \Psi_\theta(U)$. Show that such a map ios smooth if and only if its graph is a smooth manifold and if the projection from the graph to X is a submersion.

Talk no. 4: Constructions with tangential structures and the associated cobordism category.
The talk should cover Section 3.1 and Section 3.2 of [4]. The first part contains some constructions that induce homotopy equivalences between differently defined set of sumbmanifolds. Two different versions to define the bordism category (C_θ and D_θ) should be introduced and shown to be homotopy equivalent (Theorem 3.9).

Talk no. 5: The homotopy type of the space of all θ-manifolds.
The talk should cover Section 3.3 of [4]. This talk is the last remaining part of the proof of Thm. 2.

Supplementary Talk no. 1: The solution of a conjecture by Mumford
The cases $n = 1$ of Theorem 3 holds as well. It was first proven by Madsen and Weiss, and it implies a conjecture by Mumford. Section 4 of [4] gives a simpler proof for this case. This supplementary talk is not needed for the proof of Theorem 2.

Preliminary part:

2 The theorem by Galatius and Randal-Williams (Thm. 3)

Talk no. 6: Overview and outline of the proof of Theorem 3. The goal of this talk is to introduce the intermediate cobordism categories $C_{\theta,L}$, $C^{\kappa}_{\theta,L}$ and $C^{\kappa,l}_{\theta,L}$ and present an overview of the proof of Theorem 3. In addition, it should explain the first step of the proof:

$$BC_{\theta,L} \simeq BC_\theta.$$

The material for this talk is in [5, Sections 1,2 and possibly 7.1].

Talk no. 7: Surgery on morphisms (1-2 talks). This talk should cover Section 3 of [5]. The main ideas of the proof of Theorem 3.1 should be explained (up to Theorem 3.4, on the contractibility of a space of surgery data, which will be discussed later).

Talk no. 8: Surgery on objects (1-2 talks). This talk should cover Section 4 of [5] and explain the main ideas of the proof of Theorem 4.1 (up to Theorem 4.5 which will be discussed later).

Talk no. 9: Surgery on objects in the middle dimension(2 talks). Section 5 of [5].

Talk no. 10: Contractibility of surgery data (2-3 talks). Section 6 of [5]. Talk 1: Simplicial techniques. Talk 2: Proofs of Theorems 3.4 and 4.5. Talk 3: Proof of Theorem 5.14.

Talk no. 11: The group completion argument and conclusion. Section 7 of [5] - especially Subsection 7.1. Additional topics: (a) rational cohomology of $\Omega_0^\infty MT\theta$, (b) the homological stability theorem of Galatius-Randal-Williams.

3 Homotopy of $\mathrm{Riem}^+(M)$

Will come later.

4 Related literature

Related article [7]

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/psc`

Note: this seminar program took elements from a similar seminar program in Münster, organized by Johannes Ebert, see `http://wwwmath.uni-muenster.de/u/jeber_02/winter1415/grwseminar.pdf`. We will advance essentially slower, as the group in Münster already has previous experience, e.g. with Mumford's conjecture.

Literatur

[1] B. Botvinnik, J. Ebert, O. R.-W. Infinite loop spaces and positive scalar curvature. ArXiv `http://arxiv.org/abs/1411.7408`.

[2] Bär, C., and Dahl, M. Surgery and the Spectrum of the Dirac Operator. J. reine angew. Math. 552 (2002), 53–76.

[3] Galatius, S. Lectures on the Madsen-Weiss theorem. In Moduli spaces of Riemann surfaces, vol. 20 of IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2013, pp. 139–167. ArXiv version: http://www.math.ku.dk/english/research/top/topmod/notes3.pdf.

[4] Galatius, S., and Randal-Williams, O. Monoids of moduli spaces of manifolds. Geom. Topol. 14, 3 (2010), 1243–1302. ArXiv version: http://arxiv.org/abs/0905.2855.

[5] Galatius, S., and Randal-Williams, O. Stable moduli spaces of high-dimensional manifolds. Acta Math. 212, 2 (2014), 257–377.

[6] Galatius, S., Tillmann, U., Madsen, I., and Weiss, M. The homotopy type of the cobordism category. Acta Math. 202, 2 (2009), 195–239.

[7] Walsh, M. The space of positive scalar curvature metrics on a manifold with boundary. ArXiv http://arxiv.org/abs/1411.2423.

[8] Walsh, M. Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics. Proc. Amer. Math. Soc. 141, 7 (2013), 2475–2484.