Seminar: Lie groups, Lie algebras and symmetric spaces

Summer term 2015

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We continue the program from the winter term which was the following:

1 Lie groups and Lie algebras

In the first part of the seminar we mainly follow the lecture notes by Wolfgang Ziller [12]. We assume that the participants are already familiar with the fundamentals of Lie groups, as explained in the first chapter of [12]. See also [11] and [10] for general facts. The lecture notes by Ziller will be supplemented by several other sources, but for ensuring the global logic of the seminar, the speaker should always put special emphasis on Ziller's notes.

Talk no. 1: Fundamentals and Examples. Assume a solid knowledge on fundamental facts. Recall several facts from [12, 1.6–1.7]:

- the adjoint representation of Lie groups and Lie algebras,
- inner and outer automorphisms of Lie algebras and their relation to derivations
- the Killing form
- complexification and realifications of Lie algebras, real forms of complex Lie algebras
- Passage from Lie groups to real Lie algebras and then to complex algebras, and back. How unique is the way back? Example: $SU(2)$ and $SL_2\mathbb{R}$ give the same complex Lie algebra

Discuss the example of [12, Chap. 2]

Talk no. 2: Basic Structure Theorems. The main subject is [12, Chap. 3 up to Exerc. 3.30]. If time permits, present additional material from [5, Chap. 9].

Talk no. 3: Maximal tori and Weyl groups. The main subject is [12, Chap. 3 from Prop. 3.31]. However, there are many other important facts on maximal tori, and the speaker should choose some from [1, Chap. 4] and [4, IV, 1–3] as convenient for the talk. Explain the complex representation theory of tori $\mathbb{R}/\Gamma$: the irreducible representations are 1-dimensional and classified by weights, i.e. elements in $2\pi\Gamma$. (It can be wise to briefly mention some parts of [12, 5.1] for this goal). The reader should understand the map $R(G) \to R/T)^W$ in [4, IV, Cor 2.8]. Then several classical examples should be given.

Talk no. 4: Complex semi-simple Lie algebras. Cartan subalgebras, roots, Dynkin diagrams and classification of complex semi-simple Lie algebras [12, 4.1–4.2]. Additional material: [5, Chap. 21 and Appendix D] and [9, Chap. III].

Talk no. 5: Weyl groups and lattices. Explain the Weyl Chevalley normal form, the Weyl group of a complex semi-simple Lie algebra, real compact forms, maximal roots, extended Dynkin diagrams, the theorem of Borel-Siebenthal, lattices and discuss examples [12, 4.3–4.7].

2 Geometric significance of the exceptional Lie groups

The talks in the section are logically not needed for the talks in the sections afterwards. The literature provides many interesting facts about the mysterious exceptional Lie group G_2, which attracted much attention recently because of the tight relationship between metrics with holonomy G_2 and metrics with parallel spinors.
Bundles with fibers $\mathbb{O}P^2$ are important for string bordisms, but there exact significance is still under investigation. They seem to play a similar role as $\mathbb{H}P^2$-bundles for spin bordism and $\mathbb{C}P^2$-bundles for oriented bordism. However, these relationship are not the subject of this section, but possibly of a sequel to this seminar.
If time is short or if the main interest lies on other parts, any of the last two talks of this section can be skipped, or also the whole section.

Talk no. 6: The exceptional Lie group G_2. Explain the main concepts [2, Sec. 1 and 2] (see also [3] for some typos of minor importance).

Talk no. 7: Octonionic projective geometry. The geometry of $\mathbb{O}P^2$ [2, Sec. 3].

Talk no. 8: Models and Geometric significance of the exceptional Lie groups. Explain many astonishing facts around the exceptional Lie groups [2, Sec. 4 and 5].

3 Representation theory

We now want to get insight into the representation theory of Lie algebras and thus indirectly of Lie groups. It is important to see many examples as we want to be able to apply the theory to geometric applications after the seminar.

Talk no. 9: Special represention. Explain those parts of [12, 5.1] which have not been explained in previous talks. Then we turn to the representation theory of $sl_2\mathbb{C}$ and $sl_3\mathbb{C}$, see [12, 5.2] and [5, Chap. 11] (additional literature [9, II.7]). Then explain how the representations of $sl_2\mathbb{C}$ can be used to understand the representations of $sl_3\mathbb{C}$, see [5, Chap. 12 and 13] and [6, Chap. 6, Sec. 1]. This will be a motivation for the constructions in the following talks. When working out the details of [5, Chap. 12 and 13] the speaker should try to use the language (roots, weights,.., see [6, Chap. 6, Sec. 2]) we already know from previous talks (in contrast to the typical readers of [5]).

Talk no. 10: Representations of semi-simple Lie algebras. The subject of this talk is [12, 5.3–5.5]. Special emphasis should be put on the highest weight representations and the representations of classical Lie algebras, also real ones. Give many examples.

4 Symmetric spaces

Talk no. 11: Defintions, basic concepts and examples. [12, Chap. 6 up to 6.3]

Talk no. 12: Geometric properties of symmetric spaces. [12, 6.3–6.6]

Talk no. 13: Hermitian symmetric spaces. [12, 6.7] and [7, Chap. VIII]

Talk no. 14: More on Classical Lie groups. [8, Chap. 17] The author may choose some interesting examples out of this reference. Particularly interesting are the descriptions in terms of classical groups of the spin groups to $SO(3)$, $SO(1, s + 1)$, $s = 1, 2, 4$, $SO(6)$, $SO(3, 3)$ and $SO(4, 2)$.
The last two talks might be omitted if we run out of time.

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/lehre/2015s_lie/`

Literatur

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