Exercises in Differential Geometry

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Exercise Sheet no. 0

1. Problem
Let (X, d) be a metric space. Show that a compactly supported continuous function $f : X \to \mathbb{R}$ is uniformly continuous.

Partitions of Unity

Let M be a smooth manifold. Recall that to any open covering $\{V_\beta\}_{\beta\in B}$ of M there exists a locally finite refinement $\{U_\alpha\}_{\alpha\in A}$ together with an subordinated partition of unity $\{\eta_\alpha\}_{\alpha\in A}$. In other words

- $\{U_\alpha\}_{\alpha\in A}$ is an open covering of M, and for any $\alpha \in A$ we find a $\beta \in B$ such that $U_\alpha \subset V_\beta$.
- For any $p \in M$ there exists a neighbourhood W of p such that there are only finitely many $\alpha \in A$ with $W \cap U_\alpha \neq \varnothing$.
- $\operatorname{supp}(\eta_\alpha) \subset\subset U_\alpha$, $\eta_\alpha \geq 0$ and $\sum_{\alpha\in A} \eta_\alpha = 1$. Note that $\sum_{\alpha\in A} \eta_\alpha(p)$ is in fact a finite sum.

2. Problem
Let M be a smooth manifold and $A, B \subset M$ closed and disjoint subsets.

a) Show that there exists a smooth function $f : M \to \mathbb{R}$ such that $f(A) = 1$ and $f(B) = 0$.

b) Conclude that there are disjoint open subsets $U, V \subset M$ separating A and B, i.e. $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.

Recall the following:
Let K be a subset of a smooth manifold M and f a map of some subset of M containing K to $\mathbb{R}^m$. We say that f is **smooth** on K if its restriction to K is locally a restriction of a smooth map, i.e. for every point $p \in K$ there exists an open neighbourhood U of p in M and a smooth map $F : U \to \mathbb{R}^m$ that agrees with f on $U \cap K$.

3. Problem
Let K be closed subset of a smooth manifold M and $f : K \to \mathbb{R}$ a smooth function. Then f is a restriction of a smooth function on M.

4. Problem

Let M be a smooth manifold. Show the following approximation theorem.

a) Let $f : M \to \mathbb{R}^m$ be a continuous map, smooth on a closed subset K of M, and let $\varepsilon > 0$. Then there exists a smooth map $g : M \to \mathbb{R}^m$ with $f|_K = g|_K$ and $\|f(p) - g(p)\| < \varepsilon$ for all $p \in M$.

b) Now suppose that M is connected. Use a) two prove that any two points of M can be joined by a smooth curve.