Exercises in Differential Geometry

Universität Regensburg, Winter Term 2015/16
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Exercise Sheet no. 1

1. Problem (4 points)
Let $B \in \mathbb{R}^{n\times n}$ be symmetric and $A \in \mathrm{GL}(n, \mathbb{R})$. Show that the numbers of positive, zero and negative eigenvalues of $A^\top BA$ does not depend on A.

2. Problem (4 points)
We define a symmetric bilinear form $g^{(1,1)} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$g^{(1,1)}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = xx' - yy' \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2.$$

a) Show that (b_1, b_2) is a generalized orthonormal basis for $g^{(1,1)}$ if and only if there exists a $t \in \mathbb{R}$ and $\delta, \epsilon \in \{1, -1\}$ such that

$$b_1 = \delta \cdot \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} \quad \text{and} \quad b_2 = \epsilon \cdot \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

b) Determine the number of connected components of $O(1, 1) := \mathrm{Isom}_{\mathrm{lin}}\left(\mathbb{R}^2, g^{(1,1)}\right)$.

3. Problem (4 points)
Let X be a topological space. Show that the following conditions are equivalent:

a) X is compact, i.e. every open cover of X has a finite subcover.

b) Let I be an arbitrary index set and $(V_i)_{i\in I}$ a centered system of closed sets, i.e. intersections of finitely many V_i are not empty. Then the intersection of *all* V_i is not empty.

Give a counter example that condition b) does not apply for $X = \mathbb{R}^n$ endowed with the standard topology induced by the Euclidean metric.

4. Problem (4 points)
We consider the orthogonal group $O(n) = \left\{A \in \mathbb{R}^{n\times n} \mid A^\top = A^{-1}\right\}$. We set

$$s : O(n) \to O(n), \quad A \mapsto A^{-1}$$

and define for $X \in O(n)$ the left multiplication

$$L_X : O(n) \to O(n), \quad A \mapsto XA.$$

Furthermore, let $A(n) = \left\{B \in \mathbb{R}^{n\times n} \mid B^\top = -B\right\}$ be the set of skew-symmetric matrices in $\mathbb{R}^{n\times n}$.

a) Prove that $O(n)$ is a smooth submanifold of $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ and compute its dimension.

b) Show that $T_X O(n) = L_X A(n) = \{XB \mid B \in A(n)\}$ for any $X \in O(n)$.

The Euclidean scalar product $\langle X, Y\rangle = \mathrm{Trace}\left(X^\top Y\right)$ on $\mathbb{R}^{n\times n} \cong \mathbb{R}^{n^2}$ induces a Riemannian metric on $O(n)$. Let $X \in O(n)$.

c) Show that the maps s and L_X as defined above are smooth and isometries of $O(n)$.

d) * (2 bonus points) Show that $SO(n) = \{A \in O(n) \mid \det A = 1\}$ is arcwise connected, i.e. any two $A, B \in SO(n)$ can be joined by a continuous path in $SO(n)$.

- Submission deadline: Thursday 22.10.2015 at the beginning of the lecture
- Please write **your name** and the **number of your exercise class** on every sheet of your proposal for solution.
- Each participant should hand in his own solution. A joint solution of a working group is not allowed.