Exercises in Differential Geometry

Universität Regensburg, Winter Term 2015/16
Prof. Dr. Bernd Ammann / Dipl.-Math. Manuel Streil

Exercise Sheet no. 2

1. Problem (4 points)
Let M be a smooth manifold and ∇ be an affine connection on TM. Show that the torsion

$$\begin{aligned} T^{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ T^{\nabla}(X,Y) &= \nabla_X Y - \nabla_Y X - [X,Y] \end{aligned}$$

is defined by a (1,2)-tensor, i.e. there is an $H \in \Gamma(T^{1,2}M)$ with $T^{\nabla} = \mathcal{L}_H$.

2. Problem (4 points)
We consider the vector space $\mathbb{R}^{k+\ell}$ endowed with the symmetric 2-form $g^{(k,\ell)}$ defined as in the lecture by

$$g^{(k,\ell)}(X,Y) = \sum_{i=1}^{k} X^i Y^i - \sum_{i=k+1}^{k+\ell} X^i Y^i$$

for all $X = (X^1, \ldots, X^{k+\ell})^\top$, $Y = (Y^1, \ldots, Y^{k+\ell})^\top \in \mathbb{R}^{k+\ell}$.

a) Let $V \subset \mathbb{R}^{k+\ell}$ be a linear subspace. We set

$$V^{\perp} := \left\{ X \in \mathbb{R}^{k+\ell} \mid g^{(k,\ell)}(X,Y) = 0 \text{ for all } Y \in V \right\}$$

Find a necessary and sufficient condition for $V \oplus V^{\perp} = \mathbb{R}^{k+\ell}$.

b) For a submanifold $M \subset \mathbb{R}^{k+\ell}$ let $i : M \hookrightarrow \mathbb{R}^{k+\ell}$ be the inclusion. We define the first fundamental form as $g := i^* g^{(k,\ell)}$. Determine whether g is a semi-Riemannian metric on M in the following examples; and if it is, then determine its index.

   i) $k = \ell = 1$, $M = S^1 = \{(x,y)^\top \mid x^2 + y^2 = 1\}$.
   ii) $k = \ell = 1$, $M = \{(x, \sqrt{x^2+1})^\top \mid x \in \mathbb{R}\}$.
   iii) $\ell = 1$, $M = (\mathbb{R}x)^{\perp}$, $x \in \mathbb{R}^{k+1}$.

3. Problem (4 points)
We define the pseudo-sphere by

$$S^{k-1,\ell} = \{X \in \mathbb{R}^{k+\ell} \mid g^{(k,\ell)}(X,X) = 1\}.$$

a) Show that $S^{k-1,\ell}$ is a submanifold of $\mathbb{R}^{k+\ell}$. Determine a unit normal field for $S^{k-1,\ell}$, viewed as a hypersurface in $\mathbb{R}^{k+\ell}$. ("normal" here is in the sense of $g^{(k,\ell)}$.)

b) Prove that $g^{(k,\ell)}$ induces a semi-Riemannian metric on $S^{k-1,\ell}$ and compute its index.

c) Show that for every $p \in S^{k-1,\ell}$ there exists a unique surjective linear map $\pi_p : \mathbb{R}^{k+\ell} \to T_pS^{k-1,\ell}$ such that $\pi_p \circ \pi_p = \pi_p$ and $\pi_p(p) = 0$.

d) Show that the Levi-Civita connection on $S^{k-1,\ell}$ is given by

$$(\nabla_X Y)_p = \pi_p\left(\partial_{X_p} Y\right)$$

for arbitrary vector fields $X, Y \in \mathfrak{X}(S^{k-1,\ell})$ and $p \in S^{k-1,\ell}$.

4. Problem (4 points)
In the following we consider the unitary group

$$U(n) = \{A \in \mathbb{C}^{n\times n} \mid A^* = A^{-1}\}$$

and the special unitary group

$$SU(n) = \{A \in U(n) \mid \det(A) = 1\}.$$

a) Show that $U(n)$ and $SU(n)$ are both connected and compact.

b) Show that $U(n)$ is a smooth submanifold of $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$ and compute $T_xU(n)$ for every $x \in U(n)$. What is the dimension of $U(n)$?

c) Let $B : (-\varepsilon, \varepsilon) \to \mathbb{C}^{n\times n}$ be a smooth family of matrices with $B(0) = \mathrm{Id}$. Show that $\frac{d}{dt}\big|_{t=0} \det(B(t)) = \mathrm{Tr}(B'(0))$.

d) Prove that $SU(n)$ is a smooth submanifold of $U(n)$ and compute $T_xSU(n)$ for every point $x \in SU(n)$. *(2 bonus points)*
Hint: Use c) to establish that $SU(n)$ is a submanifold of $U(n)$ in a neighbourhood of $\mathrm{Id} \in SU(n)$.

- Submission deadline: Thursday 29.10.2015 at the beginning of the lecture
- Please write **your name** and the **number of your exercise class** on every sheet of your proposal for solution.
- Each participant should hand in his own solution. A joint solution of a working group is not allowed.