Exercises in Differential Geometry

Universität Regensburg, Winter Term 2015/16
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Exercise Sheet no. 3

1. Problem (4 points)
We define the hyperbolic plane as

$$\mathfrak{H} = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R},\ y \in \mathbb{R}_{>0}\}$$

endowed with the metric $g^{\mathrm{hyp}}_{x+iy} = \frac{1}{y^2} g^{\mathrm{eucl}}$.

a) Compute the Christoffel symbols with respect to the chart given by the identity $\mathfrak{H} \to \mathfrak{H} \subset \mathbb{R}^2$.

b) For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2;\mathbb{R})$ let $\Psi_A(z) = \frac{az+b}{cz+d}$ be the associated Möbius transformation. Show that Ψ_A is an isometry of $\left(\mathfrak{H}, g^{\mathrm{hyp}}\right)$.
Hint: You can use without proof that $\mathrm{SL}(2;\mathbb{R})$ *is generated by matrices of the form* $\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ *with* $a \in \mathbb{R} \setminus \{0\}$ *and* $b \in \mathbb{R}$.

2. Problem (4 points)
We consider $\mathbb{R}^3$ with the standard Euclidean scalar product g^{eucl} and introduce polar coordinates via

$$\begin{aligned} \Psi : \mathbb{R}_{>0} \times (-\pi, \pi) \times (0, \pi) &\to \mathbb{R}^3 \\ (r, \varphi, \vartheta) &\mapsto (r \sin\vartheta \cos\varphi, r \sin\vartheta \sin\varphi, r\cos\vartheta). \end{aligned}$$

Compute the Christoffel symbols of g^{eucl} with respect to the chart Ψ^{-1}.
Hint: Consider the pullback metric $\Psi^*\left(g^{\mathrm{eucl}}\right)_{(r,\varphi,\vartheta)}$.

3. Problem (4 points)
Let M, N be smooth manifolds and ∇ the Levi-Civita-connection on M. For $f \in C^\infty(N, M)$ we denote as in the lecture by ${}^f\nabla$ the induced covariant derivative for vector fields along f.

a) Let $v \in T_pN$ for some $p \in N$ and $Z, \tilde{Z} \in \mathfrak{X}(f)$. Show the following product rule:

$$\partial_v g\left(Z, \tilde{Z}\right) = g\left({}^f\nabla_v Z, \tilde{Z}\right) + g\left(Z, {}^f\nabla_v \tilde{Z}\right).$$

b) If $y : U \to V$ is a chart of N, then

$${}^f\nabla_{\frac{\partial}{\partial y^i}}\left(df\left(\frac{\partial}{\partial y^j}\right)\right) = {}^f\nabla_{\frac{\partial}{\partial y^j}}\left(df\left(\frac{\partial}{\partial y^i}\right)\right).$$

4. Problem (4 points)
Let (M, g) be a Riemannian manifold with Levi-Civita connection, $v \in T_pM$ for some $p \in M$ and $X \in \mathfrak{X}(M)$. We choose a smooth curve $c : (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = v$ and denote by

$$\mathcal{P}_{c,t} : T_{c(0)}M \to T_{c(t)}M$$

the parallel transport along c. Prove that

$$\nabla_v X = \frac{d}{dt}\bigg|_{t=0} \mathcal{P}_{c,t}^{-1}(X(c(t))).$$

- Submission deadline: Since there is no lecture on Thursday 5.11.2015, you can either give your solution to Mrs Bonn, office 217, by 12:00 or to A. Platzer if you attend his exercise class.
- Please write **your name** and the **number of your exercise class** on every sheet of your proposal for solution.
- Each participant should hand in his own solution. A joint solution of a working group is not allowed.