Exercises in Differential Geometry

Universität Regensburg, Winter Term 2015/16
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Exercise Sheet no. 4

1. Problem (4 points)
We consider again the hyperbolic plane $(\mathfrak{H}, g^{\mathrm{hyp}})$.

a) Compute explicitly the parallel transport $\mathcal{P}_{c,t} : T_{(0,1)}\mathfrak{H} \to T_{(t,1)}\mathfrak{H}$ along the curve $c : [0, 1] \to \mathfrak{H}$ with $c(s) = (st, 1)$.

b) Let $x_0 \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$. Show that $\mathbb{R} \ni t \mapsto (x_0, e^{at})$ is a geodesic of $(\mathfrak{H}, g^{\mathrm{hyp}})$.

2. Problem (4 points)
Let (M, g) be a semi-Riemannian manifold with associated Levi-Civita connection ∇. For a diffeomorphism $f : M \to M$ and a vector field $X \in \mathfrak{X}(M)$ we define $f_*X \in \mathfrak{X}(M)$ by $(f_*X)_p = d_{f^{-1}(p)}f(X_{f^{-1}(p)})$ and $f^*X \in \mathfrak{X}(M)$ by $f^*X = (f^{-1})_*X$.

a) Let $f \in \mathrm{Isom}(M, g)$. We define $\tilde{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by

$$\tilde{\nabla}_X Y = f^* \left(\nabla_{f_*X}\, f_*Y\right).$$

Show that $\tilde{\nabla} = \nabla$.

b) Let $f \in \mathrm{Isom}(M, g)$ and $c : (-\varepsilon, \varepsilon) \to M$ be geodesic. Use a) to show that $f \circ c : (-\varepsilon, \varepsilon) \to M$ is also a geodesic.

c) Show that every isometry f of $\left(\mathbb{R}^{k+\ell}, g^{(k,\ell)}\right)$ is affine, i.e there exists a linear map $A : \mathbb{R}^{k+\ell} \to \mathbb{R}^{k+\ell}$ and a constant $b \in \mathbb{R}^{k+\ell}$ such that $f(x) = Ax + b$ for all $x \in \mathbb{R}^{k+\ell}$.

3. Problem (4 points)
Let (M, g) be a semi-Riemannian manifold. For a map $\psi : M \to M$ we denote by $\mathrm{Fix}(\psi)$ the set of fixed points of ψ, i.e. $\mathrm{Fix}(\psi) = \{p \in M \mid \psi(p) = p\}$.

a) Let $\psi \in \mathrm{Isom}(M, g)$. Prove that for $p \in \mathrm{Fix}(\psi)$ and $\xi \in T_pM$ such that $d_p\psi(\xi) = \xi$ the geodesic $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$ and $c'(0) = \xi$ takes all its values in $\mathrm{Fix}(\psi)$.

b) Use a) to the determine the geodesics of

$$H^n = \{X \in \mathbb{R}^{n+1} \mid g^{(n,1)}(X, X) = -1\}.$$

4. Problem (4 points)
We continue discussing $SU(n) = \{A \in \mathbb{C}^{n\times n} \mid A^*A = \mathrm{Id} \text{ and } \det A = 1\}$. Recall that the tangential space at $X \in SU(n)$ is given by $L_X\mathfrak{su}(n)$ with

$$\mathfrak{su}(n) = \{A \in \mathbb{C}^{n\times n} \mid A + A^* = 0 \text{ and } \mathrm{Tr}(A) = 0\}$$

and $L_X : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}, Y \mapsto XY$. For $V, W \in T_X SU(n)$ we define

$$g_X(V,W) = -\frac{1}{2}\mathrm{Tr}\left(((d_X L_{X^{-1}})V) \circ ((d_X L_{X^{-1}})W)\right),$$

where $\circ$ denotes matrix multiplication.

a) Show that $g : SU(n) \ni X \mapsto g_X$ is a Riemannian metric on $SU(n)$.

b) Prove that $SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1 \right\}$.

c) Use b) to show that $\left(S^3 = \{x \in \mathbb{R}^4 \mid \|x\|_{\mathrm{eucl}} = 1\}, g^{\mathrm{eucl}}\right)$ and $(SU(2), g)$ are isometric.

- Submission deadline: Thursday 12.11.2015 at the beginning of the lecture
- Please write **your name** and the **number of your exercise class** on every sheet of your proposal for solution.
- Each participant should hand in his own solution. A joint solution of a working group is not allowed.