Constant scalar curvature metrics on sphere bundles

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This is the continuation of my previous talk at Oberseminar Globale Analysis last month, though the contents will be logically independent. The goal of this talk is to recall and prove our results on metrics of csc (constant scalar curvature) on Hirzebruch surfaces. The plan of my talk is as follows.

1. Connection metrics I briefly recall the general terminology of connection metrics on fiber bundles with finite-dimensional structure group and their relation to Riemannian submersions with totally geodesic fibers.

2. Hirzebruch surfaces I construct metrics of csc on Hirzebruch surfaces $\Sigma_{\mathbf{n}}$, $\mathbf{n} \geq 1$. These metrics are connection metrics and, interestingly, the typical fibers do *not* have csc. It is reasonable to regard our metrics as generalization of the csc product metrics on $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$.

3. Homogeneous sphere bundles I mention the most general case of our construction of csc metrics on homogeneous sphere bundles. These metrics are also connection metrics and generalize the csc metrics on Hirzebruch surfaces. This part of my talk will be sketchy, but at least I'd like to show our argument for homogeneous sphere bundles whose isotropy representation is irreducible.

4. Uniqueness and multiplicity of csc metrics I recall the reduction technique for the Yamabe PDE in the presence of harmonic Riemannian submersions and try to give a brief overview on what is known about the reduced subcritical Yamabe-type PDE. This part will include a sketch of proof of the uniqueness theorem due to Bidaut-Veron–Veron (1991), and a tiny related problem that I think yet unsettled. If time allows, I'd like to mention our recent local bifurcation results, also based on the reduction technique, that improve previous results due to de Lima–Piccione–Zedda (2012) and Bettiol–Piccione (2013).

5. Hirzebruch surfaces revisited I estimate the number of csc metrics in the conformal classes of the metrics on Hirzebruch surfaces. Using the reduction technique in the previous section, I could state a more general (but weaker) statement applicable to a more general situation. In the special case of Hirzebruch surfaces, however, one can obtain more interesting assertions. For example, we show that the equivariant Yamabe invariant of $\mathrm{U}(2) \curvearrowright \Sigma_{\mathbf{n}}$ is ∞.