Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
Prof. Dr. Bernd Ammann/ PD Dr.habil. Olaf Müller
Exercise Sheet 4, due to 17.11.2016

Exercise 1

Let V carry a fixed indefinit symmetric bilinear form. Let W be a $\mathrm{Cl}(V)$ module. A $\mathbb{Z}_2$-*grading* of W is a decomposition $W = W_+ \oplus W_-$, and let ω be the grading operator, i.e. it acts as ± 1 on $W_{\pm 1}$. We say that the action of $\mathrm{Cl}(V)$ on W is *even* iff every element of $\mathrm{Cl}(V)$ preserves the grading of W (= commutes with ω), and *odd* iff the action any element of V reverses the grading (= anti-commutes with ω). A $\mathbb{Z}_2$-*graded Clifford module* is a Clifford module with a $\mathbb{Z}_2$-grading such that the Clifford action is odd.

1. Let W be a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{r,s}$-module. We define a new Clifford multiplication by
$$X \bullet \phi = X \cdot \omega\phi.$$
Show that we obtain a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{s,r}$-module.

2. Show that if we have two $\mathbb{Z}_2$-graded actions on W, one by $\mathrm{Cl}_{r,s}$ and the other by $\mathrm{Cl}_{k,\ell}$, commuting with each other, then we can define a $\mathbb{Z}_2$-graded action of $\mathrm{Cl}_{r+\ell,k+s}$ on W.

3. Show that if there is a even action of $\mathrm{Cl}_{r,s}$ on W and a odd action of $\mathrm{Cl}_{k,\ell}$ on W commuting with each other then this defines an action of $\mathrm{Cl}_{r+k,s+\ell}$.

Exercise 2

Let M be the standard torus. Calculate the spectrum of the Dirac operator on the form bundle $W := \Lambda^* T^* M$.

Exercise 3

Let M be an n-dimensional compact oriented manifold, $k \in \{0, \dots, n\}$.

1. Show that the cup product
$$S : H^k(M) \times H^{n-k}(M) \to \mathbb{R}, \qquad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$
is well-defined, i.e. for every two representatives $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ we have
$$\int \alpha' \wedge \beta' = \int \alpha \wedge \beta.$$

2. Now let $n = 2m$ be even. Show that for $k = m$ we have
$$S([\beta], [\alpha]) = (-1)^m S([\alpha], [\beta]).$$

Exercise 4

Let $D : \Gamma(W) \to \Gamma(W)$ be a generalized Dirac operator over a compact n-dimensional manifold and let $V \subset W$ be a parallel subbundle with

$$\langle \mathcal{K}(v), v \rangle \geq \alpha \langle v, v \rangle \qquad \forall v \in V.$$

Let λ be an eigenvalue $D^2|_{\Gamma(V)}$. Show that

$$\lambda \geq \frac{n}{n-1}\alpha.$$

Hint: Use Exercise 2 on Sheet 3.

Conclude: If $\mathrm{ric}(v, v) \geq (n-1)\kappa\|v\|^2$, then the first positive eigenvalue of the Laplace-Beltrami operator $\Delta : C^\infty(M) \to C^\infty(M)$ is at least $n\kappa$.

Hint: Use Exercise 4 on Sheet 2.