Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 5, due to 24.11.2016

Exercise 1

Let a partial differential operator A of order d between vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$ be given. We want to define the principal symbol of A as a totally symmetric vector bundle homomorphism from $\bigotimes_{i=1}^d \tau^* M$ to $\pi_E^* \otimes \pi_F$. Given a frame $\partial_1, ..., \partial_n$ in $x \in M$, we use the notation $\xi_i := \partial_1^{\otimes i_1} \otimes ... \otimes \partial_n^{\otimes i_k} \in \bigotimes_{k=1}^{|i|} T_x M$ for a multiindex $i = (i_1, ..., i_k)$. Show that the three following characterizations of the term 'principal symbol' are well-defined and equivalent:

1. If A is written w.r.t. trivializing local coordinate charts κ_E of π_E, κ_F of π_F as

$$\kappa_F^{-1} \circ A \circ (\cdot \circ \kappa_E) = \sum_{i \text{ multiindex}, |i| \leq d} A_i \partial_i$$

   for matrices A_i, then the principal symbol $\sigma(A)$ of A is defined by

$$\kappa_F^{-1} \circ \sigma(A) \circ (\cdot \circ \kappa_E) = \sum_{i \text{ multiindex}, |i| = d} A_i \xi_i .$$

2. Let f be a smooth real function on M vanishing at $x \in M$. The principal symbol of A at $x \in M$ is defined as the unique multilinear bundle map $\sigma_x(A)$ from $T_x^* M$ to the endomorphism bundle between π_1 and π_2 such that $\sigma(A)(d_x f, ..., d_x f)(\psi(x)) = \frac{1}{d!}(A(f^\ell \cdot \psi))(x)$ for any local section ψ around x.
3. For $v \in \pi_1^{-1}(x)$, define $\sigma(A)(d_x f \otimes ... \otimes d_x f)(v) := i^d \cdot \lim_{a \to \infty} a^{-l} e^{-iaf} A(e^{iaf} V)(x)$, where V is any local section of π_1 with $V(x) = v$.

Show moreover that for $l = 1$ we have $\sigma(A)(d_x f)(\psi(x)) = [A, m_f](\psi)(x)$ for every smooth function f on M and every section ψ of π_1, where m_f is multiplication with f.

Exercise 2

1. Compute the principal symbols of D and of ∇_X on a Clifford bundle.
2. Show: $[\nabla_X, A]$ is of order $\leq \ell$ for any linear differential operator A of order $\leq \ell$.
3. Give a nice sufficient condition for $[\nabla_X, D]$ being of order 0.

Exercise 3

1. Let H_i be Hilbert spaces and let $A : H_1 \to H_2$ be a bounded linear operator. Show that for any sequence h_n in H_n with weak limit h, the sequence Ah_n converges weakly to Ah.
2. Now let π be a Clifford bundle. Let $S := \{\psi \in \Gamma(\pi) | (\psi, \psi)_{L^2 = 1}\}$ and let $\phi_i \in S$ with $\lim_i (D\phi_i, D\phi_i)_{L^2} = \mu := \inf\{(D\gamma, D\gamma)_{L^2} | \gamma \in S\}$. Show that there is a subsequence $\phi_{i(j)}$ converging strongly to a section ϕ such that $D\phi_i$ converges weakly to a section ψ. Show also that in this situation $D\phi = \psi$ and $D^2\phi = \mu\phi$.