Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 6, due to 1.12.2016

Exercise 1

Let W_i be Clifford modules with Clifford multiplication $\cdot_i : V_i \times W_i \to W_i$. Prove that $W_1 \otimes W_2 \otimes \mathbb{C}^2$ is a Clifford module over $V_1 \oplus V_2$ with the Clifford multiplication $\cdot$ defined by $v_1 \cdot := v_1 \cdot_1 \otimes \mathrm{id}_{\mathrm{V}_2} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for all $v_1 \in V_1$ and $v_2 \cdot := \mathrm{id}_{V_1} \otimes \mathrm{id}_{V_2} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. for all $v_2 \in V_2$.

Exercise 2

(Change of notation: Replace $\boxtimes$ by $\circledast$.)
Show that for two real vector bundles $E \to M$ and $F \to N$, we have $L^2(E \circledast_{\mathbb{R}} F) = L^2(E) \hat{\otimes}_{\mathbb{R}} L^2(F)$, where $\hat{\otimes}_{\mathbb{R}}$ is the real Hilbert space tensor product (the metric completion of the usual algebraic tensor product w.r.t. the usual product metric). What happens for the special case of E and F being the trivial real line bundle? Discuss whether similar statements also hold for complex vector bundles instead of real ones. Describe a Hilbert space orthonormal basis of $L^2(E \circledast F)$ in terms of a Hilbert space orthonormal basis $(\phi_i)_{i \in I}$ resp. $(\psi_j)_{j \in J}$ of $L^2(E)$ resp. $L^2(F)$.

Exercise 3

Assume that we have a Clifford bundles W_1 over M_1 and a Clifford module W_2 over M_2. Then the bundle $Q := (W_1 \circledast W_2) \otimes \mathbb{C}^2$ carries a natural structure of a Clifford bundle, fiberwise given as in Exercise 1. Prove that

$$D_Q|_{W_1 \circledast W_2} = D_1 \otimes \mathrm{id} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathrm{id} \otimes D_2 \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and

$$D_Q^2|_{W_1 \circledast W_2} = D_1^2 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{D}_2^2 \otimes \mathrm{id}.$$