Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 7, due to 8.12.2016

Exercise 1

Let (M, g) be a compact Riemannian manifold with a Clifford bundle $W \to M$. We assume that $\rho_M : G \times M \to M$ and $\rho_W : G \times W \to W$ are compatible properly discontinuous and free group actions, isometric on M, and preserving the connection and metric on $W \to M$. Equip $N := M/\rho_M$ with the Riemannian metric $\underline{g}$ such that the quotient map is a local isometry, and discuss the Clifford bundle structure on W/ρ_W. Express the heat kernel on $(N, \underline{g})$ by the heat kernel on (M, g).

Exercise 2

Let (M, g) be a compact Riemannian manifold. We want to show that the map λ_i assigning to a Riemannian metric g the i-th eigenvalue of the Hodge-Laplacian Δ_g on k-forms is a continuous real function on $C^\infty(\mathrm{Sym}^2_+ T^*M)$ equipped with the C^0-topology. To this aim:

1. Show that the spectrum of $\Delta_g|_{\Omega^k}$ is equal to

$$0_{b_k} \,\hat{\cup}\, \mathrm{spec}(\Delta_k^{\mathrm{exact}} := \Delta_g|_{d\Omega^{k-1}}) \,\hat{\cup}\, \mathrm{spec}(\Delta_k^{\mathrm{coexact}} := \Delta_g|_{\delta\Omega^{k+1}}),$$

   where $\hat{\cup}$ denotes the union with multiplicities, and where 0_{b_k} is zero with multiplicity b_k (= k-th Betti number). Show moreover that $d|_{delta\Omega^{k+1}} : \delta\Omega^{k+1} \to d\Omega^k$ is an isomorphism, that we have $\Delta^{\mathrm{exact}} \circ d = d \circ \Delta^{\mathrm{coexact}} : \delta\Omega^{k+1} \to d\Omega^k$, and that the second part of the spectrum equals to the third part of the spectrum with k shifted by one.

2. Prove each equality in the following chain of equations:

$$\begin{aligned}
\lambda_i(\Delta_g^{\mathrm{coexact}}) &= \min_{\substack{V \subset \delta\Omega^{k+1} \\ \dim V = i}} \max_{a \in V \setminus \{0\}} \frac{(\Delta_g^{\mathrm{coexact}} a, a)}{(a, a)} \\
&= \min_{\substack{V \subset \delta\Omega^{k+1} \\ \dim V = i}} \max_{a \in V \setminus \{0\}} \frac{(da, da)}{(a, a)} \\
&= \min_{\substack{\ker(d) \subset \widetilde{V} \subset \Omega^k \\ \dim(\widetilde{V}/\ker(d)) = i}} \max_{a \in \widetilde{V} \setminus \{0\}} \frac{(da, da)}{(a, a)}.
\end{aligned}$$

3. Show the continuity as claimed above.