Übungen zur Indextheorie

Universität Regensburg, Wintersemester 2016/17
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Exercise Sheet 8, due to 15.12.2016

Exercise 1

A $\mathbb{Z}_2$*-grading* (A^0, A^1) *of an algebra* A is a splitting of A into sub-vector spaces A^0 and A^1 such that $A^0A^0, A^1A^1 \subset A^0$ and $A^0A^1, A^1A^0 \in A^1$. Show that the $\mathbb{Z}_2$-grading $(\bigoplus_{i\in 2\mathbb{N}}(\mathbb{R}^n)^{\otimes i}, \bigoplus_{i\in 2\mathbb{N}+1}(\mathbb{R}^n)^{\otimes i})$ of $\bigoplus_{i\in\mathbb{N}}(\mathbb{R}^n)^{\otimes i}$ induces a $\mathbb{Z}_2$-grading $(\mathrm{Cl}_n^0, \mathrm{Cl}_n^1)$ of Cl_n. Show that Cl_{n+1}^0 is isomorphic to Cl_n as algebra, and discuss also the complex case.

Exercise 2

Let $\overline{\Sigma}_n$ be the Clifford module obtained from Σ_n by replacing the multiplication with i by the multiplication with $-i$. For which n mod4 is $\overline{\Sigma}_n$ isomorphic to Σ_n as $\mathbb{C}l_n$-module?

Exercise 3

Let n be even.

1. Describe the actions of $\mathbb{C}l_n$ on itself by left resp. right multiplication in terms of the identification

$$\mathbb{C}l_n = \mathrm{End}(\Sigma_n) = \Sigma_n \otimes \overline{\Sigma_n}. \tag{1}$$

2. Show that $\mathbb{C}l_n^0$ resp. $\mathbb{C}l_n^1$ are the $(+1)$- resp. (-1)-eigenspaces of the conjugation with $\omega_{\mathbb{C}}$.

3. Define $\mathbb{C}l_n^{\pm} := (\frac{1+\omega_{\mathbb{C}}}{2})\mathbb{C}l_n$. Show that $\mathbb{C}l_n = \mathbb{C}l_n^+ \oplus \mathbb{C}l_n^-$. Do the two splittings coincide? Describe the two splittings by means of the identification (1) and in terms of the splitting $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ into the eigenspaces of $\omega_{\mathbb{C}}$.