Seminar: Ricci flow, Part II

Summer term 2017

Prof. Bernd Ammann

Monday 8:30-10.00,
Dates: 29.3., 24.4, 8.5., 15.5., 22.5., 29.5., 12.6., 19.6., 26.6., 3.7., 10.7., 17.7., 24.7.?, ??, ??, next term ??, (16 dates, minus 2–3 dates with obstructions:)

Required preliminary: [1, Chap. 4]

Talk no. 1: Overview and discussion. *29.3.*

3 Geometrization in dimension 3

24.4

Talk no. 2: Geometrization of 3-manifolds. *8.5.* , *15.5.*

Describe the statement of the geometrization of 3-manifolds à la Thurston (without proofs). Roughly indicate what the Ricci flow will do to yield the decomposition in these geometric pieces. There are many publications with similar content: [2],[3],[4],[5], [6], [7].

4 Basic concepts and ideas about the Ricci flow in arbitrary dimensions

Talk no. 3: Short time existence and uniqueness in general. *24.4*
Short time existence and uniqueness of the Ricci flow, using the DeTurck trick. I suggest to follow the article [8]. Probably one has to follow some of the cited sources (e.g. Besse's book) for details. An alternative source is [1, Chap. 3]. A third alternative is [10, 5.1 und 5.2] which has the advantage that it can be well combined with some preliminary general comments on parabolic PDEs [10, Chap. 4]. The choice if left to the speaker. Depending on the choice one or two sessions will be adequate.

Talk no. 4: Neck pinch. *22.5.*
The original reference is [11]. The speaker should investigate whether he wants to follow [11] or the more recent reference [12]. Beware of the typos in the corresponding section of [1]. If time permits, it would be interesting to say some words about the degenerate neckpinch [13]. See also [21, Sec. 12].

Talk no. 5: Curvature evolution and Shi estimates. *29.5.*
Recall the scalar maximum principle from [1, 4.1, Pages 93–94]. Check that

this implies the weak and strong maximum principle of [10, 3.1]. Then obtain basic control on the curvature [10, 3.2]. The final goal of the talk are the Shi estimates [10, 3.3].

Supplementary Talk no. 1: Shi estimates for other parabolic flows It might be interesting to follow as a sideway some articles about Shi estimates for other parabolic flows. Literature will be given later. As this is a bit beside the main track, this talk might be in the Arbeitsgruppenseminar.

Talk no. 6: Long time existence until curvature blow-up. *12.6.*
This talk is comparably short, maybe only half a session. We want to show if a solution is maximally defined up to time $T < \infty$, then

$$\sup_{x \in M} |\mathrm{Rm}(x,t)| \nearrow \infty \quad \text{for } t \nearrow \infty,$$

[10, 5.3]. This behavior is called *curvature blowup.*

Talk no. 7: Gromov-Hausdorff convergence in the Riemannian setting. *19.6.* , *26.6.*
This talk contains many structures and results of central importance within Riemannian geometry. The following talk will strongly build on this talk, but in this talk we should also focus on the fact that Gromov-Hausdorff is also powerful if no flow is present. Subjects to be covered are Hausdorff distance, Gromov-Hausdorff distance, Cheeger-Gromov finiteness theorem, Cheeger-Gromov convergence for spaces and for spaces with basepoint. A good source is [14, Chap. 5, Sections 1,3,4,5]. The talk should include [10, 7.1], but go in much more detail. There are also several original references which might be helpful, by Gromov [15], Cheeger [16], Greene and Wu [18], and Peters [17], and many such facts are collected in the lecture notes of Petersen [19].

Talk no. 8: Gromov-Hausdorff convergence for Ricci flows and applications. *3.7.*

In this talk, we want to derive an analogue of the previous talk's convergence results in the world of Ricci flow. Control over derviatives of curvature gets easier because of Shi estimates, [14, Chap. 5, Sect. 1.1]. Applications for the Ricci flow require not a limit of Riemann metrics, but of whole solutions of the Ricci flow. The talk should cover in particular [10, 7.2 and 7.3] and [14, Chap. 5, 2], and extension in [25, Chapter 3 and 4] are highly welcome, but seem to involved for the purpose of our seminar. The original publications is [20, Sec. 16] and the speaker might also find interesting facts in there. An important step will be that the audience understands why lower bounds on the injectivity radius are centrally important. At the end of the talk we should have covered the results of Bamler in [21, Sec. 13], however this source does not give proofs, but refers to Bamler's Master thesis.

Talk no. 9: The role of Ricci solitons as limit solutions. *10.7.*
This talk will directly continue the last one. One analyses the situation that a

Ricci is maximally defined until $T < \infty$, the rescalings of this flow converge to an ancient complete solution of the Ricci flow. Explain (with no detailed proof, only some sketch) [25, Theorem 6.68 in 6.5.3.1]. Discuss the definition and role of κ-solutions [22] without going in too much details.
Discuss the classification of 2- and 3-dimensional κ-solutions [22], [23].
Further source for singularities in 3 dimensions: Cao [**?**].

Decision

At this point of the seminar we probaby have to make a choice. And the choice will probably depend on when we reach this point.

- Either we accept that we have now understood the basic elements of the Ricci flow proof of the geometrization without having gone through the details.
- Or we go again a step deeper into the subject. In this case I think it would be good to follow the lecture notes by Bamler [21]. In this case we have to continue the seminar in the winter term.
- There is also an intermediate way, in which we only discuss some elements of the proof, e.g. the discussion of the $\mathcal{W}$-functional with leads to non-collapsing results.

This is why the program is not worked out in detail from here. There might also be inconsistencies in the plan for next 7 talks. And we cannot fix yet a the time schedule.

Talk no. 10: Uhlenbeck's trick and curvature evolution.
[21, Sec. 4 and 5]. Alternative source for the first part: [10, 9.4].

Talk no. 11: Curvature maximum principles and vector bundle maximum principles.
Discuss those parts of [21, Sec. 6] which were not covered before. Then discuss [21, Sec. 8].

We are now entering the ideas of Perelman's spectacular articles on the Ricci flow, in particular the first one [27]. The article can be read well if one simultaneously consult the notes by Kleiner and Lott [26].

Talk no. 12: Perelman's $\mathcal{F}$- and $\mathcal{W}$-functionals.
Explain [21, Sec. 17]. Alternative/additional literature is [10, 6.1–6.4] using [26, Page 20 (?)] for (6.4.1) in [10], and [10, 8.1–8.2].

Talk no. 13: No local collapsing theorem I.
Explain [21, Sec. 18] which is a **stronger** than the original statement by Perelman [27, section 4]. The alternative (weaker) source [10, 8.3] proves the same no local collapsing theorem as [27, section 4]

Unfortunately the no local collapsing theorem is still not sufficient and one has to dig even deeper. We have to study the $\mathcal{L}$-geometry.

Talk no. 14: $\mathcal{L}$-geometry.
Explain [21, Sec. 20] $\mathcal{L}$-geometry, $\mathcal{L}$-geodesics, $\mathcal{L}$-Jacobi fields, Monotonicity of the reduced volume. Original literature: [27, section 7].

Talk no. 15: No local collapsing theorem II.
Explain [21, Sec. 21] No local collapsing theorem II. Original literature [27, section 8].

Talk no. 16: κ-solutions.
Explain [21, Sec. 22]

5 Possible extensions

Talk no. 17: Examples of quantities preserved under the Ricci flow.

Talk no. 18: More on curvature evolution.

6 Ricci flow in 3 dimensions

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/lehre/2017s_ricci2`

Important collection of links on the www

- John Lott:
  `https://math.berkeley.edu/~lott/ricciflow/perelman.html`
- Misha Kapovich:
  `https://www.math.ucdavis.edu/~kapovich/eprints.html`

Central Literature

[1] B. Chow, D. Knopf; The Ricci Flow: An Introduction, AMS 2004, E-Book link UB Regensburg

[2] J. Milnor, Towards the Poincaré Conjecture and the Classification of 3-Manifolds, Notices Amer. Math. Soc. 50 (2003), no. 10, 1226–1233. `http://www.ams.org/notices/200310/fea-milnor.pdf`

[3] M. Kapovich, Geometrization Conjecture and Ricci Flow. Talk at the MPI Arbeitstagung in Bonn, 2003. Available on `https://www.math.ucdavis.edu/~kapovich/eprints.html`.

[4] M. Kapovich, Geometrization Theorem. McGraw Hill 2010 Yearbook of Science and Technology. Available on `https://www.math.ucdavis.edu/~kapovich/eprints.html`.

[5] M. Anderson, Geometrization of 3-Manifolds via the Ricci Flow, `http://www.ams.org/notices/200402/200402-body-pdf.html`.

[6] J. Milnor, The Poincaré conjecture

[7] B. Leeb, Geometrisierung 3-dimensionaler Mannigfaltigkeiten und Ricci-Fluß: zu Perelmans Beweis der Vermutungen von Poincaré und Thurston. Mitt. Dtsch. Math.-Ver. 14 (2006), no. 4, 213–221. `https://web.archive.org/web/20080419011624/http://www.mathematik.uni-muenchen.de/%7Eleeb/preprints/pv.pdf`

[8] D. DeTurck; Deforming metrics in the direction of their Ricci tensors; J. Differential Geom. 18 (1983), no. 1, 157–162. Available also as part of a collection, namely [9, article 3].

[9] H.D. Cao, B. Chow, S.C. Chu, S.T. Yau (editeurs), Collected Papers on Ricci Flow, International Press, Series in Geometry and Topology, Volume 37

[10] P. Topping; Lectures on the Ricci flow, LMS Lecture Note series 325, Cambridge

[11] M. Simon; A class of Riemannian manifolds that pinch when evolved by Ricci flow, Manuscripta Math. **101**, *89 – 114* (2000)

[12] S. Angenent, D. Knopf; An example of neckpinching for Ricci flow on S^{n+1}. Math. Res. Lett. **11**, 493–518 (2004). DOI

[13] S. Angenent, D. Knopf; Degenerate neckpinches in Ricci flow. J. Reine Angew. Math. **709**, *81–117* (2015).

[14] J. Morgan, G. Tian, Ricci Flow and the Poincaré Conjecture, ArXiv: `math/0607607`

[15] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Based on the 1981 French original, Progress in Mathematics **152**, Birkhäuser Boston 1999.

[16] J. Cheeger; Finiteness theorems for Riemannian manifolds, Amer. J. Math. **92** (1970), 61–74.

[17] S. Peters, Convergence of Riemannian manifolds, Compositio Math. **62** (1987), 3–16.

[18] R. Greene, H. Wu, Lipschitz convergence of Riemannian manifolds, Pacific J. Math. **131** (1988), 119–141.

[19] P. Petersen, Convergence theorems in Riemannian geometry. Comparison geometry (Berkeley, CA, 1993–94), 167–202, Math. Sci. Res. Inst. Publ. **30**, Cambridge Univ. Press, Cambridge, 1997.

[20] R. Hamilton; The formation of singularities in the Ricci flow. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995. `http://www.ams.org/mathscinet-getitem?mr=1375255`. Available also as part of a collection, namely [9, article 1].

[21] R. Bamler, Ricci flow, Lecture Notes, `http://web.stanford.edu/~cmad/Papers/RicciFlowNotes.pdf`

[22] Rugang Ye, κ-solutions of the Ricci flow, Preprint, see seminar cloud

[23] Rugang Ye, Entropy Functionals, Sobolev Inequalities and kappa-Noncollapsing Estimates along the Ricci Flow, `https://arxiv.org/abs/0709.2724`

[24] H.-D. Cao, Singularities of the Ricci flow on 3-manifolds, Mathemática Contemporônea, **34** (2008), 103–133.

[25] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, The Ricci flow, techniques and applications, PartI Geometric Aspects, AMS Math. surv. and Monographs, Vol. 135.

[26] B. Kleiner, J. Lott, Notes on Perelman's Paper, Geometry & Topology, **12**, (2008), 2587–2855. The version on the arxive is updated further: `https://arxiv.org/abs/math/0605667`

[27] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, ArXiv: `math.DG/0211159`

[28] G. Perelman, Ricci flow with surgery on three-manifolds, ArXiv: `math.DG/0303109`

[29] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, ArXiv: `math.DG/0307245`

Extended Literature

[30] G. Besson; Preuve de la conjecture de Poincaré en déformant la métrique par la courbure de Ricci (d'après G. Perelman). Séminaire Bourbaki. Vol. 2004/2005. Astérisque No. 307 (2006), Exp. No. 947, ix, 309–347.

[31] L. Bessières, G. Besson, S. Maillot, M. Boileau, J. Porti, Geometrisation of 3-manifolds, EMS Tracts in Mathematics, 13. European Mathematical Society (EMS), Zürich, 2010.

[32] C. Böhm; B. Wilking; Manifolds with positive curvature operators are space forms. Ann. of Math. (2) 167 (2008), no. 3, 1079–1097.

[33] M. Boileau; Geometrization of 3-manifolds with symmetries. `http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.106.8849`

[34] S. Brendle; Ricci Flow and the Sphere Theorem, AMS Graduate Studies Vol. 111, 2010

[35] S. Brendle; R. Schoen; Manifolds with 1/4-pinched curvature are space forms. J. Amer. Math. Soc. 22 (2009), no. 1, 287–307.

[36] S. Brendle; R. Schoen; Classification of manifolds with weakly 1/4-pinched curvatures, Acta Math. 200, 1–13 (2008) See also ArXiv: `0705.3963`

[37] S. Brendle; R. Schoen; Curvature, sphere theorems, and the Ricci flow; to appear at Bull. AMS, ArXiv: `1001.2278`

[38] B. Chow; P. Lu; L. Ni; Hamilton's Ricci flow; Graduate Studies in Mathematics, 77; American Mathematical Society, Providence, RI; Science Press, New York, 2006

[39] B. Chow, J. Diff. Geom., **33**, (1991), pp. 325–334

[40] A. Hatcher. Notes on basic 3-manifold topology, Link siehe Homepage.

[41] J. Lott, www-Seite, `http://math.berkeley.edu/~lott/ricciflow/perelman.html`

[42] J. Morgan, Recent progress on the Poincaré Conjecture and the classification of 3-manifolds. Bull. Amer. Math. Soc. 42 (2005), 57-78. Link siehe Homepage

[43] T. Shioya, T. Yamaguchi, Volume collapsed three-manifolds with a lower curvature bound, Math. Ann. 333 (2005), no. 1, 131–155.

[44] T. Shioya, T. Yamaguchi, Collapsing three-manifolds under a lower curvature bound. J. Differential Geom. 56 (2000), no. 1, 1–66.

Literature on Surfaces

[45] X. Chen, P. Lu, G. Tian; A note on uniformization of Riemann surfaces by Ricci flow, Proc. Amer. Math. Soc., **134**(2006) *3391–3393 (electronic)*, DOI 10.1090/S0002-9939-06-08360-2, `https://arxiv.org/abs/math/0505163`

[46] R. Hamilton; The Ricci flow on surfaces. Available also as part of a collection, namely [9, article 6].

[47] J.A. Isenberg, R. Mazzeo, N. Sesum; Ricci flow in two dimensions, https://arxiv.org/abs/1103.4669

[48] L. Ma; Ricci-Hamilton flow on surfaces, lecture notes, http://faculty.math.tsinghua.edu.cn/~lma/lectures/ricciface2.pdf

[49] B. Stetler; The Ricci flow on surfaces and the uniformization theorem, Master-Arbeit

Alternative literature list on web site of the seminar.