The sign of a multiconformal class

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1 Background

The smooth Yamabe invariant $\sigma(M) \in \mathbb{R}$ of a closed manifold M^m is defined through the minimax procedure for the volume-normalized total scalar curvature functional (also called the Einstein–Hilbert functional) as

$$\sigma(M) = \sup_{[g]} \mu(M, [g]), \qquad \text{(the smooth Yamabe invariant)}$$

$$\mu(M, [g]) = \inf_{\tilde{g} \in [g]} \frac{\int_M R_{\tilde{g}} d\mu_{\tilde{g}}}{\left(\int_M d\mu_{\tilde{g}}\right)^{\frac{m-2}{m}}} \qquad \text{(the conformal Yamabe constant)}$$

where $[g]$ is an arbitrary conformal class on M. We remark that $\sigma(M) > 0$ if and only if M admits a metric of positive scalar curvature. Due to the saddle point structure of critical points for the Einstein–Hilbert functional, the Yamabe invariant plays a fundamental role in understanding distinguished Riemannian metrics on a manifold.

It has been elusive to determine the Yamabe invariants of such manifolds of crucial importance as compact space-forms and their direct products. For example, it is an open problem to determine $\sigma(S^2 \times S^2)$. In this talk, we study the intrinsic geometry of multiconformal classes with applications to problems of these kinds in mind.

2 Multiconformal classes

Let (M^m, g) be a Riemannian manifold with an orthogonal splitting $TM = E_1 \oplus \cdots \oplus E_l$ of the tangent bundle into its subbundles E_i of rank m_i, $1 \leq i \leq l$. We write $g = g_1 \oplus \cdots \oplus g_l$ for some fiberwise inner product $g_i \in \Gamma(E_i^* \otimes E_i^*)$.

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Definition 1. We call

$$[\![g]\!] = \{f_1^2 g_1 \oplus \cdots \oplus f_l^2 g_l \mid f_1, \ldots, f_l : M \to \mathbb{R}_{>0}\}$$

the multiconformal class of g.

We note that a multiconformal class contains not only the whole conformal classes of metrics in $[\![g]\!]$ but also their warped product deformations of all kinds. This observation leads to define the following invariant

$$\sigma(M, [\![g]\!]) := \sup_{[\tilde{g}] \subset [\![g]\!]} \mu(M, [\tilde{g}])$$

of a multiconformal class, which verifies

$$\sigma(M, [\![g]\!]) \leq \sigma(M).$$

3 The scalar curvature formula

Assume that $g = g_1 \oplus \cdots \oplus g_l$ is a product Riemannian metric defined on $M^m = M_1^{m_1} \times \cdots \times M_l^{m_l}$. We briefly explain how we calculated the scalar curvature $R_{\tilde{g}}$ of the multiconformally deformed metric $\tilde{g} = f_1^2 g_1 \oplus \cdots \oplus f_l^2 g_l$ as

$$\begin{aligned} R_{\tilde{g}} = & \sum_i \frac{R_i}{f_i^2} - 2 \sum_i (m_i - 1) \frac{\Delta_i f_i}{f_i^3} - 2 \sum_{i \neq j} m_j \frac{\Delta_i f_j}{f_i^2 f_j} \\ & - \sum_i (m_i - 1)(m_i - 4) \frac{|\mathrm{grad}_i\, f_i|^2}{f_i^4} - 2 \sum_{i \neq j} m_j (m_i - 2) \frac{\langle \mathrm{grad}_i\, f_i, \mathrm{grad}_i\, f_j \rangle}{f_i^3 f_j} \qquad (\star) \\ & - \sum_{i \neq j} m_j (m_j - 1) \frac{|\mathrm{grad}_i\, f_j|^2}{f_i^2 f_j^2} - \sum_{i \neq j, j \neq k, k \neq i} m_j m_k \frac{\langle \mathrm{grad}_i\, f_j, \mathrm{grad}_i\, f_k \rangle}{f_i^2 f_j f_k} \end{aligned}$$

where the indices i, j, k run over $\{1, \ldots, l\}$. Here, R_i, Δ_i, and grad_i are the scalar curvature, the Laplacian[1], and the gradient operator of g_i, respectively. A key to deriving this formula is to get rid of the conventional O'Neill's formalism and make use of Karcher's idea [2].

4 Illustrating examples

Instead of describing our results in full generality, we consider the following situation. For $i \in \{1, 2\}$, let (Σ_i, ds_i^2) be a closed Riemannian surface[2] of Euler characteristic χ_i.

[1] $\Delta = \frac{d^2}{ds^2}$ on $(\mathbb{R}, ds^2)$

[2] a compact Riemannian manifold of dimension 2 without boundary

Theorem 2. (1) *In the multiconformal class* $[\![ds_1^2 \oplus ds_2^2]\!]$*, there exists a metric of constant negative scalar curvature whose energy is arbitrarily large in absolute value, regardless of* (Σ_1, ds_1^2) *and* (Σ_2, ds_2^2)*.*

(2) *If* $\chi_1 = \chi_2 = 0$*, then* $\sigma(\Sigma_1 \times \Sigma_2, [\![ds_1^2 \oplus ds_2^2]\!]) = 0$ *is only attained by scalar-flat product metrics.*

(3) *If* $\chi_1 = 0$ *and* $\chi_2 < 0$ *or if* $\chi_1 < 0$ *and* $\chi_2 = 0$*, then* $\sigma(\Sigma_1 \times \Sigma_2, [\![ds_1^2 \oplus ds_2^2]\!]) = 0$ *is not attained.*

(4) *If* $\chi_1, \chi_2 < 0$*, then* $\sigma(\Sigma_1 \times \Sigma_2, [\![ds_1^2 \oplus ds_2^2]\!]) < 0$ *is attained by a unique unit-volume product metric of constant scalar curvature.*

Proof. Certain integral estimates for solutions of the underdetermined PDE ($\star$) with $R_{\tilde{g}}$ an unknown constant. □

Remark 3.
- Theorem 2 (1) implies that the space of unit-volume constant scalar curvature metrics within the multiconformal class $[\![ds_1^2 \oplus ds_2^2]\!]$ is always noncompact, with respect to an arbitrary topology finer than say C^2. In contrast, such a space within a conformal class is known to be compact in dimension 4 (cf. Druet [1], Khuri–Marques–Schoen [3]).

- The Einstein–Hilbert functional E behaves along the family

$$\{c_1^2 ds_1^2 \oplus c_2^2 ds_2^2\}_{c_1, c_2 \in \mathbb{R}_{>0}} \subset [\![ds_1^2 \oplus ds_2^2]\!]$$

of product metrics on $\Sigma_1 \times \Sigma_2$ as in Table 1, provided that both ds_1^2 and ds_2^2 have constant Gaussian curvatures. Theorem 2 (2)–(4) implies in particular that $\sigma(\Sigma_1 \times \Sigma_2, [\![ds_1^2 \oplus ds_2^2]\!]) > 0$ if and only if $\chi_1 > 0$ or $\chi_2 > 0$.

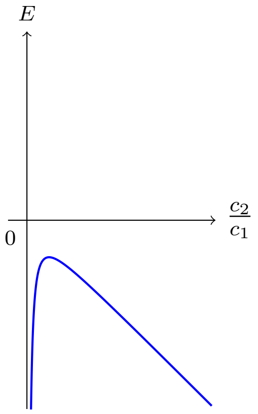
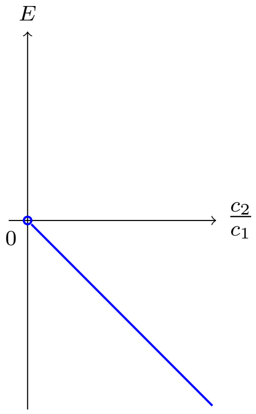
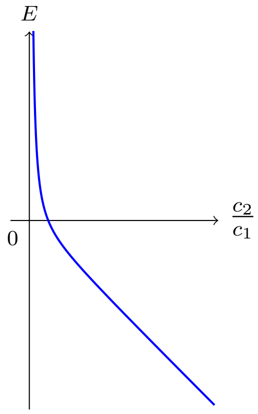
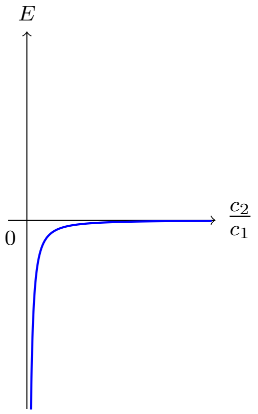
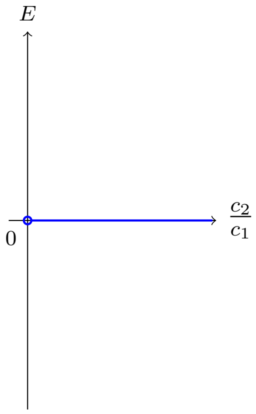
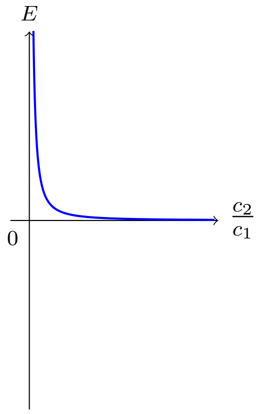
References

[1] Olivier Druet, *Compactness for Yamabe metrics in low dimensions*, Int. Math. Res. Not. (2004), no. 23, 1143–1191. MR 2041549

[2] H. Karcher, *Submersions via projections*, Geom. Dedicata **74** (1999), no. 3, 249–260. MR 1669359

[3] M. A. Khuri, F. C. Marques, and R. M. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. **81** (2009), no. 1, 143–196. MR 2477893

Table 1: The behavior of the Einstein–Hilbert functional

	$\chi_2 < 0$	$\chi_2 = 0$	$\chi_2 > 0$
$\chi_1 < 0$			
$\chi_1 = 0$			
$\chi_1 > 0$