Seminar: Positive scalar curvature and bordism

Winter term 2017/18

Prof. Bernd Ammann, Prof. Niko Naumann, Dr. Nobuhiko Otoba

Informal meeting and distribution of talks:
Monday Oct 9, 13:30 in the office of Bernd Ammann

Tuesday 14.15–16.00, SFB seminar room
first talk on Oct 17, no talk on Oct 24 and Oct 31, second talk Nov 7

The goal of this seminar is to understand the following theorem by Stolz: Assume that M is a closed simply connected spin manifold of dimension n at least 5. Then M carries a metric of positive scalar curvature if and only if $\mathrm{ind}(M) \in KO^{-n}(pt)$ vanishes.

1 Geometric preliminaries

Talk no. 1: From Bordisms to surgeries using Morse function. *Oct 17 and Nov 14.* Bernd Ammann.
Overview over Morse functions, decomposition, Morse homology, Simplification of bordisms and related issues.

Talk no. 2: Positive scalar curvature survives surgeries of codimension ≥ 3. *Nov 7.* Johannes Wittmann.
The goal of the talk is to prove the following result by Gromov and Lawson [7, theorem A]: Let M be a closed manifold, equipped with a Riemannian metric of positive scalar curvature. Let N be obtained from M by a surgery of codimension ≥ 3, then N also carries a Riemannian metric of positive scalar curvature.
Part of the talk is also to briefly recall the definition of a surgery and to explain, how surgeries are related to Morse functions on bordisms. The original reference [7] is essentially a good reference for a proof of this fact, however some simple arguments with ordinary differential inequality contain an error which is not hard to correct. The speaker should thus also consult some of the many sources which reprove this result, e.g. [15] or handouts written by Lenny Ng, Kyler Siegel, and others which can be found with standard search engines, or results by Chernysh [4] and Walsh [19, 18, 20] generalizing this construction.

Talk no. 3: Decomposing bordisms in surgeries without low codimension. *Nov 21 and 28.* Michal Marcinkowski.
This talk shall prove [9, VIII Prop. 3.1] and [9, VIII Theorem 4.1]. This will require to summarize some previous parts of[9]. Both statement explain how to

decompose a "nice" bordisms into handles of suitable dimensions. Alternative books oriented more towards the h-cobordism theorem are [11][1] or [5].

Talk no. 4: Bordism conclusion from the last talks. *Dec 5.* Nobuhiko Otoba.
The goal of this talk is to prove [7, Theorems B and C], namely:

- Closed simply-connected non-spin manifolds of dimension ≥ 5 carry psc
- Any closed simply-connected spin manifold of dimension ≥ 5 that is spin bordant to a manifold of positive scalar curvature also carries a metric of positive scalar curvature.

To build up the talk, it is recommended to follow the proof of [10, IV §4 Theorem 4.4]. This proof is densely written and uses techniques from the h-cobordism theorem. The most consistent source for these facts is probably the book of [9] which explains how to decompose a bordisms into handles. Alternative books oriented more towards the h-cobordism theorem are [11][2] or [5].

Talk no. 5: The KO_*-valued index obstruction to positive scalar curvature metrics. *Dec 19.* Jonathan Glöckle.
The main goal of this talk is to introduce the Atiyah-Milnor-Singer invariant

$$\Omega_*^{\mathrm{Spin}}(pt) \xrightarrow{\mathrm{ind}_*} KO^{-*}(pt)$$

as decribed in [10, II §7, page 144]. Note that this map ind_* is often called Hitchin's α-invariant.
Introduce Cl_n-linear Dirac operators D^{Cl} on an n-dimensional spin manifold M and construct $\mathrm{ind}(M) := [\ker D^{\mathrm{Cl}}] \in KO^{-n}(pt)$ following [10, II §7]. One probably should recall some facts from [10, I §5] for this. The facts that $\mathrm{ind}(M)$ is independent of the Riemannian metric and invariant under spin bordisms shall be explained without going into the details of its proof. The Schrödinger-Lichnerowcz formula (e.g. [1]) then yields an obstruction to positive scalar curvature.

2 Pontrjagin-Thom construction

Talk no. 6: Pontrjagin-Thom spaces. *Jan 9.* Gesina Schwalbe.
Discuss [13, §18 Thom Spaces and Transversality]. The main result is the Theorem of Thom which establishes a bijection between oriented (co)bordism groups and certain homotopy groups of the universal Thom space. For the constructions and the proofs results about differential topology are required for which [2, 3] and [8] are good references.

[1] Attention: as the main focus of Milnor's book is the h-cobordism theorem, one requires not just the main results of [11], but also their proof. This is a disadvantage of [11] compared to Kosinski's book [9].

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Talk no. 7: Pontrjagin-Thom spectra. *Jan 16.* Niko Naumann.
This talk will explain why spectra yield a better description of the Pontrjagin-Thom construction, and thus motivates the next part.
The next three talks are designed to acquire a working knowledge of the Steenrod algebra. We use the Steenrod/Epstein, Cohomology operations [6] as the main reference, but one can also profit a lot from looking at Jacob Lurie's course notes on the proof of the Sullivan conjecture here `https://ocw.mit.edu/courses/mathematics/18-917-topics-in-algebraic-topology-the-sullivan-conjecture-fall-2007/lecture-notes/`

Talk no. 8: Steenrod Squares. *Jan 30.* Nobuhiko Otoba.
[6] Chapter I: state the axioms in §1. Explain stability (2.1) and the essentialness of all suspensions of the Hopf map (2.3) as an application, mention 2.4-2.7. In §3, define $A(2)$ by generators and relations and deduce the basis of admissible monomials in §4. Determine the indecomposables of $A(2)$ and obtain a result on the Hopf invariant one problem (4.3 including the remark following it and 4.5 including the remark following it).

Talk no. 9: The dual Steenrod algebra. *Feb 6.* Gesina Schwalbe.
[6] Chapter II: Following Milnor, it's often much easier to work with the dual of $A(2)$: Observe it's a Hopf-algebra (1.1), and determine its dual (2.2 and 2.5). Explain without complete proof the fundamental calculation of Borel and Serre stated after 5.4.

Talk no. 10: The cohomology of BO. *??.* B. Ammann.
[6] Chapter IV: State the mod-2 cohomology of orthogonal Grassmannianns as modules over the Steenrod algebra (6.2), and give the proof of 7.1. which uses this to estimate the number of linearly independent vector fields on a sphere (7.1). Then do your best to explain the proof of 6.2: State without proof the cell-structure and use it to infer the Steenrod action on the cohomology, finally see how much you can prove about the cell structure. Always restrict attention tho the orthogonal case, i.e. take $F = \mathbb{R}$.

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/psc`

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