Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, winter term 2017/18
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Please hand in your solutions on Oct 27 before the lecture

Exercise Sheet 2

Exercise 1 *(4 points)*

Let $Q \subset \mathbb{R}^n$ be a k-dimensional submanifold of $\mathbb{R}^n$ with the induced metric. Show that a smooth curve $\mathbf{x}\colon (a,b) \to Q$ is a geodesic iff[1]

$$\ddot{\mathbf{x}}(t) \in (T_{\mathbf{x}(t)}Q)^\perp$$

for all $t \in (a,b)$.

Hint: For the definition of a geodesic see Prof. Garcke's script, Analysis IV, 8.6 Geodätische. You may use in your proof

$$\frac{\nabla}{dt}V(t) = \mathrm{pr}_{T_{c(t)}M}\dot{V}(t)$$

for every smooth vector field V along the smooth path c.

Exercise 2 *(4 points)*

Let $m > 0$. We consider a Newtonian system

$$m\ddot{\mathbf{x}}(t) = \mathbf{F}_0(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \mathbf{F}_{constr}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

constrained to a hypersurface $Q^n \subset \mathbb{R}^{n+1}$. Show that the constraint force[2] is given by

$$\mathbf{F}_{constr}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = mII(\dot{\mathbf{x}}(t), \dot{\mathbf{x}}(t)) - \mathbf{F}_0^\perp(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

where II is the second fundamental form of $Q \subset \mathbb{R}^{n+1}$ and $\mathbf{F}_0^\perp(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$ is the component of $\mathbf{F}_0(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$ perpendicular to Q. *For a definition of II see Prof. Ammann's script, Analysis IV, 1.5 Hyperflächen.*

Exercise 3 *(4 points)*

Let $Q \subset \mathbb{R}^n$ be a k-dimensional submanifold of $\mathbb{R}^n$ and let

$$A\colon \mathbb{R}^n \to \mathbb{R}^n$$

a linear isometry[3] with $A(Q) = Q$. Let $\mathbf{x}\colon (a,b) \to Q$, $a < 0 < b$, satisfy the Newton's law

$$m\ddot{\mathbf{x}}(t) = -\mathrm{grad}V(\mathbf{x}(t)) + \mathbf{F}_{constr}$$

with holonomic constraint Q, $m > 0$. We assume that the potential $V\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $V \circ A = V$, and that

$$A\mathbf{x}(0) = \mathbf{x}(0), \qquad A\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}(0).$$

Prove that $A\mathbf{x}(t) = \mathbf{x}(t)$ for all $t \in (a,b)$.

Hint: Show that $\tilde{\mathbf{x}}(t) := A\mathbf{x}(t)$ is also a solution of the Newton equation with holonomic constraint Q.

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[1] "iff" means "if and only if" ("genau dann, wenn").

[2] We always assume $\mathbf{F}_{constr}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) \in (T_{\mathbf{x}(t)}Q)^\perp$.

[3] I.e., $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism with $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.