Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, winter term 2017/18
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Please hand in your solutions on Nov 3 before the lecture

Exercise Sheet 3

Exercise 1. *(4 points)*

For $n \in \mathbb{N}$ we consider the orthogonal group

$$O(n) = \{A \in \mathbb{R}^{n\times n} \mid AA^T = I_n\}$$

where $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Show that $O(n)$ is a submanifold of $\mathbb{R}^{n^2} = \mathbb{R}^{n\times n}$.
Hint: Show that I_n is a regular point of the map $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{sym}$, $A \mapsto A^T A - I_n$.

Exercise 2: Planar pendulum. *(4 points)*

We consider the movement of a planar pendulum in a homogeneous gravitational field. This is a special case of the spherical pendulum of the lecture with vanishing angular momentum, i.e. $p_\theta = 0$. Then

$$\ddot{\phi} - \frac{g}{l}\sin\phi = 0.$$

Draw the orbits $t \mapsto (\phi(t), \dot{\phi}(t)) \in Z := (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \ni (\phi, \dot{\phi})$. and compare them to the curves of constant energy $E = \frac{1}{2}\dot{\phi}^2 + \frac{g}{l}\cos\phi$. Determine the equilibria (= fixed-points of the flow), discuss wether they are stable/unstable. Are there homoclinic/heteroclinic/periodic orbits? Which of the periodic orbits are contractible?
Comment: To solve the exercise you might need some definitions given here.

Exercise 3. *(4 points)*

Let Q be a smooth n-dimensional manifold. We define the *cotangent bundle*

$$T^*Q := \coprod_{x\in Q} T^*_x Q$$

as the disjoint union of the dual spaces $T^*_x Q := (T_x Q)^*$. For any chart $Q \supseteq U \xrightarrow{q} V \subseteq \mathbb{R}^k$ of Q (where $U \subseteq Q$ means that U is an open subset of Q), we define a map

$$\Phi^q : q(U) \times \mathbb{R}^n \to \coprod_{x\in U} T^*_x Q$$

by

$$\Phi^q\left(x, (p_1, \dots, p_n)^T\right) := \sum_{i=1}^n p_i dq^i|_{q^{-1}(x)}.$$

a) Show that Φ^q is a bijection.

b) Let $(\tilde{U}, \tilde{q})$ be another chart of Q s.t. $U \cap \tilde{U} \neq \emptyset$. Show that

$$(\Phi^q)^{-1} \circ \Phi^{\tilde{q}} : \tilde{q}(U \cap \tilde{U}) \times \mathbb{R}^n \to q(U \cap \tilde{U}) \times \mathbb{R}^n$$

is a smooth map.

Hint: It may be helpful to look at 1.19 of Prof. Garcke's Analysis IV script.

Remark: Combining a) and b) with Lemma 1.35 in John M. Lee's book "Introduction to Smooth Manifolds" (second edition), it follows easily that T^*Q is a smooth manifold of dimension $2n$ and each of the Φ^x is a chart of T^*Q.

Exercise 4. *(4 points)*

Let Q be a smooth manifold. Let $\pi\colon T^*Q \to Q$, be the base point point map, i.e. $\alpha \in T^*_{\pi(\alpha)}Q$ for every $\alpha \in T^*Q$. Let $Q \supseteq U \xrightarrow{q} V \Subset \mathbb{R}^k$ be a chart of Q, and $q = (q^1, \dots, q^k)^T$. We define

$$p_i\colon T^*U \to \mathbb{R}, \quad p_i(\alpha) := \alpha\Big(\frac{\partial}{\partial q^i}\Big).$$

a) Show that p_i is smooth and $\alpha = \sum_{i=1}^k p_i(\alpha)\, dq^i|_{\pi(\alpha)}$ for any $\alpha \in T^*U$.

b) Show that $\lambda_{\text{can}} := \sum_{i=1}^k p_i\, \pi^*(dq^i)$ is a well-defined 1-form on T^*U. This form is called the *canonical 1-form.*

c) Derive a formula for $\omega_{\text{can}} := -d\lambda_{\text{can}}$ in terms of the functions p_i and q^i and the exterior differential.

d) Show that ω_{can} is everywhere non-degenerate, i.e. for every $\alpha \in T^*U$ it defines a non-degenerate bilinear map $T_\alpha(T^*U) \times T_\alpha(T^*U) \to \mathbb{R}$.

e) Show that $\lambda_{\text{can}}(\alpha) = \alpha \circ (d\pi|_\alpha)$. Conclude that there is a unique 1-form λ_{can} on T^*U that coincides with the definition given in b) for each chart $Q \supseteq U \xrightarrow{q} V \Subset \mathbb{R}^k$. Why is λ_{can} smooth?