Symplectic Geometry

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Part III

Hamiltonian Group Actions

Chapter 11

Moment maps

11.1 Definition and examples

In this chapter, G will be a Lie group acting on a symplectic manifold (M, ω). The action is called *symplectic* if $g^*\omega = \omega$ for all $g \in G$. Then the infinitesimal action $\mathfrak{g} \to \mathcal{X}(M)$, $X \mapsto \underline{X} = \underline{X}^M$ satisfies

$$0 = L_{\underline{X}}\omega = di_{\underline{X}}\omega.$$

So each vector field $\underline{X}$ is locally Hamiltonian. We are interested in the case that each $\underline{X}$ is globally Hamiltonian with Hamiltonian μ_X. Using a basis for $\mathfrak{g}$, we can choose the map $X \mapsto \mu_X$ to be linear. By Lemma **??** and Lemma **??**, $\omega(\underline{X}, \underline{Y}) = \{\mu_X, \mu_Y\}$ is a global Hamiltonian for $-[\underline{X}, \underline{Y}] = \underline{[X, Y]}$. Thus

$$\mu_{[X,Y]} - \{\mu_X, \mu_Y\} = C_{X,Y}$$

for constants $C_{X,Y} \in \mathbb{R}$. A Hamiltonian action will be one for which the constants can be chosen to vanish, so that $\mu : \mathfrak{g} \to \Omega^0(M)$ is a Lie algebra homomorphism.

I prefer a "dual" point of view: The map $\mu : \mathfrak{g} \to \Omega^0(M)$ can be viewed as a map $\mu : M \to \mathfrak{g}^*$ via

$$\langle \mu(x), X \rangle = \mu_X(x), \quad x \in M, X \in \mathfrak{g}.$$

The condition that μ_X is a Hamiltonian for $\underline{X}$ translates into

$$\langle d\mu \cdot, X \rangle = i_{\underline{X}}\omega.$$

The condition that μ is a Lie group homomorphism translates into

$$\langle L_{\underline{X}}\mu, Y \rangle = L_{\underline{X}}\mu_Y = \{\mu_Y, \mu_X\} = \mu_{[Y,X]} = \langle \mu, [Y, X] \rangle = \langle -\mathrm{ad}^*_X \mu, Y \rangle,$$

or the *infinitesimal equivariance* condition

$$L_{\underline{X}}\mu = -\mathrm{ad}^*_X \mu, \quad X \in \mathfrak{g}.$$

For G connected this is equivalent to the *equivariance* condition

$$\mu(gx) = \mathrm{Ad}^*_{g^{-1}}\mu(x), \quad g \in G.$$

Definition. An action of G on (M,ω) is called *Hamiltonian* if there exists a *moment map* $\mu : M \to \mathfrak{g}^*$ satisfying

(i) $\langle d\mu, X\rangle = i_{\underline{X}}\omega$;

(ii) $\mu(gx) = \mathrm{Ad}^*_{g^{-1}}\mu(x)$.

Remark. Some authors call an action Hamiltonian if it only satisfies (i), and equivariant Hamiltonian if it satisfies (i) and (ii). Other authors require a Hamiltonian action only to satisfy infinitesimal equivariance instead of (ii). For most purposes infinitesimal equivariance is sufficient, but most of the interesting examples are equivariant anyway.

The moment map is a generalized "momentum" in the following sense:

Proposition 11.1 (Generalized Noether Theorem). *If H is a G-invariant Hamiltonian, then the moment map μ is a conserved quantity for the Hamiltonian flow of H.*

Proof.

$$X_H \cdot \langle \mu, X\rangle = \langle d\mu \cdot X_H, X\rangle = \omega(\underline{X}, X_H) = -\underline{X}\cdot H = 0, \quad X \in \mathfrak{g}.$$

□

Example 11.2 (periodic Hamiltonians). Let H be a time-independent Hamiltonian on (M,ω) whose Hamiltonian flow is 1-periodic. Then the flow defines a Hamiltonian action of S^1 on M with moment map H. Equivariance, $H(gx) = H(x)$, is just conservation of energy.

Example 11.3 (cotangent bundles). Let G act smoothly on the manifold Q. The action lifts to an action on $M = T^*Q$,

$$g(q,p) = \Big(g(q), [(T_q g)^{-1}]^* p$$

Denote by $\underline{X}^Q$, $\underline{X}^M$ the induced vector fields for $X \in \mathfrak{g}$.

Proposition 11.4. *The lifted action of G on T^*Q is Hamiltonian with moment map*

$$\langle \mu(q,p), X\rangle = \langle p, \underline{X}^Q(q)\rangle.$$

Proof. A direct computation gets surprisingly messy, so here's a more elegant way. Let us first show that the G-action preserves the canonical 1-form $\lambda = p\,dq$ on T^*Q.

Recall that $\lambda_{(q,p)}(v) = \langle p, T\tau \cdot v\rangle$, where $\tau : T^*Q \to Q$ is the projection. Using $\tau \circ g = g \circ \tau$ where g denotes the action on M and Q respectively, we get

$$\begin{aligned}(g^*\lambda)_{(q,p)}(v) &= \lambda_{\left(gq,(Tg^{-1})^*p\right)}(Tg \cdot v)\\ &= \langle (Tg^{-1})^*p, T\tau \cdot Tg \cdot v\rangle\\ &= \langle (Tg^{-1})^*p, Tg \cdot T\tau \cdot v\rangle\\ &= \langle p, T\tau \cdot v\rangle = \lambda_{(q,p)}(v).\end{aligned}$$

Now invariance of λ and $\omega = -d\lambda$ implies

$$0 = L_{\underline{X}^M}\lambda = d\,i_{\underline{X}^M}\lambda - i_{\underline{X}^M}\omega,$$

thus $\mu := i_{\underline{X}^M}\lambda$ is a Hamiltonian for $\underline{X}^M$. The Hamiltonian simplifies to

$$\mu(q,p) = \langle p, T\tau \cdot \underline{X}^M(q,p)\rangle = \langle p, \underline{X}^Q(q)\rangle.$$

Equivariance of μ follows from Lemma **??**,

$$\begin{aligned}\langle \mu\Big(g(q,p)\Big), X\rangle = \langle \mu\Big(gq,(Tg^{-1})^*p\Big), X\rangle &= \langle (Tg^{-1})^*p, \underline{X}^Q(gq)\rangle\\ &= \langle p, g^*\underline{X}^Q(q)\rangle = \langle p, \underline{(\mathrm{Ad}_{g^{-1}}X)}^Q(q)\rangle\\ &= \langle \mu(q,p), \mathrm{Ad}_{g^{-1}}X\rangle = \langle \mathrm{Ad}^*_{g^{-1}}\mu(q,p), X\rangle.\end{aligned}$$

□

By the Generalized Noether Theorem, if $H : T^*Q \to \mathbb{R}$ is G-invariant, then $\langle p, \underline{X}^Q(q)\rangle$ is a conserved quantity for the Hamiltonian flow. If H is fibrewise convex, then invariance of H is equivalent to invariance of its Legendre transform L, and we recover the classical Noether Theorem.

Example 11.5 (coadjoint orbits). The action of G on a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is Hamiltonian with moment map the inclusion $\mu(\xi) = \xi$.

To see this, recall that tangent vectors to $\mathcal{O}$ are $\underline{X}_\xi = -\mathrm{ad}^*_X\xi$, $X \in \mathfrak{g}$. Then

$$\langle d\mu \cdot \underline{Y}_\xi, X\rangle = \langle \underline{Y}_\xi, X\rangle = \langle -\mathrm{ad}^*_Y\xi, X\rangle = \langle \xi, [X,Y]\rangle = \omega_\xi(\underline{X}_\xi, \underline{Y}_\xi).$$

Equivariance of μ is clear.

11.2 Properties of moment maps

Proposition 11.6. *The moment map $\mu : M \to \mathfrak{g}^*$ is a Poisson map with respect to the symplectic Poisson structure on M and the Lie-Poisson structure on $\mathfrak{g}^*$.*

Proof. We need to show $\mu^*\{F, G\} = \{\mu^* F, \mu^* G\}$ for $F, G \in \Omega^0(\mathfrak{g}^*)$, where $\{F, G\}(\xi) = \langle \xi, [d_\xi F, d_\xi G]\rangle$.

By infinitesimal invariance, $X \mapsto \mu_X = \langle \mu, X\rangle$ is a Lie algebra homomorphism, i.e.

$$\langle \mu, [X, Y]\rangle = \{\langle \mu, X\rangle, \langle \mu, Y\rangle\}.$$

Applying this to $X = d_\mu F$, $Y = d_\mu G$ yields

$$\mu^*\{F, G\} = \langle \mu, [d_\mu F, d_\mu G]\rangle = \{\langle \mu, X\rangle, \langle \mu, Y\rangle\}.$$

Now

$$d(\mu^* F) = d_\mu F \circ d\mu = \langle d\mu\cdot, d_\mu F\rangle = \langle d\mu\cdot, X\rangle = d\langle \mu, X\rangle.$$

Since the Poisson bracket depends only on the differentials,

$$\{\mu^* F, \mu^* G\} = \{\langle \mu, X\rangle, \langle \mu, Y\rangle\} = \mu^*\{F, G\}.$$

□

By equivariance, the moment map μ maps an orbit $G \cdot x$ to a coadjoint orbit $\mathcal{O}$. So we have two natural 2-forms on $G \cdot x$: the restriction of ω, and the pullback of $\omega^{\mathcal{O}}$ under μ. Not surprisingly, they are in fact equal:

$$\begin{aligned}(\mu^*\omega^{\mathcal{O}})(\underline{X}, \underline{Y}) &= \omega^{\mathcal{O}}_\mu(d\mu \cdot \underline{X}, d\mu \cdot \underline{Y}) = \omega^{\mathcal{O}}_\mu(\mathrm{ad}^*_X \mu, \mathrm{ad}^*_Y \mu) \\ &= \langle \mu, [X, Y]\rangle = \omega(\underline{X}, \underline{Y})\end{aligned}$$

by infinitesimal equivariance. So we have proved:

Lemma 11.7.

$$\omega|_{G\cdot x} = (\mu|_{G\cdot x})^*\omega^{\mathcal{O}}.$$

Finally, let us discuss the question: When is a symplectic action Hamiltonian?

Example 11.8. $S^1 = \mathbb{R}/\mathbb{Z}$ acts on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by

$$t \cdot (x, y) = (x + t, y).$$

This action is symplectic for the symplectic form $dx \wedge dy$ but not Hamiltonian. For if it were, then the vector field $(1, 0)$ on T^2 induced by $1 \in \mathbb{R}$ would be Hamiltonian, $(1, 0) = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})$, so $H(x, y) = y + \text{const}$ which is not possible on T^2.

The next result shows that nontrivial first cohomology is the only obstruction to an action being Hamiltonian.

Proposition 11.9. *If M is compact and $H^1(M; \mathbb{R}) = 0$ then every symplectic action of a connected Lie group on M is Hamiltonian.*

Proof. The vector field $\underline{X}$ on M corresponding to $X \in \mathfrak{g}$ satisfies $0 = L_{\underline{X}}\omega = di_{\underline{X}}\omega$. Since $H^1(M;\mathbb{R}) = 0$, $i_{\underline{X}}\omega = d\mu_X$ for some function μ_X. If we fix the free constant in μ_X by requiring $\int_M \mu_X \omega^n = 0$, μ_X depends smoothly on X. For infinitesimal invariance, integrate both side of the equation $\mu_{[X,Y]} - \{\mu_X, \mu_Y\} = C_{X,Y}$ over M, and note that for $F, G \in \Omega^0(M)$,

$$\{F, G\}\omega^n = (L_{X_G}F)\omega^n = L_{X_G}(F\omega^n) = d(i_{X_G}F\omega^n).$$

So the integral of the left-hand side over M vanishes, and therefore $C_{X,Y} = 0$. □

Finally, we will derive a sufficient criterion in terms of G for every symplectic action to be Hamiltonian.

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Identify $\Lambda^k(\mathfrak{g}^*)$ with left-invariant k-forms on G. The exterior derivative $d : \Omega^k(G) \to \Omega^{k+1}(G)$ maps left-invariant forms to left-invariant forms and thus induces linear maps

$$\delta_k : \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*)$$

satisfying $\delta_{k+1} \circ \delta_k = 0$. The *Lie algebra cohomology* of $\mathfrak{g}$ is

$$H^k(\mathfrak{g}) := \ker \delta_k / \mathrm{im}\delta_{k-1}.$$

Since every left-invariant function is constant, the derivative terms in Formula (vi) of Appendix **??** vanish, and (using $\underline{[X,Y]} = -[\underline{X}, \underline{Y}]$) we get a formula for δ entirely in terms of $\mathfrak{g}$:

$$\delta_k\alpha(X_0, \ldots, X_k) = \sum_{0 \le i < j \le k} (-1)^{i+j+1}\alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$$

for $\alpha \in \Lambda^k(\mathfrak{g}^*)$ and $X_0, \ldots, X)k \in \mathfrak{g}$.

Proposition 11.10. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ then every symplectic G-action is Hamiltonian with unique moment map.*

The proof uses the following simple lemma (which has nothing to do with group actions):

Lemma 11.11. *If X, Y are symplectic vector fields on a symplectic manifold (M, ω) then their commutator $[X, Y]$ is a Hamiltonian vector field with Hamiltonian $-\omega(X, Y)$.*

Proof. Just compute $L_X i_Y \omega$ in two different ways, using $L_X\omega = L_Y\omega = 0$:

$$L_X i_Y \omega = i_{[X,Y]}\omega + i_Y L_X \omega = i_{[X,Y]}\omega,$$
$$L_X i_Y \omega = d i_X i_Y \omega + i_X d i_Y \omega = d\big(\omega(Y, X)\big).$$

□

Proof of Proposition 11.10. (i) Let G act symplectically on (M, ω). First observe that $H^1(\mathfrak{g}) = 0$ means that $\langle \xi, [X,Y] \rangle = 0$ for all X, Y implies $\xi = 0$, which is equivalent to $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. So every canonical vector field $\underline{X}^M$ is the commutator of two symplectic vector fields and therefore Hamiltonian by the lemma above. So we find a linear map $\mathfrak{g} \to \Omega^0(M)$, $X \mapsto \mu_X$, such that μ_X is a Hamiltonian for $\underline{X}$.

(ii) From the map $X \mapsto \mu_X$ we get an element $\alpha \in \Lambda^2(\mathfrak{g}^*)$,

$$\alpha(X,Y) = \mu_{[X,Y]} - \{\mu_X, \mu_Y\}.$$

Its differential is

$$\begin{aligned}
\delta_2\alpha(X,Y,Z) &= \alpha([X,Y],Z) + \text{cyclic} \\
&= \mu_{[[X,Y],Z]} - \{\mu_{[X,Y]}, \mu_Z\} + \text{cyclic} \\
&= \mu_{[[X,Y],Z]} - \{\{\mu_X, \mu_Y\}, \mu_Z\} - \{\alpha(X,Y), \mu_Z\} + \text{cyclic}.
\end{aligned}$$

The first term vanishes by the Jacobi identity on $\mathfrak{g}$, the second one by the Jacobi identity for $\{\ \}$ on M, and the third one because $\alpha(X,Y)$ is a constant function on M. So $\delta_2\alpha = 0$, and since $H^2(\mathfrak{g}) = 0$, $\alpha(X,Y) = \langle \xi, [X,Y] \rangle$ for some $\xi \in \mathfrak{g}^*$. Now $\tilde{\mu}_X := \mu_X - \langle \xi, X \rangle$ satisfies

$$\begin{aligned}
\tilde{\mu}_{[X,Y]} - \{\tilde{\mu}_X, \tilde{\mu}_Y\} &= \mu_{[X,Y]} - \{\mu_X, \mu_Y\} - \langle \xi, [X,Y] \rangle \\
&= \langle \xi, [X,Y] \rangle - \langle \xi, [X,Y] \rangle = 0.
\end{aligned}$$

This proves the existence of a moment map.

(iii) Suppose that μ and $\mu + \xi$ are both moment maps, $\xi \in \mathfrak{g}^* = \Lambda^1(\mathfrak{g}^*)$. By infinitesimal equivariance,

$$0 = \langle L_{\underline{X}}(\mu + \xi) - L_{\underline{X}}\mu, Y \rangle = \langle -\mathrm{ad}^*_X \xi, Y \rangle = \langle \xi, [X,Y] \rangle = \delta_1\xi(X,Y)$$

for all $X, Y \in \mathfrak{g}$. So $\delta_1\xi = 0$, and since $H^1(\mathfrak{g}) = 0$, $\xi = 0$. □

Problem 11.1. Show that $H^1\big(\mathfrak{so}(3)\big) = H^2\big(\mathfrak{so}(3)\big) = 0$. Thus every symplectic $SO(3)$-action is Hamiltonian with unique moment map.

A Lie algebra (and the corresponding Lie group) is called *semisimple* if the *Killing form*

$$(X,Y) \mapsto \mathrm{tr}(\mathrm{ad}_X \mathrm{ad}_Y)$$

is nondegenerate. For example, $so(3)$ is semisimple. By a theorem in Lie theory (see [21] for a proof), $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ for every semisimple Lie algebra. So Proposition 11.10 implies

Corollary 11.12. *Every symplectic action of a connected semisimple Lie group is Hamiltonian with unique moment map.*

Summary: formulae for the moment map

Let $X, Y \in \mathfrak{g}$, $g \in G$, $x \in M$, and $F, G \in \Omega^0(\mathfrak{g}^*)$.

(i) $\langle d\mu\cdot, X\rangle = i_{\underline{\mathrm{X}}}\omega$ (defining property).

(ii) $\mu(gx) = \mathrm{Ad}^*_{g^{-1}}\mu(x)$ (equivariance).

(iii) $L_{\underline{\mathrm{X}}}\mu = -\mathrm{ad}^*_X\mu$ (infinitesimal equivariance).

(iv) $\mathfrak{g}_x = (\mathrm{im}\, d_x\mu)^{\perp}$.

(v) $\mu : \mathfrak{g} \to \Omega^0(M)$ is a Lie algebra homomorphism, i.e.

$$\langle \mu, [X, Y]\rangle = \{\langle \mu, X\rangle, \langle \mu, Y\rangle\} = \omega(\underline{\mathrm{X}}, \underline{\mathrm{Y}}).$$

(vi) $\mu : M \to \mathfrak{g}^*$ is a Poisson map, i.e.

$$\mu^*\{F, G\} = \{\mu^*F, \mu^*G\}.$$

(vii) If G acts on M and M' with moment maps μ, μ', then G acts on $M \times M'$ with moment map $\mu + \mu'$.

(viii) If G acts on (M, ω) with moment map μ and $\phi : H \to G$ is a Lie group homomorphism with linearization $\phi^* : \mathfrak{g}^* \to \mathfrak{h}^*$, then the induced action of H is Hamiltonian with moment map $\phi^* \circ \mu$.

Chapter 12

Symplectic reduction

12.1 The Marsden-Weinstein quotient

In this chapter, we consider a Hamiltonian action of G on (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. Since μ is an integral of the motion for every G-invariant Hamiltonian system, we can reduce the dimension of the problem by restricting to a level set of μ and dividing out the residual action of G. This procedure is called symplectic reduction.

Reduction at zero

Proposition 12.1 (reduction at zero). *Suppose that* 0 *is a regular value of* μ *and* G *acts freely on* $\mu^{-1}(0)$. *Then there exists a unique symplectic form* $\bar{\omega}$ *on* $\mu^{-1}(0)/G$ *such that* $\pi^*\bar{\omega} = \omega|_{\mu^{-1}(0)}$, *where* $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ *is the projection.*

Definition. The symplectic manifold $M//G(0) := \mu^{-1}(0)/G$ is called the reduced space, *symplectic quotient*, or *Marsden-Weinstein quotient.*

The proof is based on the following simple lemma whose proof I omit.

Lemma 12.2. *Let* $\pi : P \mapsto B$ *be a principal bundle and* $\alpha \in \Omega^k(P)$ *be* G*-invariant and horizontal* ($i_v\alpha = 0$ *for* $v \in \ker T\pi$)*. Then there exists a unique* $\bar{\alpha} \in \Omega^k(B)$ *with* $\pi^*\bar{\alpha} = \alpha$. *Moreover,* $\bar{\alpha}$ *is closed iff* α *is closed, and* $\bar{\alpha}$ *is nondegenerate iff* $\ker\alpha = \ker T\pi$.

Proof of Proposition 12.1. (i) We first show: $\mu^{-1}(0)$ is coisotropic with

$$\left(T_x\mu^{-1}(0)\right)^{\perp_\omega} = T_xG \cdot x \subset T_x\mu^{-1}(0).$$

For $\underline{X}_x \in T_xG \cdot x$ and $v \in T_x\mu^{-1}(0) = \ker d_x\mu$,

$$\omega(\underline{X}_x, v) = \langle d_x\mu \cdot v, X \rangle = 0,$$

so $T_xG \cdot x \subset \left(T_x\mu^{-1}(0)\right)^{\perp_\omega}$. For the converse inclusion we just count dimensions: $\dim \mu^{-1}(0) = \dim M - \dim G$, so

$$\dim\Big(T_x\mu^{-1}(0)\Big)^{\perp_\omega} = \dim G = \dim T_xG \cdot x.$$

Finally, $T_xG \cdot x \subset T_x\mu^{-1}(0)$ by equivariance of μ.
(ii) Consider the G-principal bundle $\pi : \mu^{-1}(0) \to \mu^{-1}(0)/G$ with the G-invariant closed 2-form ω on $\mu^{-1}(0)$. By (i),

$$\ker(\omega_{\mu^{-1}(0)}) = T_xG \cdot x = \ker Tx\pi.$$

So the existence and uniqueness of $\bar{\omega}$ follows from Lemma 12.2. □

The reduction at zero immediately generalizes to reduction at elements in the center of $\mathfrak{g}^*$,

$$center(\mathfrak{g}^*) := \{\eta \in \mathfrak{g}^* \mid \mathrm{ad}^*_X\eta = 0 \text{ for all } X \in \mathfrak{g}\} :$$

Simply consider $\tilde{\mu}(x) := \mu(x) - \eta$.

Corollary 12.3. *Let η be a regular value of μ in center$(\mathfrak{g}^*)$ such that G acts freely on $\mu^{-1}(\eta)$. Then there exists a unique symplectic form $\bar{\omega}$ on $M//G(\eta) := \mu^{-1}(\eta)/G$ such that $\pi^*\bar{\omega} = \omega|_{\mu^{-1}(0)}$.*

Reduction at coadjoint orbits

For η not in the center of $\mathfrak{g}^*$, $\mu^{-1}(\eta)$ is not invariant under G but only under

$$G_\eta = \{g \in G \mid \mathrm{Ad}^*_g\eta = \eta\}.$$

Alternatively, $\mu^{-1}(\mathcal{O})$ is invariant onder G, where $\mathcal{O}$ is the coadjoint orbit of η.

Theorem 12.4. *Suppose $\mathcal{O} \subset \mathfrak{g}^*$ is a coadjoint orbit such that G acts freely on $\mu^{-1}(\mathcal{O})$, and $\eta \in \mathcal{O}$. Then there exists a unique symplectic form $\bar{\omega}$ on*

$$M//G(\mathcal{O}) := \mu^{-1}(\mathcal{O})/G = \mu^{-1}(\eta)/G_\eta$$

such that $\pi^*\bar{\omega} = \omega|_{\mu^{-1}(\mathcal{O})}$.

Proof. Since this result is central to all that follows, I will present 3 different proofs.

First proof. Show $\left(T_x\mu^{-1}(\eta)\right)^{\perp_\omega} = T_xG\cdot x$ as for the reduction at zero (the proof there didn't use $\eta = 0$). But now $\mu^{-1}(\eta)$ is not coisotropic: $T_xG\cdot x \not\subset T_x\mu^{-1}(\eta)$ for $\eta \notin \mathit{center}(\mathfrak{g}^*)$. Instead we have

$$T_x\mu^{-1}(\eta) \cap T_xG\cdot x = T_xG_\eta \cdot x.$$

For if $\underline{\mathrm{X}}_x \in T_xG\cdot x$ satisfies $0 = d_x\mu\cdot\underline{\mathrm{X}}_x = -\mathrm{ad}_X^*\eta$, then $X \in T_eG_\eta$, so $\underline{\mathrm{X}}_x \in T_xG_\eta\cdot x$. The converse inclusion is clear.

Now consider the G_η-principal bundle $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/G_\eta$. The form $\omega|_{\mu^{-1}(\eta)}$ is invariant and closed, and

$$\ker(\omega|_{\mu^{-1}(\eta)}) = T_x\mu^{-1}(\eta) \cap \Big(T_x\mu^{-1}(\eta)\Big)^{\perp_\omega} = T_xG_\eta = \ker T\pi.$$

So the existence and uniqueness of $\bar{\omega}$ follows from Lemma 12.2.

Second proof. By Example 11.5, G acts on $(M\times\mathcal{O}, \omega\oplus -\omega^{\mathcal{O}})$ with moment map

$$\tilde{\mu}(x,\eta) = \mu(x) - \eta.$$

Then

$$\tilde{\mu}^{-1}(0) = \{(x,\eta) \mid \mu(x) = \eta \in \mathcal{O}\} \cong \{\mu^{-1}(\mathcal{O}),$$

and the result follows from Proposition 12.1.

Third proof. We will construct a closed G-invariant 2-form ω^μ on $\mu^{-1}(\mathcal{O})$ with $\ker\omega^\mu = T_xG\cdot x$, and then the result follows from Lemma 12.2.

Note first that $\omega^\mu = \omega$ doesn't work: By Lemma 11.7,

$$\omega|_{T_xG\cdot x} = \mu|^*_{T_xG\cdot x}\omega^{\mathcal{O}},$$

$$\omega(\underline{\mathrm{X}}_x, \underline{\mathrm{Y}}_y) = \langle\mu(x), [X,Y]\rangle,$$

which does not vanish in general. But

$$\omega^\mu := (\omega - \mu^*\omega^{\mathcal{O}})|_{\mu^{-1}(\mathcal{O})}$$

vanishes on $T_xG\cdot x$, and I claim that it is the 2-form we are after. All the properties are obvious except $\ker\omega^\mu \subset T_xG\cdot x$.

Note first that $v \in T_x\mu^{-1}(\mathcal{O})$ iff $v \in T_xM$ and there exists an $X \in \mathfrak{g}$ such that $d\mu\cdot v = -\mathrm{ad}_X^*\mu = d\mu\cdot\underline{\mathrm{X}}_x$, so

$$T_x\mu^{-1}(\mathcal{O}) = \{v \in T_xM \mid v - \underline{\mathrm{X}}_x \in \ker d\mu \text{ for some } X \in \mathfrak{g}\}.$$

Now for $v, w \in T_x\mu^{-1}(\mathcal{O})$ with $\tilde{v} = v - \underline{\mathrm{X}}_x, \tilde{w} = w - \underline{\mathrm{Y}}_x \in \ker d\mu$,

$$\begin{aligned}\omega(v,w) &= \omega(\tilde{v},\tilde{w}) + \omega(\tilde{v},\underline{\mathrm{Y}}_x) + \omega(\underline{\mathrm{X}}_x,\tilde{w}) + \omega(\underline{\mathrm{X}}_x,\underline{\mathrm{Y}}_y)\\ &= \omega(\tilde{v},\tilde{w}) - \langle d\mu\cdot\tilde{v}, Y\rangle + \langle d\mu\cdot\tilde{w}, X\rangle + \langle\mu(x),[X,Y]\rangle\\ &= \omega(\tilde{v},\tilde{w}) + \mu^*\omega^{\mathcal{O}}(v,w).\end{aligned}$$

This shows that

$$\omega^\mu(v,w) = \omega(v - \underline{X}_x, w - \underline{Y}_x) = \omega(v,w) - \langle \mu, [X,Y] \rangle.$$

If $v \in \ker \omega^\mu$ then $\omega(v,w) = \langle \mu, [X,Y] \rangle$ for all $w \in T_x\mu^{-1}(\mathcal{O})$. In particular pick $w \in \ker d\mu$, so $Y = 0$, and therefore $\omega(v,w) = 0$. Thus $v \in (\ker d\mu)^{\perp_\omega}$. But

$$(\ker d\mu)^{\perp_\omega} = \Big(T_x\mu^{-1}(\eta)\Big)^{\perp_\omega} = T_xG \cdot x$$

as in the first proof, and we have shown $\ker \omega^\mu \subset T_xG \cdot x$. □

12.2 Examples

Linear actions

A large and interesting class of symplectic manifolds arise as Marsden-Weinstein quotients of $\mathbb{C}^n$ by linear actions.

We start with the standard action of $U(n)$ on $\mathbb{C}^n$, $(A,z) \mapsto Az$. The infinitesimal action is given by

$$\underline{X}_z = Xz, \quad X \in \mathfrak{u}(n), z \in \mathbb{C}^n.$$

Fix the inner product

$$\langle X, Y \rangle := -\mathrm{tr}(XY)$$

on $\mathfrak{u}(n)$, and use it to identify $\mathfrak{u}(n)$ with $\mathfrak{u}(n)^*$. Two vectors $v, w \in \mathbb{C}^n$ induce a complex linear map $vw^* : \mathbb{C}^n \to \mathbb{C}^n$,

$$vw^*(z) := v \otimes w^*(z) := (z,w)v,$$

where $(\ ,\ )$ is the Hermitian product on $\mathbb{C}^n$. Since $(vw^*)^* = wv^*$,

$$\frac{i}{2}(vw^* + wv^*) \in \mathfrak{u}(n).$$

Recall that the standard symplectic structure is $\omega = -\Im(\cdot,\cdot)$.

Lemma 12.5. *For $X \in \mathfrak{u}(n)$ and $v, w \in \mathbb{C}^n$,*

$$\mathrm{tr}\Big(X\frac{i}{2}(vw^* + wv^*)\Big) = \omega(Xv, w).$$

Proof. Is suffices to show this for v, w orthonormal with respect to the Hermitian product. Then

$$\begin{aligned}\mathrm{tr}\Big(X\frac{i}{2}(vw^* + wv^*)\Big) &= \frac{i}{2}\Big\{(Xv,w) + (Xw,v)\Big\} \\ &= \frac{i}{2}\Big\{(Xv,w) - \overline{(Xv,w)}\Big\} \\ &= -\Im(Xv,w) = \omega(Xv,w).\end{aligned}$$

□

The defining equation for a moment map is

$$\langle d\mu(z)v, X\rangle = \omega(Xz, v) = -\langle X, \frac{i}{2}(zv^* + vz^*)\rangle \tag{$*$}$$

for all $X \in \mathfrak{u}(n)$ and $v \in \mathbb{C}^n$, thus

$$d\mu(z)v = -\frac{i}{2}(zv^* + vz^*)$$

for all $v \in \mathbb{C}^n$, which is satisfied by

$$\mu(z) = -\frac{i}{2}zz^*.$$

Now let $G \subset U(n)$ be a subgroup and $\pi_{\mathfrak{g}} : \mathfrak{u}(n) \to \mathfrak{g}$ the orthogonal projection. Then equation $(*)$ must hold for all $X \in \mathfrak{g}$, which is satisfied by

$$\mu(z) = \pi_{\mathfrak{g}}\Big(-\frac{i}{2}zz^*\Big).$$

Since $(Az)(Az)^* = Azz^*A^* = \mathrm{Ad}_A(zz^*)$, we have shown

Proposition 12.6. *A linear action of $G \subset U(n)$ on $\mathbb{C}^n$ is Hamiltonian with moment map*

$$\mu(z) = \pi_{\mathfrak{g}}\Big(-\frac{i}{2}zz^*\Big).$$

Example 12.7 (projective space). Let S^1 act on $\mathbb{C}^n$ by

$$e^{i\theta}(z_1, \ldots, z_n) = (e^{i\theta}z_1, \ldots, e^{i\theta}z_n).$$

This action is Hamiltonian with moment map

$$\mu(z) = -\frac{i}{2}|z|^2.$$

The reduction at a value $-\frac{i}{2}t$, $t > 0$, is

$$\mathbb{C}^n//S^1(-\frac{i}{2}t) \cong \mathbb{C}P^{n-1}.$$

Note that this S^1-action commutes with the $U(n)$-action on $\mathbb{C}^n$. So every linear action of a subgroup $G \subset U(n)$ induces an action on $\mathbb{C}P^{n-1}$ which is Hamiltonian with moment map as in Proposition 12.6, normalized by $|z|^2$.

Problem 12.1. Show that the reduced symplectic form ω^t on $\mathbb{C}^n//S^1(-\frac{i}{2}t)$ equals $t\omega_{FS}$, where ω_{FS} is the Fubini-Study form. This linear dependence of ω_t on t is our first encounter with the Duistermaat-Heckmann Theorem.

Solution. We must check the normalization on a line in $\mathbb{C}P^{n-1}$. For this line we take the quotient $\mathbb{C}P^1$ of $\mathbb{C}^2 \times 0 \subset \mathbb{C}^n$, so the question reduces to $n = 2$. Using $\mathrm{vol}(S^3) = 2\pi^2$ we get

$$\begin{aligned}\int_{\mathbb{C}P^1} \omega^t &= \frac{\text{volume of sphere of radius } \sqrt{t}}{\text{length of great circle on this sphere}} \\ &= \frac{2\pi^2 t^{3/2}}{2\pi\sqrt{t}} = t\pi \\ &= \int_{\mathbb{C}P^1} t\omega_{FS}.\end{aligned}$$

□

Example 12.8. Consider the action of S^1 on $\mathbb{C}P^3$ by

$$\lambda[z_0 : z_1 : z_2 : z_3] = [z_0 : \lambda z_1 : \lambda z_2 : \lambda^{-1} z_3]$$

with moment map

$$\mu([z]) = -\frac{i}{2|z|^2}(|z_1|^2 + |z_2|^2 - |z_3|^2).$$

Show that the symplectic quotient is diffeomorphic to

$$\mathbb{C}P^3//S^1\Big(-\frac{i}{2}t\Big) \cong \begin{cases} \mathbb{C}P^2 & \text{for } t < 0, \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{for } t > 0. \end{cases}$$

Compute the cohomology class of the reduced symplectic form as a function of t. This is a special case of a theorem of Guillemin and Sternberg [20].

Example 12.9 (toric varieties). Generalizing the previous example, consider a diagonal action of the torus T^k on $\mathbb{C}^n$ by

$$(e^{i\theta_1}, \dots, e^{i\theta_k})(z_1, \dots, z_n) = \Big(e^{i\sum_{j=1}^k \alpha_{1j}\theta_j} z_1, \dots, e^{i\sum_{j=1}^k \alpha_{nj}\theta_j} z_n\Big).$$

This action is Hamiltonian with moment map

$$\mu(z) = -\frac{i}{2}\Big(\sum_{i=1}^n \alpha_{i1}|z_i|^2, \dots, \sum_{i=1}^n \alpha_{ik}|z_i|^2\Big).$$

The easiest way to see this is by first considering the standard action of T^n on $\mathbb{C}^n$ and then aplying Property viii of the moment map to a homomorphism $T^k \to T^n$.

The quotients of $\mathbb{C}^n$ by T^k at regular values are *toric varieties* about which we will have much more to say in a later chapter.

Example 12.10 (Grassmannians). The action of $U(k)$ on $\mathbb{C}^{k\times n}$ via

$$(U, A) \mapsto UA$$

is Hamiltonian with moment map

$$\mu(A) = -\frac{i}{2} AA^*.$$

The level set $\mu^{-1}(\frac{i}{2}\mathbb{1})$ is the space of unitary k-frames in $\mathbb{C}^n$ (by taking the rows of A), and the quotient

$$\mathbb{C}^{k\times n}//U(k)\Big(\frac{i}{2}\mathbb{1}\Big) \cong Gr(k,n)$$

is the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$.

Example 12.11 (diagonalizable matrices). The action of $U(n)$ on $\mathbb{C}^{n\times n}$ by conjugation

$$(U, A) \mapsto UAU^{-1}$$

is Hamiltonian with moment map

$$\mu(A) = -\frac{i}{2}[A, A^*].$$

The level set $\mu^{-1}(0)$ is the space of normal, hence diagonalizable, matrices, and the quotient

$$\mathbb{C}^{n\times n}//U(n)\Big(0\Big)$$

is the space of complex diagonal matrices modulo permutations of their entries.

Multiplication on Lie groups

A Lie group G acts on itself by left and right multiplications, $L_g$and $R_{g^{-1}}$, giving rise to Hamiltonian actions on T^* with the canonical symplectic structure. Left multiplication has the infinitesimal action

$$\frac{d}{dt}\Big|_{t=0} \exp(tX)\cdot g = R_{g*}X$$

and therefore the moment map

$$\langle \mu_L(g,p), X\rangle = \langle p, R_{g*}X\rangle = \langle R_g^* p, X\rangle,$$

$$\mu_L(g,p) = R_g^* p.$$

Right multiplication (but as a *left* action!) has the infinitesimal action

$$\frac{d}{dt}\Big|_{t=0} g\cdot \exp(-tX) = -L_{g*}X$$

and therefore the moment map

$$\mu_R(g,p) = -L_g^* p.$$

The canonical Poisson structure on T^*G is invariant under left / right multiplication and therefore induces Poisson structures on the quotients $T^*G/G \cong \mathfrak{g}^*$. How are these structures related to the Lie-Poisson structure on $\mathfrak{g}^*$?

Proposition 12.12 (Lie-Poisson Reduction Theorem). *The map $T^*G \to \mathfrak{g}^*$, $(g,p) \mapsto R_g^* p$ (respectively $L_g^* p$) is a Poisson map (respectively anti-Poisson map).*

Proof. By the discussion above, $(g,p) \mapsto R_g^* p$ is the moment map μ_L which is a Poisson map by Proposition 11.6. Similarly, $(g,p) \mapsto L_g^* p$ is the map $i \circ \mu_R$, where $i(\xi) = -\xi$. Since μ_R is a Poisson map, for $F, G \in \Omega^0(\mathfrak{g}^*)$,

$$\{F \circ i \circ \mu_R, G \circ i \circ \mu_R\} = \{F \circ i, G \circ i\} \circ \mu_R = -\{F, G\} \circ i \circ \mu_R.$$

□

Corollary 12.13. *The induced Poisson structure on the quotient T^*/G by left multiplication corresponds to minus the Lie-Poisson structure under the canonical diffeomorphism $T^*G/G \cong \mathfrak{g}^*$. The corresponding symplectic quotient $\mu^{-1}(\mathcal{O})/G$ at a coadjoint orbit is the orbit $\mathcal{O}$ itself with the reduced symplectic form $\bar{\omega} = -\omega^{\mathcal{O}}$.*

Analogous results hold for the right action without the minus signs.

Proof. The map $T^*G \to \mathfrak{g}^*$, $(g,p) \mapsto L_g^* p$, is left-invariant and therefore induces the canonical diffeomorphism $T^*G/G \cong \mathfrak{g}^*$. Now the result follows from the Lie-Poisson Reduction Theorem. □

Example 12.14 (rigid body). For the rigid body the reduced space at angular momentum $\eta \neq 0$ is the sphere $\mathcal{O}_\eta$ of radius $\|\eta\|$ with respect to the Euclidean metric in in $so(3)^* \cong \mathbb{R}^3$. In Section **??** we computed the coadjoint orbit form to be $\omega^{\mathcal{O}_\eta} = \frac{\sigma_\eta}{\|\eta\|}$, where σ_η is the standard area form on $\mathcal{O}_\eta$. Thus the reduced symplectic form is

$$\bar{\omega} = -\frac{\sigma_\eta}{\|\eta\|}.$$

12.3 Reduced dynamics and reconstruction

Let $(\bar{M} = \mu^{-1}(\eta)/G_\eta, \bar{\omega})$ be a smooth symplectic quotient and $\pi : P := \mu^{-1}(\eta) \to \bar{M}$ the associated G_η-principal bundle.

Let $H : M \to \mathbb{R}$ be a G-invariant Hamiltonian. Then the Hamiltonian vector field X_H is tangent to P and G_μ-invariant, so it induces a vector field $\pi_* X_H$

on $\bar{M}$ which describes the reduced dynamics. On the other hand, $H|_P$ is G_η-invariant and therefore $H|_P = \pi^* \bar{H}$ for a unique $\bar{H} : \mathbf{M} \to \mathbb{R}$. The reduced dynamics is just the Hamiltonian flow of $\bar{H}$,

$$X_{\bar{H}} = \pi_* X_H.$$

This follows because π_* is surjective and for $v \in T_x P$,

$$\bar{\omega}(\pi_* X_H, \pi_* v) = \omega(X_H, v) = dH \cdot v = d\bar{H} \cdot \pi_* v.$$

Example 12.15 (free rigid body). Here the reduced system on the sphere $\mathcal{O}_\eta \subset \mathbb{R}^3$ of radius $\|\eta\|$ is given by the symplectic form $\bar{\omega} = -\frac{\sigma_\eta}{\|\eta\|}$ and the Hamiltonian

$$\bar{H}(\xi) = \frac{1}{2}\langle I^{-1}\xi, \xi\rangle,$$

where I is the inertia tensor. The orbits are described by the familiar picture of the level curves of $\bar{H}$ on $\mathcal{O}_\eta$. Almost all orbits of the reduced system are periodic, but we have seen that the corresponding orbits of the original system need not be periodic.

Let us now address the question: How do we reconstruct the motion in M from the reduced motion in $\bar{M}$?

Consider a curve $\bar{c}(t)$ in $\bar{M}$ and any lift $d(t)$ of $\bar{c}(t)$ to $P = \mu^{-1}(\eta)$. The Hamiltonian trajectory over $\bar{c}$ can be written as $c(t) = g(t)d(t)$ for a path $g(t)$ in G_η. So given $d(t)$ we need to find $g(t)$ such that $\dot{c} = X_H(c)$. Now

$$\frac{d}{dt}(g(t)d(t)) = g(t)_* \dot{d}(t) + g(t)_* \underline{X}(t)_{d(t)},$$

where $X(t) = g(t)_*^{-1}\dot{g}(t) \in \mathfrak{g}_\eta$. Setting this equal to $X_H(c)$ we obtain

$$\dot{d}(t) + \underline{X}(t)_{d(t)} = g_*^{-1} X_H(gd) = X_H\big(d(t)\big).$$

So we need to solve the equation in G_η,

$$\dot{g}(t) = g_* X(t), \quad \text{where } \underline{X}(t)_{d(t)} = X_H\big(d(t)\big) - \dot{d}(t).$$

As we will see below, in classical examples the principal bundle P comes with a natural connection $A \in \Omega^1(P, \mathfrak{g}_\eta)$. In this case we can choose $d(t)$ horizontal, $A_d(\dot{d}) \equiv 0$, and recover $X(t)$ from

$$X(t) = A_d\big(\underline{X}(t)_d\big) = A_d\big(X_H(d)\big) - A_d(\dot{d}) = A_d\big(X_H(d)\big).$$

So the problem becomes solving the two equations

$$A_{d(t)}\big(\dot{d}(t)\big) \equiv 0, \quad \pi d(t) = \bar{c}(t);$$
$$\dot{g}(t) = g(t)_* X(t), \quad \text{where } X(t) := A_{d(t)}\Big(X_H\big(d(t)\big)\Big).$$

The first one is a purely geometric equation on the principal bundle P, while the second one is a dynamic equation on the Lie group G_η.

Let us now specialize to the following situation:

(i) The isotropy group is $G_\eta \cong S^1$ and $\eta \neq 0$.

(ii) The curve $\bar{c}$ is periodic of period T, $\bar{c}(0) = \bar{c}(T)$, and it bounds a 2-chain $\bar{\sigma}$ in $\bar{M}$.

(iii) $M = T^*Q$ with the canonical symplectic form $dq \wedge dp$ and the lifted G-action from an action on Q.

(iv) The motion is free with Hamiltonian $H(q,p) = \frac{1}{2}|p|^2$.

The canonical connection

Let $X_\eta \in \mathfrak{g}_\eta$ be the unique smallest element satisfying $\exp(2\pi X_\eta) = 1$ and $\langle \eta, X_\eta \rangle > 0$. I claim that the canonical 1-form $\lambda = p\,dq$ induces a *canonical connection* A on $P \to \bar{M}$ by

$$A := \lambda \otimes \frac{X_\eta}{\langle \eta, X_\eta \rangle}.$$

Since λ is G-invariant and X_η is G_η-invariant, A is G_η-invariant. The normalization follows from $\lambda_{(q,p)}(v) = \langle p, \tau_* v\rangle$ and $\langle p, \underline{\mathrm{X}}_q \rangle = \langle \mu(q,p), X \rangle = \langle \eta, X \rangle$,

$$A_{(q,p)}\left(\underline{\mathrm{X}}_{(q,p)}\right) = \langle p, \underline{\mathrm{X}}_q \rangle \frac{X_\eta}{\langle \eta, X_\eta \rangle} = \langle \eta, X \rangle \frac{X_\eta}{\langle \eta, X_\eta \rangle} = X.$$

The curvature F_A of A is obtained from $dA = -\omega \otimes \frac{X_\eta}{\langle \eta, X_\eta \rangle}$,

$$F_A = -\bar{\omega} \otimes \frac{X_\eta}{\langle \eta, X_\eta \rangle}.$$

Let us split the *total phase* $\Delta\theta$ into the *geometric phase* $\Delta\theta_{geom}$ and the *dynamical phase* $\Delta\theta_{dyn}$,

$$\Delta\theta = \Delta\theta_{geom} + \Delta\theta_{dyn},$$

where the phases are defined by

$$\begin{aligned} c(T) &= \exp(\Delta\theta\, X_\eta) c(0), \\ d(T) &= \exp(\Delta\theta_{geom}\, X_\eta) d(0), \\ g(T) &= \exp(\Delta\theta_{dyn}\, X_\eta) g(0). \end{aligned}$$

The geometric phase

The geometric phase measures the holonomy of the connection A along the loop $\bar{c}$. It is related to the curvature by the following lemma:

Lemma 12.16. *Let $P \to B$ be an S^1-principal bundle, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Let $A \in \Omega^1(P, \mathbb{R})$ be a connection 1-form on P normalized such that $A(\underline{1}) = 1$, and F_A be its curvature. Let c be a loop which bounds a 2-chain σ in B. Then the holonomy around c equals*

$$-\int_\sigma F_A.$$

Proof. Let $f : \Sigma \to \sigma$ be a representing map, where Σ is a surface with boundary (see the remark below). In a trivialization $f^*P \cong \Sigma \times S^1$ with coordinates (x, θ) we can write

$$f^*A = d\theta + a, \quad f^*F_A = da$$

for a 1-form a on Σ. Now a lift $\big(x(t), \theta(t)$ of $c \cong \partial\Sigma$ is horizontal iff $\dot\theta + a(\dot x) \equiv 0$. So the holonomy equals

$$\Delta\theta = -\int_0^T a(\dot x)dt = -\int_{\partial\Sigma} a = -\int_\Sigma da = -\int_\sigma F_A.$$

□

Remark. In the preceding proof I used the folklore theorem that every (relative) 2-homology class can be represented by a map from a surface. More generally, one can ask whether any k-dimensional integer homology class in an n-dimensional manifold can be represented by a smooth map from a closed k-manifold to M. As John Etnyre explained me, the answer is generally "no", but is "yes" for $k = 1, 2, n-2, n-1$. Let me sketch the arguments.

For $k = 1$ this is obvious. For $k = 2$ it follows abstractly from the fact that the 2-dimensional cobordism group of a point is trivial and therefore the natural map from cobordism to homology is an isomorphism in dimension 2. For a more concrete proof, take the cell decomposition corresponding to a Morse function on M. Any 2-dimensional cellular homology class σ can first be pushed off the handles of index > 2. Then replace σ by a homologous σ' whose boundary does not run over the 1-cells. Then fill in the boundary of σ' with a surface in the 0-handle (a ball) to get a closed surface.

For large k the statement should become wrong because the cobordism groups become nontrivial.

For small codimension there is a different type of argument. For $k = n-1$ show that every 1-dimensional integral cohomology class (the Poincaré dual of σ) can be represented by a map $f : M \to S^1$ and take the preimage of a regular value of f. For $k = n-2$ show that every 2-dimensional integral cohomology class is the Chern class of a complex line bundle $L \to M$ and take the zero set of a smooth transverse section in L.

I would like to learn about references for all this.

From the lemma we immediately obtain the geometric phase

$$\Delta\theta = \frac{1}{\langle \eta, X_\eta \rangle} \int_{\bar\sigma} \bar\omega.$$

The dynamical phase

Note that up to now we have not needed the assumption on the Hamiltonian. Now we use $H = \frac{1}{2}|p|^2$ to compute

$$\lambda_{(q,p)}(X_H) = \langle p, \tau_* X_H \rangle = \langle p, \frac{\partial H}{\partial p} \rangle = |p|^2 \equiv 2E,$$

where E is the total energy. The dynamical phase is obtained from the equation on S^1, $\dot{\theta}(t) X_\eta = X(t)$, where

$$X(t) = A_d \Big(X_H(d) \Big) \equiv \frac{2E\, X_\eta}{\langle \eta, X_\eta \rangle}.$$

Integrating this constant function yields the dynamical phase

$$\Delta\theta_{dyn} = \frac{2ET}{\langle \eta, X_\eta \rangle},$$

and the have proved

Theorem 12.17 (Montgomery's formula). *For a free motion on T^*Q of energy E, if the reduced motion $\bar{c}$ is periodic of period T and $\bar{c} = \partial \bar{\sigma}$ in $\bar{M}$, the total phase is $\Delta\theta = \Delta\theta_{geom} + \Delta\theta_{dyn}$ (mod 2π) with*

$$\Delta\theta_{geom} = \frac{1}{\langle \eta, X_\eta \rangle} \int_{\bar{\sigma}} \bar{\omega}, \qquad \Delta\theta_{dyn} = \frac{2ET}{\langle \eta, X_\eta \rangle}.$$

The free rigid body

Let us apply Montgomery's formula to the free rigid body. Equip $\mathfrak{so}(3)$ with the inner product $-\frac{1}{2}\mathrm{tr}(XY)$ which corresponds to the Euclidean product under the identification $\mathfrak{so}(3) \cong \mathfrak{so}(3)^* \cong \mathbb{R}^3$. The condition $\exp(2\pi X_\eta) = 1$ is equivalent to $\|X_\eta\| = 1$, so $\langle \eta, X_\eta \rangle = \|\eta\|$. As shown earlier, the reduced symplectic form is $\bar{\omega} = -\frac{\sigma_\eta}{\|\eta\|}$ where σ_η is the area form on the sphere $\mathcal{O}_\eta$. Thus the geometric phase equals $-\frac{1}{\|\eta\|^2}\int_{\bar{\sigma}} \sigma_\eta = -\Lambda$, where Λ is the oriented solid angle enclosed by the reduced curve on $\mathcal{O}_\eta$, and we have shown:

Corollary 12.18. *The total phase of a free rigid body is*

$$\Delta\theta = -\Lambda + \frac{2ET}{\|\eta\|},$$

where Λ is the oriented solid angle enclosed by the reduced motion.

Chapter 13

Convexity

13.1 Fixed points of Hamiltonian group actions

Consider a Hamiltonian action of the compact group G on (M, ω) with moment map μ.

Lemma 13.1. *Let $x \in M$ be a fixed point of the G-action. Then there exists a diffeomorphism from a G-invariant neighbourhood of x to a neighbourhood of 0 in $\mathbb{C}^n$ under which ω corresponds to $\omega_{\mathrm{st}} = dq \wedge dp$, and the G-action corresponds to a linear action of $G \subset U(n)$ with moment map*

$$\mu(z) = \mu(x) + \pi_{\mathfrak{g}}(-\frac{i}{2}zz^*), \quad z \in \mathbb{C}^n.$$

Proof. Let $\exp : T_xM \to M$ be the exponential map of a G-invariant Riemannian metric. Since $t \mapsto g \cdot \exp(tv)$ and $t \mapsto \exp(t\, T_xg \cdot v)$ are both geodesics starting at x with velocity $T_xg \cdot v$, we have

$$\exp(T_xg \cdot v) = g \cdot \exp(v).$$

So after pulling back under exp, we have a linear G-action on $\mathbb{C}^n$ with a G-invariant symplectic form ω. Next apply a equivariant version of Darboux' Theorem (see [17]) to transform ω to a constant symplectic form ω_0.

Finally, pick a G-invariant complex structure J_0 compatible with ω_0. By the linear normal form, there exists a linear isomorphism mapping (ω_0, J_0) to $(\omega_{\mathrm{st}}, J_{\mathrm{st}})$. After conjugating the G-action under this isomorphism, we have $G \subset U(n)$ acting linearly on $\mathbb{C}^n$, and its moment map is given by Proposition 12.6. □

Lemma 13.2. *For each subgroup $H \subset G$, the set $M_H := \{x \in M \mid G_x = H\}$ is a symplectic submanifold, and $T_xM_H = \{v \in T_xM \mid H \cdot v = v\}$. In particular, the fixed point set $Fix(G)$ is a symplectic manifold.*

Proof. Consider the Hamiltonian action of H on M. By the previous lemma, the fixed point set $Fix(H)$ is locally the fixed point set of a complex linear action, which is complex and hence symplectic. So $Fix(H)$ is a symplectic submanifold. But M_H is just the open subset of $Fix(H)$ where the stabilizer of the G-action is not larger than H, and the lemma follows. □

Remark. That M_H is a manifold can also be seen from the Local Slice Theorem. In a local tube $G \times_H V$ around an orbit of type (H),

$$M_H \cong \{[g,v] \mid v \in V_{(H)}, g \in N(H)\} \cong N(H)/H \times V_{(H)},$$

where $N(H)$ is the normalizer of H in G.

13.2 The Convexity Theorem

Now we specialize to $G = T$ a k-dimensional torus with Lie algebra $\mathfrak{t}$. Define the lattice and dual lattice

$$\Lambda := \{\tau \in \mathfrak{t} \mid \exp(\tau) = \mathbb{1}\} \subset \mathfrak{t},$$
$$\Lambda^* := \{\xi \in \mathfrak{t}^* \mid \langle \xi, \tau \rangle \in 2\pi\mathbb{Z} \text{ for all } \tau \in \Lambda\} \subset \mathfrak{t}^*.$$

A basis for Λ yields an identification $\mathfrak{t} \cong t^* \cong \mathbb{R}^k$ such that the duality pairing corresponds to the Euclidean product, $\exp(\tau) = (e^{i\tau_1}, \ldots, e^{i\tau_k})$ for $\tau = (\tau_1, \ldots, \tau_k) \in \mathfrak{t}$, $\Lambda \cong 2\pi\mathbb{Z}^k$, and $\Lambda^* \cong \mathbb{Z}^k$.

Let $x \in M$ be a fixed point of the T-action. The linearized action on T_xM yields unitary representation of T in $U(n)$. Now all elements of $\mathfrak{t} \subset \mathfrak{u}(n)$ are diagonalizable, thus so are all elements of $T \subset U(n)$. Moreover, since T is commutative, all elements of T can be simultaneously diagonalized. This proves that the representation decomposes into 1-dimensional representations (characters) $T \to U(1)$. Each one-dimensional representation can be uniquely written as $\exp(\tau) \mapsto e_{-i\langle w_j, \tau\rangle}$ with *weights* $w_1, \ldots, w_n \in \Lambda^*$. So Lemma 13.1 implies:

Proposition 13.3. *Near a fixed point x a Hamiltonian T-action is conjugate to the diagonal linear action*

$$\exp(\tau) \cdot z = (e^{-i\langle w_1, \tau\rangle} z_1, \ldots, e^{-i\langle w_n, \tau\rangle} z_n), \quad z \in \mathbb{C}^n$$

with moment map

$$\mu(z) = \mu(x) + \frac{1}{2}\sum_{j=1}^{n} |z_j|^2 w_j,$$

where $w_1, \ldots, w_n \in \Lambda^*$ are the weights of the linear representation on T_xM.

This local normal form reveals the local picture of the image of the moment map near a fixed point. Denote by

$$\text{cone}_p\{w_1, \ldots, w_n\} := \{p + \sum \lambda_j w_j \mid \lambda_j \geq 0\}$$

the cone in $\mathfrak{t}^*$ with vertex p spanned by $w_1, \dots, w_n$. Note that not all the w_j will correspond to edges of the cone. Some of them may lie in the interior of the cone, and we even allow w_j to be zero. Then Proposition 13.3 implies:

Corollary 13.4. *There exist neighbourhoods U of the fixed point x and V of $\mu(x)$ such that*

$$\mu(U) = \mathrm{cone}_{\mu(x)}\{w_1, \dots, w_n\} \cap V.$$

Now consider a point $x \in M$ with stabilizer $T_x \subset T$. Let $\pi : \mathfrak{t}^* \to \mathfrak{t}_x^*$ be the dual map to the inclusion $\mathfrak{t}_x \to \mathfrak{t}$ (under the identification $\mathfrak{t} \cong \mathfrak{t}^*$, π is the orthogonal projection onto $\mathfrak{t}_x$). Let $w_1, \dots, w_n$ be the weights of the linear representation of T_x on $T_x M$.

Proposition 13.5. *[local convexity] There exist neighbourhoods U of x and V of $\mu(x)$ such that*

$$\mu(U) = \left(\pi^{-1}\mathrm{cone}_{\pi\mu(x)}\{w_1, \dots, w_n\}\right) \cap V.$$

Proof. By Corollary 13.4 there exist neighbourhood U of x and V' of $\pi\mu(x)$ such that

$$\pi \circ \mu(U) = \mathrm{cone}_{\pi\mu(x)}\{w_1, \dots, w_n\} \cap V',$$

hence

$$\mu(U) \subset \pi^{-1}\left(\pi\mu(U)\right) = \pi^{-1}\left(\mathrm{cone}_{\pi\mu(x)}\{w_1, \dots, w_n\}\right) \cap \pi^{-1}(V').$$

So it remains to show that μ maps a neighbourhood of x onto a neighbourhood of $\mu(x)$ in

$$\pi^{-1}\left(\pi\mu(U)\right) = \{\xi \in \mathfrak{t}^* \mid \pi\big(\xi - \mu(y)\big) = 0 \text{ for some } y \in U\} = \mu(U) + \mathfrak{t}_x^\perp,$$

where $\mathfrak{t}_x^\perp \subset \mathfrak{t}^*$ is the annihilator of $\mathfrak{t}_x$.

Now consider the action of T on the T-invariant submanifold

$$W := \{y \in M \mid T_y = T_x\}$$

By formula (iv) for the moment map, $\mathfrak{t}_x = \mathfrak{t}_y = \big(d_y\mu(T_y W)\big)^\perp$ for $y \in W$, so $d_y\mu : T_y W \to \mathfrak{t}_x^\perp$ is an isomorphism. This shows that $\mu|_W : W \to \mu(x) + \mathfrak{t}_x^\perp$ is a submersion.

Suppose that U is chosen small enough so that the T_x-action is linear on $M = \mathbb{C}^n$,

$$\exp(\tau) \cdot z = (e^{-i\langle w_1, \tau\rangle} z_1, \dots, e^{-i\langle w_n, \tau\rangle} z_n),$$

$$\pi\mu(z) = \pi\mu(x) + \frac{1}{2}\sum_{j=1}^{n} |z_j|^2 w_j.$$

Suppose that $w_1, \ldots, w_m \neq 0$ and $w_{m+1} = \cdots = w_n = 0$. Then W is the linear subspace $\{z_1 = \cdots = z_m = 0\}$ and $\pi\mu(z+w) = \pi\mu(z)$ for $w \in W$. So $\pi\mu$ is constant on each affine subspace $z + W$, and

$$\mu(z+W) \subset \pi^{-1}\Big(\pi\mu(z)\Big) = \mu(z) + t_x^{\perp}.$$

By the previous step, $\mu : z + W \to \mu(z) + t_x^{\perp}$ is a submersion for z near 0, and the proposition follows. □

Proposition 13.5 shows that near its boundary points, the image of the moment map imμ looks locally like a convex polyhedron. If M is compact is indeed true globally:

Theorem 13.6. *[Atiyah-Guillemin-Sternberg Convexity Theorem] The image of the moment map of a Hamiltonian torus action on a compact connected symplectic manifold is the convex polyhedron spanned by the image of the fixed point set.*

Proof. Consider a point $x \in M$ with $\mu(x) \in \partial(\mathrm{im}\mu)$. Then $\mathfrak{t}_x \neq \{0\}$ because otherwise μ would be a submersion near x. Let $w_1, \ldots w_n \in \mathfrak{t}_x^*$ be the weights at x and $\pi : \mathfrak{t}^* \to \mathfrak{t}_x^*$ the projection. By Proposition 13.5, near $\mu(x)$ the the image of μ looks like $S := \pi^{-1}\mathrm{cone}_{\pi\mu(x)}\{w_1, \ldots, w_n\}$. Let S_i be a boundary face of S. Pick $X \in \mathfrak{t}$ and $c \in \mathbb{R}$ such that $\langle \xi, X\rangle = c$ for $\xi \in S_i$ and $\langle \xi, X\rangle \leq c$ for $\xi \in S$. So $\langle \mu(y), X\rangle \leq \langle \mu(x), X\rangle$ for $y \in M$ near x, i.e. the function $\mu_X = \langle \mu, X\rangle$ has a local maximum at x.

Now we insert the result that is at the core of the Convexity Theorem and will be proved in the next section: μ_X *has a unique local maximum.* It implies that the local maximum is global, so $\langle \mu(y), X\rangle \leq \langle \mu(x), X\rangle$ for all $y \in M$. Let P be the convex polyhedron which is the intersection of all the half spaces $\{\xi \mid \langle \xi, X\rangle \leq c\}$ over all faces S_i corresponding to all boundary points of imμ (there are only finitely many such subspaces). By the preceding argument, im$\mu \subset P$. Moreover, by Proposition 13.5, $\mu : M \to P$ is a submersion. So im$\mu = P$ because M is connected.

Finally, note that the vertices of P are images of points x with $\mathfrak{t}_x = \mathfrak{t}$, i.e. of fixed points. □

13.3 Morse-Bott functions

Let M^n be a manifold.

Definition. A function $f : M \to \mathbb{R}$ is called *Morse-Bott* if its set of critical points $Crit(f)$ is a manifold (with different components having possibly different dimensions) and the Hessian $D^2 f$ is nondegenerate in the directions transverse to $Crit(f)$. The *index* of a component of $Crit(f)$ is the number of negative eigenvalues of $D^2 f$, and the *coindex* is the number of positive eigenvalues.

Every Morse function is Morse-Bott. At the other extreme, the zero function is Morse-Bott. An intermediate example is the function $f(x, y, z) = z^2$ on the unit sphere $\{x^2 + y^2 + z^2 = 1\}$.

We say that a connected set C is a *local minimum* of a function f if $f \equiv c$ on C and $f > c$ on $U \setminus C$ for some neoghbourhood U.

Proposition 13.7. *Let M^n be compact and connected and $f : M \to \mathbb{R}$ a Morse-Bott function which has no critical components of index or coindex 1. Then every level set of f is connected. In particular, f has a unique local minimum and maximum.*

Proof. We first prove that f has a unique local minimum. Pick a Riemannian metric and let $\phi_t : M \to M$ the the negative gradient flow of f. For every component C of $Crit(f)$ let

$$W^s(C) := \{x \in M \mid \phi_t(x) \to C \text{ as } t \to +\infty\},$$
$$W^u(C) := \{x \in M \mid \phi_t(x) \to C \text{ as } t \to -\infty\}$$

be the stable and unstable manifolds. Note that

$$\dim W^s(C) = n - \operatorname{ind}(C), \quad \dim W^u(C) = n - \operatorname{coind}(C)$$

and

$$M = \bigcup_C W^s(C) = \bigcup_C W^u(C).$$

Since $\operatorname{ind}(C) \neq 1$, $\bigcup_{\operatorname{ind}(C)>0} W^s(C)$ is a union of manifolds of codimension at least 2 which cannot disconnect M, so $\bigcup_{\operatorname{ind}(C)=0} W^s(C)$ is connected. But this set retracts onto $\bigcup_{\operatorname{ind}(C)=0} C$, so there is only one component $C_{\min}$ of index zero. Similarly, there is only one component $C_{\max}$ of coindex zero.

Now consider the open set

$$U := W^s(C_{\min}) \cap W^u(C_{\max}).$$

Since $\operatorname{ind}(C) \neq 1$ and $\operatorname{coind}(C) \neq 1$, the complement of U is contained in the union $\bigcup_{\operatorname{ind}(C)>0} W^s(C) \cup \bigcup_{\operatorname{coind}(C)>0} W^u(C)$ of manifolds of codimension at least 2, so U is connected. Moreover, U contains no critical points except $C_{\min}$ and $C_{\max}$, and it intersects every level set $\min f < c < \max f$ in a relatively open set. Now given two points $x, y \in f^{-1}(c)$, connect them in $f^{-1}(c)$ to points x', y' in $U \cap f^{-1}(c)$. Connect x', y' by a path γ in U. Then push γ into $f^{-1}(c)$ using the gradient flow, and the connectedness is proved. □

We will apply the preceding proposition to the map $\mu_X = \langle \mu, X \rangle : M \to \mathbb{R}$, where $\mu : M \to \mathfrak{t}$ is the moment map of a Hamiltonian torus action and $X \in \mathfrak{t}$ is arbitrary.

Proposition 13.8. *(i) $Crit(\mu_X)$ is a symplectic submanifold of M.*

(ii) μ_X *is a Morse-Bott function.*

(iii) All critical components of μ_X *have even index and coindex.*

Proof. After replacing T by the closure of $\{\exp tX \mid t \in \mathbb{R}\}$ (which is a subtorus), we may (and will) assume the this closure equals T. Then we say that X *generates* T. Under the identification $\mathfrak{t} \cong \mathbb{R}^k$ this is equivalent to the components of X being rationally independent.

(i) At a critical point x of μ_X, $d_x\mu_X = i_{\underline{X}_x}\omega = 0$, so $\underline{X}_x = 0$. This implies $\exp(tX) \cdot x = 0$ for all t, and since X generates T, $x \in Fix(T)$. So we have shown

$$Crit(\mu_X) = Fix(T).$$

But $Fix(T)$ is a symplectic submanifold by Lemma 13.2.

(ii) By Proposition 13.3, the local normal form near a critical point $x \in Fix(T)$ is

$$\exp(\tau) \cdot z = (e^{-i\langle w_1, \tau\rangle} z_1, \ldots, e^{-i\langle w_n, \tau\rangle} z_n),$$

$$\mu(z) = \mu(x) + \frac{1}{2}\sum_{j=1}^{n} |z_j|^2 w_j.$$

Suppose that $w_1, \ldots, w_l \neq 0$ and $w_{l+1} = \ldots w_n = 0$. Then

$$Crit(\mu_X) = Fix(T) = \{z \mid z_1 = \cdots = z_l = 0\} = o \times \mathbb{C}^{n-l}.$$

In the function

$$\mu_X(z) = \mu_X(x) + \frac{1}{2}\sum_{j=1}^{l} \langle w_j, X\rangle |z_j|^2$$

all coefficients $\langle w_j, X\rangle$ are nonzero because the components of X and rationally independent and $0 \neq w_j \in \mathbb{Z}^k$. Thus μ_X is nondegenerate in the directions $\mathbb{C}^l \times 0$ transverse to $Crit(\mu_X)$.

(iii) From the local form of μ_X in (ii) we read off that

$$\mathrm{ind}_x(\mu_X) = 2\#\{j \mid \langle w_j, X\rangle < 0\}.$$

□

Propositions 13.7 and 13.8 yield the missing ingredient in the proof of the Convexity Theorem:

Corollary 13.9. *Let* μ *be the moment map of a Hamiltonian torus action on a compact connected symplectic manifold, and* $X \in \mathfrak{t}$. *Then every level set of the function* $\mu_X = \langle \mu, X\rangle$ *is connected. In particular, this function has a unique local minimum and maximum.*

The following corollary will be useful in the next chapter.

Corollary 13.10. *Let μ be the moment map of a Hamiltonian torus action on a compact connected symplectic manifold. Then every level set $\mu^{-1}(\eta)$ is connected.*

Proof. I give a proof under the additional assumption that the action has no finite isotropy groups. To drop this assumption, one can either do the same proof with manifolds replaced by orbifolds, or use the argument in the proof of Theorem 5.47 in [31] which avoids symplectic reduction.

Let $\mu = (\mu_1, \dots, \mu_k)$ be the components with respect to a splitting of the torus $T = T_1 \times \cdots \times T_k$ into circles. After passing to a quotient torus, we may assume that the action is effective. By Sard's Theorem the regular values for μ_i are open and dense in $\mathrm{im}\mu_i$. Let P_{reg} be the open dense set of $\eta \in \mathrm{im}\mu$ for which each component η_i is regular for μ_i.

Take an $\eta \in P_{reg}$. By Corollary 13.9, the level set $\mu_1^{-1}(\eta_1)$ is connected, and T_1 acts freely on it. So the quotient $\mu_1^{-1}(\eta_1)/T_1$ is connected, and $T_2 \times \cdots \times T_k$ acts on the quotient with moment map $(\mu_2, \dots, \mu_k)$. Again by Corollary 13.9, $(\mu_1, \mu_2)^{-1}(\eta_1, \eta_2)/T_1$ is connected. But this is the base space of the fibration

$$(\mu_1, \mu_2)^{-1}(\eta_1, \eta_2) \to (\mu_1, \mu_2)^{-1}(\eta_1, \eta_2)/T_1$$

with connected fibre T_1, so the total space $(\mu_1, \mu_2)^{-1}(\eta_1, \eta_2)$ is connected. Now continue by induction to conclude that $\mu^{-1}(\eta)$ is connected for every $\eta \in P_{reg}$.

For a nonregular $\eta \in \mathrm{im}\mu$, suppose that $\mu^{-1}(\eta)$ is disconnected. Let U_1, U_2 be disjoint open sets such that $\mu^{-1}(\eta) \subset U_1 \cup U_2$ has points $x_i \in \mu^{-1}(\eta) \cap U_i$ for $i = 1, 2$. By the Slice Theorem, the image $\mu(U_i)$ contains an open neighbourhood V_i of η. Take a regular $\eta' \in V_1 \cap V_2 \cap P_{reg}$. By a compactness argument, for η' sufficiently close to η, $\mu^{-1}(\eta') \subset U_1 \cup U_2$. But by construction, $\mu^{-1}(\eta')$ has points in both U_i, so it is disconnected, contradicting what we just proved. □

Example 13.11. Let S^1 act on $\mathbb{C}P^2$ by

$$e^{i\theta}[z_0 : z_1 : z_2] = [z_0 : e^{-i\theta} z_1 : e^{-ik\theta} z_2],$$
$$\mu([z]) = \frac{1}{|z|^2}(|z_1|^2 + k|z_2|^2),$$

for some integer $k > 1$. The subset with stabilizer $\mathbb{Z}_k$ is the sphere

$$\{[z] \mid z_1 = 0\}.$$

The image of this sphere is the whole moment polytope, the interval $[0, k/2]$. This example shows that in general one does not find values in the moment polytope on whose preimage the torus acts freely, so new arguments are needed if we allow finite stabilizers in the proof above.

Remark. Suppose that M is connected and the T^k-action on M is effective. Then the weights $w_1, \dots, w_n$ at each fixed point must span $\mathfrak{t}^*$. For otherwise we would find a nonzero $\tau \in \mathfrak{t}$ annihilated by $w_1, \ldots, w_n$, so $\exp(\tau)$ acts as the identity on a neighbourhood of x and therefore on all of M.

In particular, $n \geq k$, so $\dim T \leq \frac{1}{2} \dim M$.

Moreover, $\dim Fix(T) \leq n - k$. E.g. on a toric manifold ($k = n$) fixed points are isolated.

Remark. For a toric manifold, by the previous remark and Proposition 13.8, μ_X is a *perfect Morse function.* This is a Morse function for which the Morse inequalities are equalities, i.e. the k-th Betti number is equal to the number of critical points of index k. So the homology of a toric manifold id determined by the indices of the fixed point set.

By the previous remark, the weights $w_1, \ldots, w_n$ at each fixed point x are linearly independent. So all the w_j span edges of the moment polytope P. For a vector $X \in \mathfrak{t}$ generating T, the index of μ_X at x equals

$$\mathrm{ind}_x(\mu_X) = 2\#\{\text{edges } e \text{ at } \mu(x) \text{ with } \langle e, X\rangle < 0\}.$$

So the homology of M can be computed from the moment polytope alone: Assign to every vertex p the number

$$h_X(p) := \#\{\text{edges } e \text{ at } p \text{ with } \langle e, X\rangle < 0\}.$$

Then the Betti numbers of M are given by $b_{2i+1}(M) = 0$ and

$$b_{2i}(M) = 2\#\{\text{vertices } p \text{ of } P \text{ with } h_X(p) = i\}.$$

Example 13.12. For a Morse-Bott function with even dimensional critical manifolds of even index and coindex it is in general *not* true that the rational homology equals the direct sum of the homologies of the critical manifolds, shifted by their indices. Consider, for example, $S^3 \times S^1$ with coordinates $(x_1, \ldots, x_4, y_1, y_2)$, $\sum x_i^2 = \sum y_j^2 = 1$, and the Morse-Bott function

$$f(x, y) = x_1^2 + x_2^2.$$

It has two critical manifolds, its maximal and minimal set

$$f^{-1}(0) = \{x_1 = x_2 = 0\} \cong T^2, \quad f^{-1}(1) = \{x_3 = x_4 = 0\} \cong T^2$$

of index 0 and 2 respectively. However, the rational homology of $S^3 \times S^1$ is not the direct sum of the homologies of two 2-tori, one shifted by 2.

The above statement *is* true if in addition each critical manifold admits a Morse function with only even indices.

Problem 13.1. Let $P \subset \mathbb{R}^n$ be a convex polytope with the property that precisely n edges meet at every vertex. By the previous remark, the moment polytope of a $2n$-dimensional toric manifold has this property. For each vector $X \in \mathbb{R}^n$ such that $\langle e, X\rangle \neq 0$ for all edges define the numbers

$$\beta_i := \#\{\text{vertices } p \text{ of } P \text{ with } h_X(p) = i\}$$

with $h_X(p)$ as in the previous remark.

(i) Show that the numbers β_i are independent of the vector X. Is this still true for polytopes with more than n edges meeting at a vertex?

(ii) Prove "Poincaré duality" $\beta_{n-i} = \beta_i$.

(iii) If P is the moment polytope of a toric variety, what happens to the Morse-Bott functions μ_X as X becomes orthogonal to an edge of P? Discuss this for the standard T^2-action on $\mathbb{C}P^2$.

13.4 Examples

Example 13.13 ($\mathbb{C}P^n$). Consider the Hamiltonian action of T^k on $(\mathbb{C}P^n, \omega_{FS})$,

$$\exp(\tau)[z] = [z_0 : e^{-i\langle w_1,\tau\rangle} z_1 : \cdots : e^{-i\langle w_n,\tau\rangle} z_n]$$

with weights $w_1, \ldots, w_n \in \mathbb{Z}^k$ and moment map

$$\mu([z]) = \frac{1}{2|z|^2} \sum_{j=1}^{n} |z_j|^2 w_j.$$

The image of the moment map is the convex hull of 0 and the $w_j/2$,

$$\mathrm{im}\mu = \mathrm{conv}\{0, w_1/2, \ldots, w_n/2\}.$$

For example, suppose that T^n acts on $\mathbb{C}P^n$ by

$$(e^{i\tau_1}, \ldots, e^{i\tau_n})[z] = [z_0 : e^{-ik_1\tau_1} z_1 : \cdots : e^{-ik_n\tau_n} z_n],$$

$$\mu([z]) = \frac{1}{2|z|^2}(k_1|z_1|^2, \ldots, k_n|z_n|^2).$$

If all $k_j \in \mathbb{Z}$ are nonzero, then there are $n+1$ isolated fixed points $[1 : 0 \ldots 0], \ldots, [0 : \ldots 0 : 1]$, and the image of the moment map is the simplex with vertices $(0, \ldots, 0), (k_1/2, 0, \ldots, 0), \ldots, (0, \ldots, k_n/2)$.

Problem 13.2. Draw the moment polytopes for the following T^2-actions:

(i) On $\mathbb{C}P^2$,

$$(e^{i\tau_1}, e^{i\tau_2})[z_0 : z_1 : z_2] := [z_0 : e^{-i\tau_1} z_1 : e^{-3i\tau_2} z_2].$$

(ii) On $\mathbb{C}P^3$,

$$(e^{i\tau_1}, e^{i\tau_2})[z_0 : z_1 : z_2 : z_3] := [z_0 : e^{-i\tau_1} z_1 : e^{-3i\tau_2} z_2 : e^{i(\tau_1+\tau_2)} z_3].$$

(iii) On $\mathbb{C}P^1 \times \mathbb{C}P^1$,

$$(e^{i\tau_1}, e^{i\tau_2})\Big([z_0 : z_1], [w_0 : w_1]\Big) := \Big([z_0 : e^{-i\tau_1} z_1], [w_0 : e^{-3i\tau_2} w_1]\Big).$$

Example 13.14. Consider the Hamiltonian action of $T^n \subset U(n)$ on the space $\mathcal{H}_\lambda$ of Hermitian matrices of eigenvalues $\lambda = (\lambda_1, \dots, \lambda_n)$ by conjugation, with moment map

$$\mu(A) = \mathrm{diag(A)} := (a_{11}, \dots, a_{nn}) \in \mathbb{R}^n .$$

The fixed points of the action are the diagonal matrices in $\mathcal{H}_\lambda$. Their image under the moment map is $\{\lambda_\sigma \mid \sigma \in S_n\}$, where $\lambda_\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ for a permutation σ. So the Convexity Theorem implies

Corollary 13.15 (Schur). *For every Hermitian matrix $A \in \mathcal{H}_\lambda$,*

$$\mathrm{diag}(A) \in \mathrm{conv}\{\lambda_\sigma \mid \sigma \in S_n\}.$$

Conversely, every point in $\mathrm{conv}\{\lambda_\sigma \mid \sigma \in S_n\}$ *is the diagonal of some Hermitian matrix $A \in \mathcal{H}_\lambda$.*

Problem 13.3. Prove Schur's Theorem directly for 2×2 matrices.

Problem 13.4. Draw the set $\mathrm{conv}\{\lambda_\sigma \mid \sigma \in S_n\}$ for 3×3 Hermitian matrices in the cases $\lambda 1 < \lambda_2 < \lambda_3$ and $\lambda_1 = \lambda_2 < \lambda_3$.

Chapter 14

Toric manifolds

14.1 Delzant's Theorem

Definition. A *toric manifold* is a connected compact symplectic manifold (M^{2n}, ω) with an effective Hamiltonian action of an n-dimensional torus T^n.

By Remark 13.3, n is the maximal possible dimension of a torus acting effectively and Hamiltonian. Let $\mu : M \to \mathfrak{t}^* \cong (\mathbb{R}^n)^*$ be the moment map and $P := \mathrm{im}\mu$ its image. The action is free on the open dense subset $\mu^{-1}(\int P)$ which is diffeomorphic to the open torus $T^n \times \int P$. So a toric manifold can be thought of as a suitable compactification of an open torus. Since all the reduced spaces are just points, one expects that all the information about (M, ω) is encoded in the moment polytope. The goal of this section is to prove that this is indeed the case: A toric variety can be uniquely reconstructed from its moment polytope.

To formulate the main result, we need the following

Definition. A convex polytope $P \subset (\mathbb{R}^n)^*$ with the following properties is called *Delzant polytope*:

(i) Precisely n edges meet at every vertex p.

(ii) The edges at p are $\{p + tw_i \mid t \geq 0\}$ with integer vectors $w_i \in (\mathbb{Z}^n)^*$.

(iii) The vectors $w_1, \ldots, w_n$ at each vertex form an integer basis of $(\mathbb{Z}^n)^*$.

The main result of this chapter is

Theorem 14.1 (Delzant [9]). *The moment polytope of a toric manifold is a Delzant polytope, and every Delzant polytope is the moment polytope of a toric variety. Two toric manifolds with the same moment polytope are equivariantly symplectomorphic.*

In short, the moment map establishes a one-to-one correspondence

$$\frac{\{\text{toric manifolds}\}}{\{\text{equivariant symplectomorphisms}\}} \longleftrightarrow \frac{\{\text{Delzant polytopes}\}}{\{\text{translations}\}}.$$

The proof of the theorem will also show:

Corollary 14.2. *Every toric manifold M^{2n} is the symplectic quotient of some $\mathbb{C}^d$ by a linear torus action of T^{d-n}. The T^n action on M is induced by the linear action of a complementary torus to T^{n-d} in T^d.*

Let us first show that the moment polytope $P = \mathrm{im}\mu$ of a toric manifold is Delzant. Let $p = \mu(x)$ be a vertex, so x is a fixed point. Let $w_1, \ldots, w_n$ be the weights of the linearized action on T_xM. We have seen in the previous chapter that the w_i generate the edges at p, and they lie in the dual lattice $(\mathbb{Z}^n)^*$. It remains to show that they form a $\mathbb{Z}$-basis. This means that the matrix $W := (w_1^T, \ldots, w_n^T)^T$ with rows w_i defines an isomorphism $W : \mathbb{Z}^n \to \mathbb{Z}^n$, or equivalently, $\det W = \pm 1$.

Suppose this is not true. Then there exists a vector $\tau \in \mathbb{R}^n$ with $W \cdot \tau \in \mathbb{Z}^n$ but $\tau \notin \mathbb{Z}^n$. This means that $\langle w_i, \tau \rangle \in \mathbb{Z}$ for all i, so $\exp(\tau) \neq \mathbb{1}$ acts trivially on a neighbourhood of x, contradicting effectiveness of the action.

14.2 Constructing toric manifolds from Delzant polytopes

Let a Delzant polytope $P \subset (\mathbb{R}^n)^*$ be given. Write the $(n-1)$-dimensional faces as $\{\xi \mid \langle \xi, u_i \rangle = \lambda_i\}$ with $u_1, \ldots, u_d \in \mathbb{Z}^n$ and $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$. The u_i are uniquely determined if we choose them minimal and inward pointing so that

$$P = \cup_{i=1}^d \{\xi \mid \langle \xi, u_i \rangle \geq \lambda_i\}.$$

Explicitly, the u_i can be found as follows: Take a vertex p with edges generated by $w_1, \ldots, w_n$ and the corresponding matrix $W = (w_1^T, \ldots, w_n^T)^T$. Define $u_i \in \mathbb{Z}^n$ by $W \cdot u_i = e_i$ where e_i are the unit vectors. This means

$$\langle w_j, u_i \rangle = \delta_{ij},$$

so u_i is the inward pointing normal vector to the face spanned be the w_j, $j \neq i$. By uniqueness, we get the same u_i for whichever vertex we choose on a face.

Here is the main construction. Define the linear map

$$\pi : \mathbb{R}^d \to \mathbb{R}^n, \quad e_i \mapsto u_i.$$

This map restricts to $\pi : \mathbb{Z}^d \to \mathbb{Z}^n$ and thus induces a quotient map $\pi : T^d \to T^n$. Define the torus $H \cong T^{d-n}$ by the exact sequence

$$0 \to H \xrightarrow{i} T^d \xrightarrow{\pi} T^n \to 0.$$

Consider the standard action of T^d on $\mathbb{C}^d$ by

$$\exp(\tau)z = (e^{i\tau_1}z_1, \dots, e^{i\tau_d}z_d)$$

with moment map

$$\mu(z) = \frac{1}{2}(|z_1|^2, \dots, |z_d|^2) + (\lambda_1, \dots, \lambda_d).$$

Note that this differs from the ordinary moment map by a constant that will be useful later. This action restricts to an action of H with moment map

$$i^* \circ \mu : \mathbb{C}^d \to \mathfrak{h}^*,$$

where i^* is defined by the exact sequences of vector spaces

$$0 \to \mathfrak{h} \xrightarrow{i} \mathbb{R}^d \xrightarrow{\pi} \mathbb{R}^n \to 0,$$
$$0 \to (\mathbb{R}^n)^* \xrightarrow{\pi^*} (\mathbb{R}^d)^* \xrightarrow{i^*} \mathfrak{h}^* \to 0.$$

Lemma 14.3. *$(i^* \circ \mu)^{-1}(0)$ is compact and H acts freely on it.*

Proof. Since $\ker i^* = \mathrm{im}\pi^*$,

$$(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\ker i^*) = \mu^{-1}(\mathrm{im}\pi^*) = \mu^{-1}(\mathrm{im}\pi^* \cap \mathrm{im}\mu).$$

Since $\mathrm{im}\mu$ is just the set of vectors with nonnegative components, $\pi^*\xi$ lies in $\mathrm{im}\pi^* \cap \mathrm{im}\mu$ iff for all i,

$$0 \leq \langle \pi^*\xi, e_i \rangle = \langle \xi, \pi e_i \rangle = \langle \xi, u_i \rangle,$$

i.e. iff $\xi \in P$. So $\mathrm{im}\pi^* \cap \mathrm{im}\mu = \pi^* P$, and

$$(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\pi^* P)$$

is compact because μ is proper.

Next we look at the stabilizer of $z \in \mathbb{C}^d$ with $\mu(z) = \pi^*\xi$, $\xi \in P$. Let

$$I := \{i \mid \langle \xi, u_i \rangle = \lambda_i\}.$$

Note that I is empty if ξ is an interior point, $|I| = n - k$ if ξ lies on a k-dimensional face, and $|I| = n$ at a vertex. From the definition of μ, $\frac{1}{2}|z|^2 + \lambda_i = \lambda_i$ for $i \in I$, so $z_i = 0$ for $i \in I$ and $z_i \neq 0$ for $i \notin I$.

Consider an element $h \in H = \ker(\pi : T^d \to T^n)$ which stabilizes z. Write $h = \exp(X)$ with $X \in \mathbb{R}^d$ such that $\pi X \in \mathbb{Z}^n$. It stabilizes z iff the components $X_i \in \mathbb{Z}$ for $i \notin I$. Let $\tilde{X} := X - \sum_{i \notin I} X_i e_i$. Then $\pi\tilde{X} = \sum_{i \in I} X_i u_i \in \mathbb{Z}^n$. But the u_i, $i \in I$ are a subset of the normal vectors to the $(n-1)$-dimensional faces meeting at a vertex p. Since $\langle w_i, u_i \rangle = \delta_{ij}$ for the corresponding edge vectors w_i, $X_i = \langle w_i, \pi\tilde{X} \rangle \in \mathbb{Z}$ for all $i \in I$. This proves $X \in \mathbb{Z}^n$ and thus $h = \mathbb{1}$. □

In view of the lemma, we get a compact symplectic quotient manifold

$$M := (i^* \circ \mu)^{-1}(0)/H = \mathbb{C}^n//H(0).$$

This manifold is connected by Corollary 13.10.

Now choose a group homomorphism $j : T^n \to T^d$ such that $\pi \circ j = \mathbb{1}$. T^n acts via j on $\mathbb{C}^d$ with moment map $j^* \circ \mu$, where $j : \mathbb{R}^n \to \mathbb{R}^d$ and $j^* : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$ are the corresponding linear maps. Since the T^n-action commutes with the H-action, it descends to the quotient M with moment map μ_M satisfying

$$\mu_M \circ \pi_H = j^* \circ \mu \circ i_H.$$

Here $i_H : (i^* \circ \mu)^{-1}(0) \hookrightarrow \mathbb{C}^d$ is the inclusion and $\pi_H : (i^* \circ \mu)^{-1}(0) \to M$ the quotient projection.

Since $(i^* \circ \mu)^{-1}(0) = \mu^{-1}(\pi^* P)$, the image of the moment map μ_M is

$$\mathrm{im}\mu_M = j^* \circ \mu \Big(\mu^{-1}(\pi^* P) \Big) = j^*(\pi^* P) = P.$$

The discussion of the stabilizers above showed that the T^d-action is free on the open dense set $\mu^{-1}(\pi^* \mathrm{int}\, P) \subset (i^* \circ \mu)^{-1}(0)$. In particular, the T^n-action is free on this set and thus effective on the quotient.

So M is the desired toric manifold with moment polytope P.

Remark. In the proof we have modified the moment map on $\mathbb{C}^d$ by the constant $\lambda = (\lambda_1, \ldots, \lambda_d)$ and reduced at zero. In later discussions it will be more convenient to choose μ with constant zero. Then $M = \mathbb{C}^d//H(-i^*\lambda)$ is the reduction at the value $-i^*\lambda \in \mathfrak{h}^*$.

14.3 Constructing a semi-isomorphism

The following definition will save us words in what follows.

Definition. Let G be a compact Lie group. A *Hamiltonian G-space* (M, ω, μ) is a symplectic manifold with a Hamiltonian G-action with moment map $\mu : M \to \mathfrak{g}^*$. An *isomorphism* of G-spaces is a diffeomorphism which intertwines the symplectic forms, the G-actions and the moment maps. A *semi-isomorphism* is a diffeomorphism which intertwines the G-actions and the moment maps, but not necessarily the symplectic forms.

The goal of this section if the following

Proposition 14.4. *Any two toric manifolds with the same moment polytope are semi-isomorphic as Hamiltonian T-spaces.*

We first derive a semi-local normal form at a face of the moment polytope.

Lemma 14.5 (action-angle coordinates). *Let (M^{2n}, ω, μ) be a toric manifold, and F an (open) k-dimensional face of the moment polytope* $\mathrm{im}\mu$*. Split the torus as $T = T^k \times T^{n-k}$ with coordinates (τ, σ), where T^{n-k} is the stabilizer at $\mu^{-1}(F)$.*

Then for every relatively compact $U \subset F$, a neighbourhood of $\mu^{-1}(U)$ is isomorphic, as a Hamiltonian T-space, to a neighbourhood of $T^k \times U \times 0$ in $T^k \times F \times \mathbb{C}^{n-k}$, where in coordinates $(\theta, \eta, z = x + iy)$ the symplectic form, T-action and moment map are given by

$$\omega = \sum d\theta_i \wedge d\eta + \sum dx_j \wedge dy_j,$$
$$(\tau, \sigma) \cdot (\theta, \eta, z) = (\theta + \tau, \eta, e^{-i\langle w_j, \sigma\rangle} z_j),$$
$$\mu(\theta, \eta, z) = (\eta, \frac{1}{2} \sum |z_j|^2 |w_j) + \text{const.}$$

Proof. T^k acts freely on $\mu^{-1}(F)$, and the moment map of this action is a submersion onto F. Moreover, every preimage $\mu^{-1}(\eta)$ is connected by Corollary 13.10. So the moment map defines a T^k-principal bundle $\mu^{-1}(F) \to F$, which is necessarily trivial:

$$\mu^{-1}(F) \cong T^k \times F$$

with the T^k-action $\tau \cdot (\theta, \eta) = (\theta + \tau, \eta)$ and moment map $\mu(\theta, \eta) = \eta$.

The restriction of ω to $T^k \times F$ is invariant and satisfies $i_{\frac{\partial}{\partial \theta_i}} \omega = d\eta_i$ by the definition of the moment map. This shows that the form $\omega - \sum d\theta_i \wedge d\eta_i$ is invariant and horizontal, so

$$\omega - \sum d\theta_i \wedge d\eta_i = \mu^* d\beta$$

for a 1-form β on F. Since the forms $\sum d\theta_i \wedge d\eta_i + t\mu^* d\beta$ are symplectic and cohomologous, Moser's Stability Theorem yields and equivariant isomorphism between $\omega|_F$ and $\sum d\theta_i \wedge d\eta_i$. (The proof of Moser's Theorem works here although $T^k \times F$ is noncompact because the constructed flow will move only in the T^k-direction. Check!).

It follows from the Slice Theorem and the fact that T splits into $T^k \times T^{n-k}$ (see also Appendix A) that the symplectic normal bundle to $\mu^{-1}(F)$ is trivial. By an equivariant version of the Symplectic Neighbourhood Theorem (here we need to restrict to $U \subset F$), $\mu^{-1}(U)$ is isomorphic to $T^k \times U \times \mathbb{C}^{n-k}$ with the symplectic form $\sum d\theta_i \wedge d\eta + \sum dx_j \wedge dy_j$. □

Proof of Proposition 14.4. Let (M_i, ω_i, μ_i), $i = 0.1$, be toric manifolds with the same moment polytope $\mathrm{im}\mu_0 = \mathrm{im}\mu_0 = P$. Let $F_k^{(j)}$ be the k-dimensional (open) faces of P. Starting with the vertices, construct inductively open subsets $V_k^{(j)} \subset P$ with the following properties for $i = 0, 1$ (see Figure):

(i) $U_k^{(j)} := V_k^{(j)} \cap F_k^{(j)}$ is relatively compact.

(ii) $\mu_i^{-1}(V_k^{(j)}) \cong T^k \times U_k^{(j)} \times B^{2n-2k}(\varepsilon)$ with action-angle coordinates as in Lemma 14.5.

(iii) $\mu_i\Big(T^k \times U_k^{(j)} \times \partial B^{2n-2k}(\varepsilon)\Big) \subset \cup_{a>k,b} V_a^{(b)}$.

Property (i) provides semi-isomorphisms between $\mu_0^{-1}(V_k^{(j)}$ and $\mu_1^{-1}(V_k^{(j)}$ for all k,j. From them we build the semi-isomorphism $\phi : M_0 \to M_1$ inductively, starting with the *highest*-dimensional faces. So suppose we already have a semi-isomorphism

$$\phi : \mu_0^{-1}\Big(\bigcup_{a>k,b} V_a^{(b)}\Big) \to \mu_1^{-1}\Big(\bigcup_{a>k,b} V_a^{(b)}\Big).$$

Use property (i) to identify $\mu_0^{-1}\big(\cup_{a>k,b}V_a^{(b)}\big)$ with $T^k \times U_k^{(j)} \times B^{2n-2k}(\varepsilon)$ for $i = 0,1$. By induction hyppothesis and property (ii), ϕ induces a semi-isomorphism of $T^k \times U_k^{(j)} \times \partial B^{2n-2k}(\varepsilon)$ with the standard T-action and moment map. So we are done if we can extend this semi-isomorphism to the whole $T^k \times U_k^{(j)} \times B^{2n-2k}(\varepsilon)$, which is precisely Lemma 14.6 below. □

Lemma 14.6. *Every semi-isomorphism ϕ of $T^k \times U_k^{(j)} \times \partial B^{2n-2k}(\varepsilon)$ (with the standard T-action and moment map) extends to a semi-isomorphism of $T^k \times U_k^{(j)} \times B^{2n-2k}(\varepsilon)$.*

Proof. Set $l := n-k$ and $S^{2l-1} := \partial B^{2l}(\varepsilon)$. The condition that ϕ is a semi-isomorphism implies that

$$\phi(\theta,\eta,z) = \Big(\theta + \theta_0(\eta,z), \eta, \zeta(\eta,z)\Big).$$

So we are done if we can extend the function $\zeta(z) = \zeta(\eta,z)$ as a semi-isomorphism from S^{2l-1} to B^{2l} for every η (and depending smoothly on η). The condition that ζ is a semi-isomorphism implies

$$|\zeta_j|^2 = |z_j|^2,$$
$$\zeta_j(e^{i\theta_1}z_1,\dots,e^{i\theta_l}z_l) = e^{i\theta_j}\zeta_j(z_1,\dots,z_l).$$

Consider the restriction of ζ_1 to the real sphere $S^{l-1} = \Re(S^{2l-1})$. The above conditions become

$$|\zeta_1|^2 = x_1^2,$$
$$\zeta_1(-x_1,\pm x_2,\dots,\pm x_l) = -\zeta_1(x_1,\dots,x_l).$$

The function $f(x) := \frac{\zeta_1(x)}{x_1}$, $x_1 \neq 0$, extends via its derivatives to a smooth function $f : S^{l-1} \to S^1$ such that

$$\zeta_1(x) = x_1 f(x).$$

Moreover, the function f is even,

$$f(\pm x_1, \ldots, \pm x_l) = f(x_1, \ldots, x_l).$$

Write $f = e^{ig}$ for a smooth even function $g : S^{l-1} \to \mathbb{R}$ satisfying

$$g(\pm x_1, \ldots, \pm x_l) = g(x_1, \ldots, x_l).$$

(For this we need that f is homotopically trivial. This is clear for $l \neq 2$; for $l = 2$ it follows from the evenness of f).

By a theorem attributed to Whitney in [9] (see the problem below), the even function g can be written as

$$g(x) = h(x_1^2, \ldots, h_l^2)$$

for a smooth function $h : \{a \mid a_j \geq 0, \sum a_j = 1\} \to \mathbb{R}$. Unwrapping the definitions, we see that

$$\zeta_1(z) = z_1 e^{ih(|z_1|^2, \ldots, |z_l|^2)}, \quad z \in S^{2l-1}.$$

Now wxtend ζ_1 to B^{2l} by the same formula, and do the same for $\zeta_2, \ldots, \zeta_l$. □

Problem 14.1. Prove the theorem of Whitney used in the preceding proof. Hint: First prove that every even function $g : \mathbb{R} \to \mathbb{R}$ can be written as $g(x) = h(x^2$ for a smooth function h.

14.4 Constructing an isomorphism

Now we wish to prove:

Proposition 14.7. *Any two toric manifolds with the same moment polytope are isomorphic as Hamiltonian T-spaces.*

The proof follows the lines of [25] (see also [24]). We first show how the cohmology class $[\omega]$ is determined by the moment polytope. Define the "length" of a vector $v \in \mathbb{R}^n$ which points in a rational direction as follows: Write $v = ru$ where $r \geq 0$ is a real number and $u \in \mathbb{Z}^n$ is minimal (i.e. not a multiple of an integer vector) and set $\|v\|_{\mathbb{Z}} := r$.

Let $S_{p,q}$ be the preimage under μ of an edge of P between two vertices p and q. We will see in the proof below that $S_{p,q}$ is an embedded 2-sphere in M.

Lemma 14.8. *The value of the cohomology class $[\omega]$ on the 2-sphere $S_{p,q}$ equals*

$$\int_{S_{p,q}} \omega = 2\pi \|\mu(p) - \mu(q)\|_{\mathbb{Z}}.$$

Proof. Write the edge as $q + tw_i$, $t \geq 0$, and let $u_i \in \mathbb{Z}^n$ be dual to w_i such that $\langle w_i, u_i \rangle = 1$. The function $H := \langle \mu, u_i \rangle : S_{p,q} \to \mathbb{R}$ satisfies $dH = \langle d\mu \cdot, u_i \rangle = i_{\underline{u_i}}\omega$. Since $\underline{u_i}$ generates an S^1-action on $S_{p,q}$ with 2 fixed points, we have the following situation on $S_{p,q} \cong S^2$:

$H : S^2 \to \mathbb{R}$ is a Morse function with a unique maximum and minimum whose Hamiltonian vector field X_H (with respect to some symplectic form ω on S^2) generates an S^1-action.

Let (θ, H) be coordinates on S^2 such that $\frac{\partial}{\partial\theta} = X_H$. Then $i_{\frac{\partial}{\partial\theta}}\omega = dH$ shows $\omega = d\theta \wedge dH$, and we get

$$\int_{S^2} \omega = \int_0^{2\pi} \int_{\min H}^{\max H} d\theta \wedge dH = 2\pi(\max H - \min H).$$

Translating back this means

$$\int_{S_{p,q}} \omega = 2\pi\langle \mu(p) - \mu(q), u_i \rangle = 2\pi\|\mu(p) - \mu(q)\|_{\mathbb{Z}}.$$

□

Remark. This lemma and its proof is a special case of the Duistermaat-Heckmann Theorem.

Pick an irrational vector $X \in \mathfrak{t} = \mathbb{R}^n$ and consider the perfect Morse (here it is really Morse, not Morse-Bott!) function $\mu_X : M \to \mathbb{R}$. Recall that the second homology of M has as basis the unstable manifolds (with respect to some invariant metric) of the critical points of μ_X of index 2. These unstable manifolds are the preimages $S_{p,q}$ of the descending edge from a vertex p of P with one descending edge to another vertex q. So the preceding Lemma determines the cohomology class $[\omega]$ from the moment polytope. By the way, it also shows:

Corollary 14.9. *The cohomology class $[\omega/2\pi]$ is integral if and only if all difference vectors between vertices of P are integral.*

The last ingredient in the proof of Proposition 14.7 is a linear algebra lemma.

Lemma 14.10. *The only linear symplectic form on $\mathbb{C}^n$ invariant under the standard T^n-action and with moment $\mu(z) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2)$ is the standard form $\sum dx_j \wedge dy_j$.*

Proof. Write the real form ω as

$$\omega = i\sum_j A_j dz_l \wedge d\bar{z}_j + \frac{i}{2}\sum_{j\neq k}(b_{jk}dz_j \wedge dz_k - \bar{B}_{jk}d\bar{z}_j \wedge d\bar{z}_k) + i\sum_{j\neq k} C_{jk}dz_j \wedge d\bar{z}_k$$

with complex coefficients satisfying

$$A_j = \bar{A}_j, \quad B_{jk} = -B_{kj}, \quad C_{jk} = \bar{C}_{kj}.$$

The vector field generating the action of the j-th circle is

$$X_j = -\frac{\partial}{\partial \theta_j} = -i\Big(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\Big).$$

By the definition of the moment map, the differential of its j-th component $\mu_j = \frac{1}{2}|z_j|^2$,

$$d\mu_j = \frac{1}{2}(\bar{z}_j dz_j + z_j d\bar{z}_j),$$

equals the interior product

$$\begin{aligned} i_{X_j}\omega = &\sum_j A_j(z_j d\bar{z}_j + \bar{z}_j dz_j) + \sum_{k\neq j}(B_{jk} z_j dz_k + \bar{B}_{jk}\bar{z}_j d\bar{z}_k) \\ &+ \sum_{k\neq j}(C_{jk} z_j d\bar{z}_k + \bar{C}_{jk}\bar{z}_j dz_k). \end{aligned}$$

Comparing coefficients yields

$$\frac{\partial \mu_j}{\partial z_j} = A_j z_j, \quad \frac{\partial \mu_j}{\partial z_k} = B_{jk} z_j + \bar{C}_{jk}\bar{z}_j, \quad k \neq j.$$

Hence the second derivatives of the μ_j determine all the coefficients of ω,

$$\frac{\partial^2}{\partial z_j \partial \bar{z}_j} = A_j, \quad \frac{\partial^2}{\partial z_k \partial z_j} = B_{jk}, \quad \frac{\partial^2}{\partial z_k \partial \bar{z}_j} = \bar{C}_{jk}.$$

□

Now we are ready to prove the following result, which immediately implies Proposition 14.7 by Moser's Theorem:

Proposition 14.11. *Let (M, ω, μ) be a toric manifold and ω' a symplectic form on M which is invariant with the same moment map. Then*

$$\omega_t := (1-t)\omega + t\omega', \quad t \in [0,1],$$

are cohomologous invariant symplectic forms with the same moment map.

Proof. Let $x \in \mu^{-1}(F)$ where F is a k-dimensional face of P. By Lemma 14.5, a neighbourhood of x is isomorphic to $T^k \times U \times B^{2n-2k}(\varepsilon)$ for $U \subset F$. The proof of Lemma 14.5 also showed that $\omega' - \omega_{T^k \times U} = \mu^* d\beta$ for $\beta \in \Omega^1(U)$, so the restriction of ω_t to $T^k \times U$ is symplectic for all t. By Lemma 14.10 the restrictions of ω_x and ω'_x to $T_x B^{2n-2k}(\varepsilon)$ agree. So $(\omega_t)_x$ is symplectic for all t, and this holds for all $x \in M$.

The forms ω_t are clearly invariant and have the same moment map, and they are cohomologous by Lemma 14.8. □

14.5 Examples

Example 14.12 ($\mathbb{C}P^n$). Let $P \subset (\mathbb{R}^n)^*$ be the standard simplex with vertices $0, e_1, \ldots, e_n$. The normal vectors to the $(n-1)$-dimensional faces are $u_j = e_j$, $j = 1, \ldots, n$, and $u_{n+1} = (-1, \ldots, -1)^T$. P is the intersection of the half spaces $\langle \xi, u_j \rangle \geq 0$, $j = 1, \ldots, n$, and $\langle \xi, u_{n+1} \rangle \geq -1$. The kernel of $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is generated by $(1, \ldots, 1)^T$, so $H \cong S^1$ acting on $\mathbb{C}^{n+1}$ by the standard action

$$e^{i\theta} \cdot z = (e^{-i\theta} z_1, \ldots, e^{-i\theta} z_{n+1}),$$
$$\mu(z) = \frac{1}{2} \sum |z_j|^2.$$

The toric manifold is the reduction at $-i^*\lambda = 1$,

$$M = \mathbb{C}^{n+1} // S^1(1) \cong \mathbb{C}P^n,$$

with the symplectic form $\omega = 2\omega_{FS}$ (see Problem 12.1). Note that the symplectic form can also be seen from Lemma 14.8: The preimage of the edge $[0,1] \times 0 \times 0$ is a line $\mathbb{C}P^1$ on which $[\omega]$ takes the value

$$\int_{\mathbb{C}P^1} \omega = 2\pi = 2 \int_{\mathbb{C}P^1} \omega_{FS}.$$

Example 14.13 (products). If $P_i \subset (\mathbb{R}^{n_i})^*$ are moment polytopes of toric manifolds (M_i, ω_i, μ_i), $i = 1, 2$, then $P_1 \times P_2$ is the moment polytope of the toric manifold

$$M_1 \times M_2, \omega_1 \oplus \omega_2, \mu_1 \times \mu_2).$$

For example, the standard cube $[0,1]^n$ is the moment polytope of

$$\mathbb{C}P^1 \times \ldots, \times \mathbb{C}P^1, 2\omega_{FS} \oplus \cdots \oplus 2\omega_{FS}).$$

Problem 14.2. Show that $\mathbb{C}P^1$ as constructed in Example 14.12 is isomorphic, as a Hamiltonian S^1-space, to S^2 with half the area form and rotation around the z-axis as circle action. Find an explicit isomorphism.

Example 14.14 ($\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$). Let P be the quadrangle in $(\mathbb{R}^2)^*$ with vertices $(0,0)$, $(1,0)$, $(1-a, a)$ and $(0, a)$ with $0 < a < 1$. It is obtained from the standard simplex by cutting off the vertex $(0,1)$ at height a. The data for the Delzant construction are

$$\pi = (u_1, \ldots, u_4) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix} : T^4 \to T^2,$$
$$\lambda = (0, 0, -1, -a),$$
$$i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} : H \cong T^2 \to T^4.$$

$H \cong T^2$ acts on $\mathbb{C}^4$ by

$$(\alpha, \beta) \cdot (z_1, \ldots, z_4) = (\alpha z_1, \alpha\beta z_2, \alpha z_3, \beta z_4),$$
$$\mu(z) = (|z_1|^2 + |z_2|^2 + |z_3|^2, |z_2|^2 + |z_4|^2),$$

and the toric manifold M is the reduction at $-i^*\lambda = (1, a)$,

$$M = \mathbb{C}^4//H\Big((1, a)\Big).$$

Note that on the level set $\mu^{-1}(1, a)$, $|z_2|^2 \leq a$ and $|z_1|^2 + |z_3|^2 \geq 1 - a > 0$. So M can be written as

$$M = \{(z_1, \ldots, z_4) \mid (z_1, z_3) \neq 0, (z_2, z_4) \neq 0\}/(\mathbb{C}^*)^2,$$

where $(\alpha, \beta) \in (\mathbb{C}^*)^2$ acts as above. This is the projectivization $M = \mathbb{P}[L_1 \oplus L_2]$ of the sum of the two line bundles

$$L_1 = \{(z_1, z_2, z_3) \mid (z_1, z_3) \neq 0\}/\mathbb{C}^* \to \{[z_1 : z_3]\} = \mathbb{C}P^1,$$
$$L_2 = \{(z_1, z_3, z_4) \mid (z_1, z_3) \neq 0\}/\mathbb{C}^* \to \{[z_1 : z_3]\} = \mathbb{C}P^1.$$

Here the projection is the quotient by the action of $\alpha \in \mathbb{C}^*$. Since α does not act on z_4, $L_2 \cong \mathbb{C}$ is the trivial line bundle. I claim that $L_1 \cong H$ is the canonical (hyperplane) bundle. To see this, consider the tautological bundle

$$L = \{\Big([z_1 : z_3], (w_1, w_3)\Big) \Big| z_1 w_3 = z_3 w_1\}.$$

Set $\lambda := w_1/z_1$ if $z_1 \neq 0$ and w_3/z_3 if $z_3 \neq 0$. Since the map $(z, w) \mapsto (z, \lambda)$ transforms under the $\mathbb{C}^*$-action as $(\alpha z, w) \mapsto (\alpha z, \alpha^{-1}\lambda$, it establishes (with $\lambda = z_2$) an isomorphism from $L^* = H$ to L_1. So

$$M = \mathbb{P}[H \oplus \mathbb{C}] \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$$

is the blow-up of $\mathbb{C}P^2$ at a point.

14.6 Deformation and blow-up

Consider a continuous family (ω_t, μ_t) of toric structures on a manifold M^{2n}. Then the moment images $\text{im}\mu_t$ form a continuous family of Delzant polytopes in $(\mathbb{R}^n)^*$. By Lemma 14.8, these polytopes depend (up to translation) only on the cohomology classes $[\omega_t] \in H^2(M; \mathbb{R})$.

What are continuous deformations of Delzant polytopes? By the integrality conditions, the directions of the edges and faces cannot change. So the only possible deformations are parallel translations of $(n-1)$-dimensional faces as in the Figure below. This means that in the description of the faces $\{\langle \xi, u_i \rangle =$

$\lambda_i\}$ the vector λ can move in $(\mathbb{R}^d)^*$ inducing toric symplectic forms ω_λ on M. Lemma refL:toric-cohomology shows that the map

$$(\mathbb{R}^d)^* \to H^2(M;\mathbb{R}), \quad \lambda \mapsto [\omega_\lambda]$$

is *linear*: The value of $[\omega_\lambda]$ on the preimage of each edge is a linear function of λ. Recall that ω_λ is the reduced symplectic form on $\mathbb{C}^d//H(-i^*\lambda)$. The Duistermaat-Heckmann Theorem states that for any reduction by a torus action, the reduced symplectic form depends linearly on the moment value.

Example 14.15. The symplectic forms on $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ corresponding to the rectangles $[0,\lambda_1] \times \cdots \times [0,\lambda_n]$ have area $2\pi\lambda_i$ on the i-th factor.

What we said about deformations is true as long as the topology of the polytope does not change. But consider Example 14.14 as $a \to 1$. Then one of the faces degenerates to a vertex. In the example, we know how the toric manifolds change: For $a = 1$ it is $\mathbb{C}P^2$, while for $a < 1$ it is the blow-up of $\mathbb{C}P^2$ at a point. This turns out to be the general picture as a face degenerates to a vertex. To discuss this, let us first give a more convenient description of blow-up.

Symplectic cuts and blow-up

The idea of symplectic cuts is due to E. Lerman [27] and contained in the following

Lemma 14.16. *Let (M,ω,H) be a Hamiltonian S^1-space such that S^1 acts freely on $H^{-1}(c)$. Then the space obtained from $\{H \geq c\}$ by collapsing the S^1-orbits in $\{H = c\}$ is naturally a smooth symplectic manifold.*

Proof. Consider the S^1-action on $M \times \mathbb{C}$ with moment map K,

$$e^{i\theta} \cdot (x,z) = (e^{i\theta}x, e^{i\theta}z),$$
$$K(x,z) = H(x) - \frac{1}{2}|z|^2.$$

The hypothesis implies that S^1 acts freely on $K^{-1}(c)$, so $K^{-1}(c)$ is a smooth symplectic manifold. Now note that the part of $K^{-1}(c)$ where $H > c$ is symplectomorphic to $\{H > c\} \subset M$, and the part where $H = c$ is symplectomorphic to $H^{-1}(c)/S^1$. ∎

Applying this to $M = \mathbb{C}^n$ with $H(x) = \frac{1}{2}|x|^2$ yields a nice description of the blow-up:

Corollary 14.17. *The blow-up of $\mathbb{C}^n$ on a ball $B^{2n}(\delta)$ of radius δ is obtained from $\mathbb{C}^n \subset B^{2n}(\delta)$ by collapsing the boundary of the ball to $\mathbb{C}P^{n-1}$.*

Blow-up of toric manifolds

Let (M, ω, μ) be a toric manifold and $x \in M$ a fixed point with $\mu(x) = p$. A neighbourhood of x is isomorphic to a neighbourhood of the origin in $\mathbb{C}^n$ with the standard T^n-action and moment map

$$\mu(z) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2) + p.$$

The moment map maps a small ball of radius δ around x onto a standard simplex at the vertex p,

$$\mu\Big(B^{2n}(\delta)\Big) = p + \Delta_\varepsilon,$$

where $\varepsilon = \frac{1}{2}\delta^2$ and

$$\Delta_\varepsilon = \{\xi \mid \xi_i \geq 0, \sum \xi_i \leq \varepsilon\}.$$

Now blow up the ball $B^{2n}(\delta)$, i.e. cut it out and collapse its boundary under the diagonal S^1-action to the exceptional divisor $E \cong \mathbb{C}P^{n-1}$. Since the T^n-action commutes with this S^1-action, the blow-up inherits a T^n-action which agrees with the old one outside the exceptional divisor. So the blow-up is a toric manifold whose moment polytope is the old one minus the interior of the simplex $p + \Delta_\varepsilon$. The image of the exceptional divisor is precisely the new face,

$$\mu(E) = p + \{\xi \mid \xi_i \geq 0, \sum \xi_i = \varepsilon\}.$$

Note that the isotropy group of this face is the diagonal S^1 in T^n, which fits with the normal vector to the face being $(1, \ldots, 1)$. So we have shown:

Proposition 14.18. *The blow-up of a toric manifold at a fixed point is again a toric manifold. Its moment polytope is the old one minus the simplex* $p + \{\xi \mid \xi_i \geq 0, \sum \xi_i < \varepsilon\}$ *at the vertex* $p = \mu(x)$*. Here* $\varepsilon = \frac{1}{2}\delta^2$ *where* δ *is the radius of the ball blown up. The image of the exceptional divisor is the new face* $p + \{\xi \mid \xi_i \geq 0, \sum \xi_i = \varepsilon\}$.

Example 14.19. Consider the regular octagon in $(\mathbb{R}^2)^*$ with vertices $(\pm 1, \pm\frac{1}{2})$, $(\pm\frac{1}{2}, \pm 1)$. It is obtained from the quadrangle with vertices $(\pm 1, \pm 1)$ by cutting off triangles at all four vertices. The quadrangle corresponds to

$$(\mathbb{C}P^1 \times \mathbb{C}P^1, 4\omega_{FS} \oplus 4\omega_{FS}).$$

The triangles cut off have $\varepsilon = \frac{1}{2}$, corresponding to $\delta = 1$. So the toric manifold M corresponding to the octagon is obtained from $\mathbb{C}P^1 \times \mathbb{C}P^1$ by blowing up balls of radius 1 at the four fixed points.

Since $\mathrm{vol} B^4(1) = \pi^2/2$, the volume of M is

$$\mathrm{vol} M = 16\pi^2 - 4\mathrm{vol} B^4(1) = 14\pi^2.$$

Note that this equals $(2\pi)^2$ times the area of the octagon. The following problem shows that this is no accident.

Problem 14.3. Prove that the $2n$-dimensional symplectic volume of a toric manifold equals $(2\pi)^n$ times the n-dimensional Euclidean volume of its moment polytope,

$$\int_M \frac{\omega^n}{n!} = (2\pi)^n \mathrm{vol}(\mathrm{im}\mu).$$

In the language of Chapter 17, this says that the Duistermaat-Heckmann measure of a toric manifold equals $(2\pi)^n$ times the Euclidean measure.

Hint: Use action-angle coordinates on the set where the action is free.

Problem 14.4. Show that $\mathbb{C}P^1 \times \mathbb{C}P^1$ is obtained from $\mathbb{C}P^2$ by two blow-ups followed by one blow-down.

Problem 14.5. Prove that every toric 4-manifold is obtained from $\mathbb{C}P^2$ by a sequence of blow-ups and blow-downs.

Problem 14.6. Let P be the quadrangle in $(\mathbb{R}^2)^*$ with vertices $(0,0)$, $(k+1,0)$, $(1,1)$ and $(0,1)$ for an integer $k \geq 1$. Arguing as in Example 14.14, show that the corresponding toric manifold equals

$$M = \mathbb{P}[H^k \oplus \mathbb{C}] \cong \begin{cases} \mathbb{C}P^1 \times \mathbb{C}P^1 & \text{for } k \text{ even}, \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} & \text{for } k \text{ odd}. \end{cases}$$

Here H is the hyperplane bundle over $\mathbb{C}P^1$, and $\cong$ means diffeomorphic.

Find an explicit sequence of k alternating blow-ups and blow-downs that takes M to $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Chapter 15

Kähler quotients

In this chapter we study in more detail symplectic reduction on Kähler manifolds.

15.1 Kähler structure on the quotient

Proposition 15.1. *Suppose a compact Lie group G acts on a Kähler manifold (M, ω, J) with moment map μ and preserving J, and G acts freely on $\mu^{-1}(0)$. Then the symplectic quotient $M//G(0)$ inherits a natural Kähler structure.*

Corollary 15.2. *Every toric manifold is Kähler.*

Proof. Consider the principal G-bundle

$$M_0 := \mu^{-1}(0) \xrightarrow{\pi} \bar{M} := M//G(0).$$

Recall that M_0 is coisotropic with

$$(T_x P)^{\perp_\omega} = T_x(G \cdot x) \subset T_x M_0.$$

We invoke a simple linear algebra lemma:

For a coisotropic subspace W of a Hermitian vector space (V, ω, J),

$$V = W \oplus JW^{\perp_\omega}, \quad \text{and} \quad W = W^{\perp_\omega} \oplus (W \cap JW).$$

So the J-invariant subbundle $E := TM_0 \cap JTM_0$ of TM_0 yields a decomposition

$$T_x M_0 = T_x G \cdot x \oplus E_x.$$

Since E is also G-invariant, it defines a connection on M_0.

Now define an almost complex structure $\bar{J}$ on $\bar{M}$ by

$$\bar{J}\bar{v} := \pi_* Jv,$$

where v is the horizontal lift of $\bar{v} \in T_x\bar{M}$ to some point in $\pi^{-1}(x)$. This is clearly well-defined and compatible with ω. It only remains to show that if J is integrable then so is $\bar{J}$.

Here we use an idea from [18]. Every almost complex structure J on a manifold defines a complex subbundle

$$F := \{v + iJv \mid v \in TM\} \subset TM \otimes \mathbb{C}$$

of the complexified tangent bundle. Its crucial property is the following: *J is integrable iff $[F, F] \subset F$.* In this case F is called a *polarization.* The proof of the property is immediate:

$$[v + iJv, w + iJw] = \Big([v, w] - [Jv, Jw]\Big) + i\Big([Jv, w] + [v, Jw]\Big) \in F$$

iff $[v, w] - [Jv, Jw] = J\big([Jv, w] + [v, Jw]\big)$, i.e. the Nijenhuis tensor of J vanishes. Now let $v, w \in E$ be horizontal lifts of $\bar{v}, \bar{w} \in T\bar{M}$. Then by integrability of J,

$$[v + iJv, w + iJw] = u + iJu$$

with

$$u = [v, w] - [Jv, Jw] \in TM_0, \quad Ju = [Jv, w] + [v, Jw] \in TM_0,$$

so $u \in E$. Projecting down under π_* yields

$$[\bar{v} + i\bar{J}\bar{v}, \bar{w} + i\bar{J}\bar{w}] = \bar{u} + i\bar{J}\bar{u},$$

showing that $\bar{J}$ is integrable. □

Remark. A connection 1-form on M_0 with horizontal space E can be given as follows: Each $x \in M_0$ induces an invariant inner product $\langle X, Y\rangle_x := \omega(\underline{X}_x, J\underline{Y}_x)$ on $\mathfrak{g}$ and a corresopnding isomorphism $\phi_x : \mathfrak{g}^* \to \mathfrak{g}$. Then

$$\phi_x \circ d\mu \circ J \in \Omega^1(M_0, \mathfrak{g})$$

is a connection 1-form on M_0 with horizontal space E. It may be interesting to study this connection in more depth.

Problem 15.1 ([18]). Let (V^{2n}, ω) be a (real) symplectic vector space and extend ω complex linearly to $V \otimes \mathbb{C}$. Show:

(i) $J \mapsto F := \{v + iJv \mid v \in V\}$ sets up a bijection between complex structures on V and n-dimensional complex subspaces $F \subset V \otimes \mathbb{C}$ with $F \cap \bar{F} = \{0\}$. Note that F is the $(-i)$-eigenspace of J.

(ii) $\omega(\cdot, J\cdot)$ is J-invariant and symmetric iff $\omega(v, w) = 0$ for $v, w \in F$, i.e. F is Lagrangian.

(iii) $\omega(\cdot, J\cdot)$ is positive definite iff the Hermitian form $i\omega(v, \bar{w})$ is positive definite on F.

15.2 Action of the complexified group

In this section G is a compact connected Lie group.

The exposition follows [18]. Let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ denote the complexified Lie algebra. For the proof of the following result see e.g. [6].

Proposition 15.3. *There exists a unique connected complex Lie group $G^{\mathbb{C}}$, the* complexification of *G, with the following two properties:*

(i) Its Lie algebra is $\mathfrak{g}^{\mathbb{C}}$.

(ii) G is a maximal compact subgroup of $G^{\mathbb{C}}$.

Moreover, there exists a subset $P \subset G^{\mathbb{C}}$, homeomorphic to a ball, such that $P \times G \to G^{\mathbb{C}}$, $(p, g) \mapsto pg$ is a homeomorphism, and exp *maps $i\mathfrak{g}$ onto P.*

Example 15.4.

$$(T^n)^{\mathbb{C}} = (\mathbb{C}^*)^n, \quad U(n)^{\mathbb{C}} = GL(n, \mathbb{C}), \quad SU(n)^{\mathbb{C}} = SL(n, \mathbb{C}).$$

Proposition 15.5. *An action of G on a Kähler manifold (M, ω, J) preserving J can be extended to an action of $G^{\mathbb{C}}$ on M preserving J (but not ω!).*

Proof. We first extend the action $X \mapsto \underline{X}$ of the Lie algebra in the obvious way to

$$\tau : \mathfrak{g}^{\mathbb{C}} \to \mathcal{X}(M), \quad X + iY \mapsto \underline{X} + J\underline{Y}.$$

Claim: τ is a Lie algebra homomorphism to vector fields preserving J with minus the commutator.

To see this, let F be the polarization of J as above. Recall that a vector field X preserves J iff it preserves F, i.e. $[X, Y + iJY] \in F$ for every section $Y + iJY$ in F. This means that

$$[X, JY] = J[X, Y]$$

for all vector fields Y. Using integrability of J, this also implies

$$[JX, JY] = J[JX, Y]$$

for all vector fields Y. But this just says that JX preserves J, so the image of τ consists of vector fields preserving J.

If both X and Y preserve J, then the two equations combine to

$$[JX, JY] = -[X, Y],$$

and it follows that τ is a homomorphism.

Now the group $Diff(M, J)$ of biholomorphic maps of a Kähler manifold is a finite dimensional Lie group (see []), and τ is a homomorphism into its Lie algebra. If $G^{\mathbb{C}}$ is simply connected, τ can be uniquely extended to a homomorphism

$G^{\mathbb{C}} \to Diff(M,J)$. If not let $\tilde{G}$ be the universal covering group of G and $K \subset \tilde{G}$ a discrete subgroup such that

$$G = \tilde{G}/K, \quad G^{\mathbb{C}} = \tilde{G}^{\mathbb{C}}/K.$$

Then τ extends to a homomorphism $\tilde{G}^{\mathbb{C}} \to Diff(M,J)$. But since the action of $\tilde{G}$ factors through G, K is in the kernel of this homomorphism, which therefore factors through $G^{\mathbb{C}}$. □

Now suppose that the action of G on the Kähler manifold (M,ω,J) is Hamiltonian and preserves J, and is free on $M_0 := \mu^{-1}(0)$. Define the set of *stable points* for the $G^{\mathbb{C}}$-action,

$$M_s := \{gx \mid x \in M_0, g \in G^{\mathbb{C}}\}.$$

Recall that a group action $G \times X \to X$ is called *proper* if the following holds: If x_k and $g_k x_k$ are converging sequences, then g_k has a converging subsequence. Also recall that the quotient of a manifold under a free proper action is a manifold (see [1]).

Proposition 15.6. *M_s is open and $G^{\mathbb{C}}$ acts freely and properly on it. The inclusion $M_0 \subset M_s$ induces a diffeomorphism*

$$M//G(0) \cong M_s/G^{\mathbb{C}}.$$

Proof. The tangent space at $x \in M_0$ decomposes as

$$T_xM = T_xM_0 \oplus JT_xG \cdot x,$$

where the second factor consists of the vector fields associated to $i\mathfrak{g}$. Since the infinitesimal action of $\mathfrak{g}$ on M_0 is free, this shows that M_s contains an open neighbourhood U of M_0, so $M_s = \bigcup_{g \in G^c} U$ is open.

The proof that the action is free rests on the following crucial observation which follows directly from the definitions: *The vector field associated to $iX \in i\mathfrak{g}$ is the gradient of $\mu_X = \langle \mu, X\rangle$ with respect to the metric $\omega(\cdot, J\cdot)$.*

So let $g \in G^{\mathbb{C}}$ be in the stabilizer group of $x \in M_0$. Write g as

$$g = \exp(iX)k$$

with $X \in \mathfrak{g}$ and $k \in G$, and consider the curve

$$t \mapsto \exp(tiX)kx, \quad t \in [0,1]$$

from kx to x. By the observation above, this is an integral curve of the gradient of μ_X, so if $X \neq 0$, the value of μ_X is strictly increasing along this curve. But this contradicts $\mu_X(x) = \mu_X(kx) = 0$, showing that $X = 0$. Then $g = k \in G$, and $g = \mathbb{1}$ because G acts freely on M_0.

That the $G^{\mathbb{C}}$-action is proper follows from the Lemma below. So the quotient $M_s/G^{\mathbb{C}}$ inherits a smooth structure. The map $M_0/G \to M_s/G^{\mathbb{C}}$ induced by the inclusion is clearly a homeomorphism. The proof that it is a diffeomorphism is left to the reader. □

Lemma 15.7. *For every compact subset $K \subset M_s$ there exists a constant T such that* $\exp(iX)x \in K$, $x \in M_0$, $X \in \mathfrak{g}$ *implies* $|X| \leq T$.

Proof. Let K_0 be a compact neighbourhood of M_0 in M_s. Fix $x_0 \in M_0$ and $X_0 \in \mathfrak{g}$ with $|X_0| = 1$. The curve

$$\gamma_0(t) := \exp(tiX_0)x_0$$

is the gradient trajectory of the function μ_{X_0} starting at x_0. As $t \to \infty$, it tends to the critical set of μ_{X_0}, which lies outside M_s. So there exists a T_0 such that $\gamma_0(t) \notin K$ for $t \geq T_0$. By compactness of M_0 and the unit sphere in $\mathfrak{g}$, we are done if we can find such a constant T_0 uniformly for a neighbourhood of x_0 and X_0.

Arguing by contradiction, suppose there is no uniform constant. I.e., there are sequences $x_k \to x_0$, $X_k \to X_0$ and $t_k \to \infty$ such that $\gamma_k(t_k) \in K_0$, where $\gamma_k(t) := \exp(tiX_k)x_k$. Choose a compact subset $K_1 \subset M_s$ such that $K_0 \subset \mathrm{int} K_1$, and the gradient line γ_0 does not get back to K_0 once it has reached ∂K_1 for positive time.

I claim that for k large, the curve γ_k must be outside K_1 for some time prior to t_k. For otherwise $\gamma_k([0, t_k]) \subset K_1$, and since $\|\nabla \mu_{X_k}\| \geq \varepsilon > 0$ on K_1 for some uniform ε,

$$\mu_{X_k}(t_k) - \mu_{X_k}(0) = \int_0^{t_k} \langle \nabla \mu_{X_k}, \dot{\gamma}_k \rangle dt \geq \varepsilon^2 (t_k - 0),$$

which is a contradiction since μ_{X_k} is uniformly bounded.

So let $0 < s_k < t_k$ be times with $\gamma_k(s_k) \in \partial K_1$ and $\gamma_k([s_k, t_k]) \subset K_1$. By the same argument as above, since $\|\nabla \mu_{X_k}\|$ is uniformly bounded above and below on K_1, there exists a uniform constant C such that

$$\frac{1}{C} \leq t_k - s_k \leq C.$$

Thus the curves $\gamma_k(t + s_k)$, $t \in [0, t_k - s_k]$, converge to a curve $\gamma : [0, a] \to K_1$ with the following properties: It is a gradient line of μ_{X_0} with $\gamma(0) \in \partial K_1$ and $\gamma(a) \in K_0$. So $\gamma(t) = \gamma_0(t + b)$ for some $b \in \mathbb{R}$. Since γ arose as a limit of the gradient lines γ_k starting at $x_k \to x_0$, it is connected to x_0 in backward time by a sequence of broken gradient lines. So it must satisfy $\mu_{X_0}\big(\gamma(0)\big) \geq \mu_{X_0}(x_0)$, hence $b \geq 0$. But this contradicts the choice of K_1. $\square$

Chapter 16

Cohomology of toric manifolds

16.1 Morse theory on polytopes

Suppose $P \subset (\mathbb{R}^n)^*$ is the moment polytope of a toric manifold (M, ω, μ). Fixing an irrational direction $X \in \mathbb{R}^n$ we get a Morse function $\mu_X = \langle \mu, X \rangle$ on M with only critical points of even index. This implies immediately that the integral (co-)homology of M has no torsion, and is the free $\mathbb{Z}$-module generated by the critical points of μ_X. Moreover, the *Betti numbers*

$$b_i(M) := \dim_{\mathbb{Q}} H_k(M; \mathbb{Q})$$

satisfy: The odd Betti numbers are zero, and the even Betti number $b_{2k}(M)$ equals the number of critical points of index k.

In order to derive an explicit formula for the Betti numbers, we will do some "Morse theory" on the moment polytope. The discussion essentially follows [11].

Definition. A convex polytope $P \subset (\mathbb{R}^n)^*$ is called *simple* if precisely n edges emanate from every vertex. It is called *rational* if all edges are generated by integer vectors.

So a Delzant polytope is a rational simple convex polytope for which, in addition, the edges at every vertex form an integer basis. More generally, Delzant's construction associates a *toric orbifold* to every rational simple convex polytope. This orbifold has finite order singularities at the preimage of vertices where the edges fail to form a $\mathbb{Z}$-basis. All results about cohomology proved in this chapter continue to hold for toric orbifolds in rational cohomology.

Let P be a simple polytope, and fix a vector $X \in \mathbb{R}^n$ which is not perpendicular to any face (of positive dimension). We view the linear function

$$f_X : P \to \mathbb{R}, \quad \xi \mapsto \langle \xi, X \rangle$$

as a “Morse function” on P. Its “critical points” are the vertices, and the *index* of a vertex p is

$$\mathrm{ind}(p) := \#\{\text{edges } e \text{ at } p \text{ with } \langle e, X\rangle < 0\}.$$

Here all edges are oriented away from p. The *unstable manifold* at p is

$$W^-(p) := \{\text{faces } F \text{ at } p \text{ along which } f_X \text{ decreases}\}.$$

The index of p is the maximal dimension of a face in $W^-(p)$. Faces in $W^-(p)$ are called *descending faces at p.*

For any face F there is a unique vertex p at which f_X attains its maximum, so we have the following crucial fact:

Every face F of P is contained in a unique unstable manifold $W^-(p)$, $p \in F$.

Clearly all these notions reflect the corresponding notions for the Morse function μ_X if P is a moment polytope.

Now we express the the “Betti numbers”

$$h_k := \#\{\text{vertices of index } k\}$$

in terms of the numbers f_k of k-dimensional faces of P.

Proposition 16.1. *For a simple convex polytope,*

$$\begin{aligned} f_k &= \sum_{i=k}^{n} \binom{i}{k} h_i, \\ h_k &= \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i. \end{aligned}$$

Proof. Take a vertex p of index i. Since P is simple, the number of k-dimensional faces contained in the unstable manifold $W^-(p)$ equals $\binom{i}{k}$. Summing over all vertices p, since every face is contained in a unique unstable manifold, the first equation follows.

The second formula follows from the first one by descending induction over k. So suppose it holds for $k' > k$. Then

$$\begin{aligned} h_k &= f_k - \sum_{i=k+1}^{n} \binom{i}{k} h_i \\ &= f_k - \sum_{i=k+1}^{n} \binom{i}{k} \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} f_j \\ &= f_k - \sum_{j=k+1}^{n} \Big(\sum_{i=k+1}^{j} (-1)^i \binom{i}{k} \binom{j}{i} \Big) (-1)^j f_j. \end{aligned}$$

Since $\binom{i}{k}\binom{j}{i} = \binom{j}{k}\binom{j-k}{i-k}$, the term in brackets becomes

$$\binom{j}{k} \sum_{i=k+1}^{j} (-1)^i \binom{j-k}{i-k} = \binom{j}{k} \sum_{i=1}^{j-k} (-1)^{i+k} \binom{j-k}{i} = -(-1)^k \binom{j}{k},$$

and the second equation follows. □

Corollary 16.2. *The integral (co-)homology of a toric manifold has no torsion. Its odd Betti numbers are zero, and the even Betti numbers are given in terms of the numbers f_k of k-dimensional faces of the moment polytope by*

$$b_{2k}(M) = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i.$$

The statement on the Betti numbers also holds for a toric orbifold.

Conversely, the formulae give us information about the face numbers f_i of a simple polytope. For example, Poincaré duality implies (after perturbing the polytope to make it rational)

Corollary 16.3. *For any simple polytope, the numbers h_k defined by Proposition 16.1 satisfy the* Dehn-Sommerville equations

$$h_k = h_{n-k}.$$

Example 16.4. For $n = 3$ the h_k are given by

$$h_3 = f_3, \quad h_2 = f_2 - 3f_3, \quad h_1 = f_1 - 2f_2 + 3f_3, \quad h_0 = f_0 - f_1 + f_2 - f_3.$$

Since $f_3 = 1$, the Dehn-Sommerville equations read

$$f_0 - f_1 + f_2 = 2, \quad 3f_2 = f_1 + 6.$$

The first equation is Euler's formula. The second equation expresses the simplicity condition, which becomes clear if we combine it with Euler's equation to obtain the equivalent equations

$$f_2 = \frac{f_0}{2} + 2, \quad f_1 = \frac{3f_0}{2}.$$

Remark. Already fo $n = 3$, the numbers of faces f_k do not determine the type of a simple polytope. Cutting off from a cube two adjacent vertices, or two opposite vertices, yields two combinatorially different simple polytopes with the same face numbers.

16.2 Line bundles on toric manifolds

This section is compiled from [13] and [4]. Recall the construction of the toric manifold M corresponding to the Delzant polytope P. The polytope is described by equation $\langle \xi, u_j \rangle \geq \lambda_j$, $j = 1, \ldots, d$. The integral vectors u_j define a map π that fits into the exact sequence of tori

$$0 \to H \xrightarrow{i} T^d \xrightarrow{\pi} T^n \to 0.$$

Then $M = Z/H$ where

$$Z := (i^* \mu_0)^{-1}(-i^* \lambda) \subset \mathbb{C}^n$$

and μ_0 is the standard moment map on $\mathbb{C}^n$ (with constant zero). Z is compact and a principal bundle

$$H \to Z \to M.$$

So every character $\chi : H \to S^1$ induces a line bundle

$$L_\chi := Z \times_H \mathbb{C} \to M,$$

where H acts on $\mathbb{C}$ via χ. Note that the set of characters is canonically identified with the integral lattice $\mathfrak{h}^*_{\mathbb{Z}}$ in $\mathfrak{h}^*$. Also recall that complex line bundles are classified by their first Chern class $c_1(L)$.

Lemma 16.5. *The map*

$$\mathfrak{h}^*_{\mathbb{Z}} \to H^2(M; \mathbb{Z}), \quad \chi \mapsto c_1(L_\chi),$$

is an isomorphism.

Proof. By the discussion in Chapter 15, M can be described as a complex quotient $M = \mathbb{C}^d_s / H^{\mathbb{C}}$. Here the set $\mathbb{C}^d_s$ of stable points can be described explicitly: Let $z \in Z$ whose moment image lies on the face described by equations $\langle \xi, u_i \rangle = \lambda_i$, $i \in I$. Then $z_i = 0$ for $i \in I$ and $z_i \neq 0$ for $i \notin I$. The imaginary part of the complexified group $H^{\mathbb{C}}$ acts by rescaling the coordinates $z_i \mapsto r_i z_i$, $r_i \in \mathbb{R}_+$. So

$$\mathbb{C}^d_s = \cup_I \{z \mid z_i = 0 \text{ iff } i \in I\},$$

where the union is taken over all multi-indices $I \subset \{1, \ldots, d\}$ corresponding to faces of P. Now if $z \in \mathbb{C}^d$ has all components nonzero it lies in the set with $I = \emptyset$ corresponding to the interior of P. If z has precisely one component z_i equal to zero it lies in the set with $I = \{i\}$ corresponding to the i-th codimension 1 face. This shows that $\mathbb{C}^d_s$ is obtained from $\mathbb{C}^d$ by taking out some complex subspaces of complex codimension at least 2, so its first two homotopy groups vanish. Since $\mathbb{C}^d_s$ deformation retracts onto Z, we have shown

$$\pi_1(Z) = \pi_2(Z) = 0.$$

The homotopy exact sequence of the fibration $H \to Z \to M$ yields $\pi_1(M) = 0$, and $\pi_2(M) \to \pi_1(H)$ is an isomorphism. The dual of this map on cohomology, $H^1(H;\mathbb{Z}) \to H^2(M;\mathbb{Z})$, is just the map $\chi \mapsto c_1(L_\chi)$ under the natural identification $\mathfrak{h}_\mathbb{Z}^* \cong H^1(H;\mathbb{Z})$ (verify this!). Alternatively, one could argue in the last step that the bundle $Z \to M$ is universal for the group H up to dimension 2, see [4]. □

Note that a function $f : Z \to \mathbb{C}$ defines a section in the line bundle L_χ iff

$$f(hz) = \chi(h)^{-1} f(z), \quad h \in H.$$

In particular, consider the character $i^* e_j^* \in \mathfrak{h}_\mathbb{Z}^*$, where $e_j^* \in (\mathbb{Z}^d)^*$ is the projection onto the j-th component, and the associated line bundle $L_j \to M$. Since $(hz)_j = e_j^* \circ i(h) z_j$, the dual bundle L_j^* has the canonical section z_j induced by

$$Z \to \mathbb{C}, \quad z \mapsto z_j.$$

This section is transverse to the zero section, and its zero set $\{z_j = 0\}$ is precisely the preimage $\mu^{-1}(U_j) \subset M$ of the codimension 1 face

$$U_j := \{\langle \xi, u_j\rangle = \lambda_j\} \subset P.$$

So the homology class $[\mu^{-1}(U_j)] \in H_{2n-2}(M;\mathbb{Z})$ is Poincaré dual to $c_1(L_j^*)$, where μ is the moment map of M.

The relations between these classes are also easy to see: $\sum_j a_j[\mu^{-1}(U_j)] = 0$ for an integer vector $a \in (\mathbb{Z}^d)^*$ iff the line bundle $\sum_j a_j L_j^*$ is trivial. As this is the line bundle associated to the character i^*a, it is trivial iff $i^*a = 0$, i.e. $a = \pi^* w$ for some $w \in (\mathbb{Z}^n)^*$. By definition of π this means $a_j = \langle w, u_j\rangle$, so we have shown:

Lemma 16.6. *$H^2(M;\mathbb{Z}) \cong H_{2n-2}(M;\mathbb{Z})$ is the free abelian group on d generators $U_1, \dots, U_d$ with relations (n independent ones)*

$$\sum_j \langle w, u_j\rangle U_j = 0, \quad w \in (\mathbb{Z}^n)^*.$$

16.3 The cohomology ring of a toric manifold

Now we will compute the ring structure of $H^*(M;\mathbb{Z})$ for a toric manifold. This section follows [12]. The otherwise nice exposition in [11] has a gap in the proof that the given relations are complete.

We assign to every codimension k face F of P the (co-)homology class

$$[\mu^{-1}(F)] \in H_{2n-2k}(M;\mathbb{Z}) \cong H^{2k}(M;\mathbb{Z}).$$

Extend the map $U \mapsto [\mu^{-1}U_i]$ of the codimension 1 faces to a ring homomorphism

$$\phi : \mathbb{Z}[U_1, \dots, U_d] \to H^*(M;\mathbb{Z}),$$

from the polynomial ring on d generators to the cohomology ring with the cup product $\cup$. If $U_{i_1}, \dots, U_{i_k}$ are pairwise distinct codimension 1 faces, then

$$\mu^{-1}(U_{i_1} \cap \dots \cap U_{i_k}) = \mu^{-1}(U_{i_1}) \cap \dots \cap \mu^{-1}(U_{i_k}).$$

Since this is a transverse intersection, it immediately implies

$$[\mu^{-1}U_{i_1}] \cup \dots \cup [\mu^{-1}U_{i_k}] = \begin{cases} [\mu^{-1}(U_{i_1} \cap \dots \cap U_{i_k})] & \text{if } U_{i_1} \cap \dots \cap U_{i_k} \text{ is a codimension } k \text{ face,} \\ 0 & \text{otherwise.} \end{cases}$$

As every face arises as such an intersection, and the preimages of faces generate the homology (by Morse theory), it follows that ϕ is surjective.

It remains to determine the kernel of ϕ. By the previous section, the kernel contains the ideal I generated by the linear relations

$$\sum_j \langle w, u_j \rangle U_j, \quad w \in (\mathbb{Z}^n)^*.$$

By the discussion above, the it also contains the ideal J generated by the non-linear relations

$$U_{i_1} \dots U_{i_k}$$

for pairwise distinct $U_{i_1}, \dots, U_{i_k}$ such that $U_{i_1} \cap \dots \cap U_{i_k}$ is not a codimension k face. It comes as a pleasant surprise that these two ideals generate the kernel:

Theorem 16.7. *For a toric manifold,*

$$\phi : \mathbb{Z}[U_1, \dots, U_d]/(I + J) \to H^*(M; \mathbb{Z})$$

is a ring isomorphism. The same is true with rational coefficients for a toric orbifold.

For the proof we need an algebraic lemma. Identify a face of $F = U_{i_1} \cap \dots \cap U_{i_k}$ with the monomial $U_{i_1} \dots U_{i_k}$, and let $[F]$ be its class in $\mathbb{Z}[U_1, \dots, U_d]/(I + J)$.

Lemma 16.8 (Shifting Away Lemma). *Let $E \subseteq F \subsetneq G$ be faces of P. Then there exist faces F_i of G, not containing E, and integers a_i such that*

$$[F] = \sum a_i [F_i] \in \mathbb{Z}[U_1, \dots, U_d]/(I + J).$$

Proof. Renumber the U_i such that for $0 \le k < l \le m \le n$,

$$G = U_1 \dots U_k, \quad F = U_1 \dots U_l, \quad E = U_1 \dots U_m.$$

By the Delzant condition, there exists a $w \in (\mathbb{Z}^n)^*$ satisfying $\langle w, u_l \rangle = 1$ and $\langle w, u_i \rangle = 0$ for $i \neq l$, $1 \le i \le m$. It yields the linear relation

$$[U_l] = \sum_{i=m+1}^{d} \langle -w, u_i \rangle U_i.$$

Multiply this relation by $U_1 \dots U_{l-1}$,

$$[F] = [U_1 \dots U_l] = \sum_{i=m+1}^{d} \langle -w, u_i \rangle [U_1 \dots U_{l-1} U_i].$$

Since $k < l$, $F_i := U_1 \dots U_{l-1} U_i$ is a face of G, and it does not contain E. □

Lemma 16.9. *$\mathbb{Z}[U_1, \dots, U_d]/(I + J)$ is generated as a $\mathbb{Z}$-module by monomials $U_{i_1} \dots U_{i_k}$ without multiple factors.*

Proof. Suppose by induction that the statement holds for degree up to k. Let $E = U_1 \dots U_k$ be a square-free monomial (after renumbering). We must show that $U_j E$ can be expressed through square-free monomials for $1 \leq j \leq k$. By the Shifting Away Lemma (with $F = U_j$ and $G = P$), $[U_j] = \sum_{i>k} a_i [U_i]$, and the induction step follows. □

Now we use Morse theory on P. Let $f_X : P \to \mathbb{R}$ be the linear Morse function associated to an irrational vector X. Order the vertices $p_1, \dots, p_m$ by increasing value of f_X. Let $F_k \in W^-(p_k)$ be the descending face of the maximal dimension, $\dim F_k = \operatorname{ind}(p_k)$.

Lemma 16.10. *$[F_1], \dots, [F_m]$ generate $\mathbb{Z}[U_1, \dots, U_d]/(I + J)$ as a $\mathbb{Z}$-module.*

Proof. We prove by induction over k: *Every descending face $F \subset W^-(p_k)$ is a linear combination of $[F_1], \dots, [F_k]$.* Since every face is a descending face at some vertex, this proves the lemma.

So assume the statement holds up to $k - 1$, and let F be a descending face at p_k other than F_k. By the Shifting Away Lemma (with $E = p_k$ and $G = F_k$), $F = \sum_i a_i \tilde{F}_i$, where each $\tilde{F}_i$ is a face of F_k not containing p_k. Thus $\tilde{F}_i$ is a descending face at some vertex p of F_k. Since the value of f_X on F_k is maximal at p_k, $f_X(p) < f_X(p_k)$. So p is one of the p_i, $i < k$, and the statement follows by induction hypothesis. □

Proof of Theorem 16.7. It only remains to show that $\phi : \mathbb{Z}[U_1, \dots, U_d]/(I + J) \to H^*(M; \mathbb{Z})$ is injective. Let $F_1, \dots, F_m$ be as in the preceding lemma. By Morse theory on M, the $[\mu^{-1}(F_i)]$ form a basis of $H^*(M; \mathbb{Z})$ as a $\mathbb{Z}$-module. So ϕ has a surjective inverse, sending $[\mu^{-1}(F_i)]$ to $[F_i]$, and is therefore injective. □

Remark. The Shifting Away Lemma also gives us a recipee for computing the intersection $[\mu^{-1}(F) \cap \mu^{-1}(F')]$ for faces F, F' of P: If F and F' do not lie on the same codimension 1 face, the intersection is the cycle $[\mu^{-1}(F \cap F')]$ (possibly zero if it is of higher codimension). Otherwise express the monomial $F F'$ as a linear combination of square-free monomials, whose cycles can be computed as before.

Problem 16.1 ([13], Exercise 3.17). Let F_-, F, F_+ be consecutive edges of a 2-dimensiona Delzant polytope P. Let $w_-, w, w_+ \in \mathbb{Z}^2$ span the edges (oriented counterclockwise) and $u_-, u, u_+ \in \mathbb{Z}^2$ be inward pointing normal vectors.

(i) Show that the self-intersection number of $E := \mu^{-1}(F)$ equals

$$E \cap E = -\langle w_+, u_- \rangle = \langle w_-, u_+ \rangle.$$

In particular, $E \cap E$ is zero if the angle between F_- and F_+ is right, positive if is is acute and negative it is obtuse.

(ii) Show that every integer arises as such a self-intersection number.

(iii) Show that the sphere $E \subset M$ can be blown down to yield a toric manifold if and only if $E \cap E = -1$.

16.4 The number of faces of a simplicial convex polytope

The title of this section is the title of a 3-page paper by R. Stanley [36] in which he proves a conjecture by McMullen which I now explain.

A convex polytope $P \subset \mathbb{R}^n$ is called *simplicial* if all its proper faces are simplices. The vector $(f_0, \ldots, f_{n-1})$ of the numbers of faces if called the *f-vector*. Set $f_{-1} = f_n = 1$. Define the *h-vector* $(h_0, \ldots, h_n)$ by

$$h_k := \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i.$$

The problem is to characterize those f-vectors, or equivalently h-vectors, which can arise for a simplicial polytope.

These notions are related to simple polytopes by duality. If $P \subset \mathbb{R}^n$ is a convex polytope which contains the origin in its interior, then its *dual (polar) polytope* $P^* \subset (\mathbb{R}^n)^*$ is the intersection of the hyperplanes

$$\{\xi \mid \langle \xi, p \rangle \leq 0\}$$

over all vertices p of P. P^* has the following properties (see [11]):

(i) P^* is a convex polytope which contains the origin in its interior.

(ii) Each k-dimensional face of P corresponds to a $(n-k-1)$-dimensional face of F^*. In particular, the numbers of faces are related by $f_k^* = f_{n-k-1}$.

(iii) P is simplicial iff P^* is simple, and vice versa.

(iv) $(P^*)^* = P$.

Example 16.11. The octahedron is dual to the cube, the icosahedron to the dodecahedron, and the tetrahedron is self-dual.

The h-vector of the simplicial polytope P is just the h-vector of its dual as defined in Section 16.1. We have seen that it must satisfy the Dehn-Sommerville equations $h_k = h_{n-k}$.

Problem 16.2 ($n = 3$). Show that for $n = 3$ the Dehn-Sommerville equations are equivalent to

$$f_1 = 3(f_0 - 2), \quad f_2 = 2(f_0 - 2).$$

Show that every triple (f_0, f_1, f_2) satisfying these two equations and the condition $f_0 \geq 4$ is realized by some simplicial convex polytope.

In higher dimensions the Dehn-Sommerville equations and the condition $f_0 \geq n + 1$ do not suffice. For example, the quadratic inequality

$$f_1 \leq \frac{f_0(f_0 - 1)}{2},$$

which expresses the fact that two vertices can be joined by at most one edge, becomes independent of the other equations from dimension 4 on.

Let $m = [n/2]$ and

$$g_k := h_k - h_{k-1}, \quad k = 1, \ldots, m.$$

Theorem 16.12 (McMullen Conjecture [5],[36]). *$(f_0, \ldots, f_{n-1}$ is the f-vector of a simplicial convex polytope if and only if its h-vector satisfies the* McMullen conditions*:*

(i) Dehn-Sommerville equations $h_k = h_{n-k}$, $0 \leq k \leq n$.

(ii) $g_k \geq 0$, $1 \leq k \leq m$.

(iii) Write g_k uniquely as

$$g_k = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_r}{r}$$

with $n_k > n_{k-1} > \cdots > n_r \geq r \geq 1$. Then

$$g_{k+1} \leq \binom{n_k + 1}{k + 1} + \binom{n_{k-1} + 1}{k} + \cdots + \binom{n_r + 1}{r + 1}, \quad 1 \leq p \leq m - 1.$$

A vector $(g_1, \ldots, g_m)$ satisfying the last two conditions is called a *Macaulay vector.*

Proof. Let P^* be the dual polytope to the simplicial polytope P. Perturb P^* to make it rational, without changing its combinatorics. Let (M, ω) be the toric orbifold associated to P^*. Recall that $H^*(M; \mathbb{Q})$ vanished in odd dimensions, and $\dim H^{2k}(M; \mathbb{Q}) = h_k$ is the h-vector of P. We have noted before that the Dehn-Sommerville equations are just Poincaré duality. For the other conditions, e use the following facts about the rational cohomology ring of $H^*(M; \mathbb{Q})$:

(i) $H^*(M;\mathbb{Q})$ is generated by $H^2(M;\mathbb{Q})$.

(ii) (Hard Lefschetz Theorem). Multiplication by ω^{n-2k} yields isomorphisms $H^{2k}(M;\mathbb{Q}) \to H^{2n-2k}(M;\mathbb{Q})$ for $0 \le k \le m = [n/2]$.

For toric manifolds, the first fact has been proved in Section 16.3, and the Hard Lefschetz Theorem is proved, e.g., in [15]. For the extension to orbifolds see the references in [36].

By the Hard Lefschetz Theorem, multiplication by ω yields an injection $H^{2k-2} \to H^{2k}$ for $1 \le k \le m$, which implies the inequalities $g_k = h_k - h_{k-1} \ge 0$. For the last inequality, consider the commutative graded $\mathbb{Q}$-algebra

$$R^* := \bigoplus_{k=0}^{m} R^k, \quad \text{where}$$
$$R^0 := \mathbb{Q}, \quad R^k := H^{2k}(M;\mathbb{Q})/\omega \cup H^{2k-2}(M;\mathbb{Q}), 1 \le k \le m.$$

In other words, R^* is the quotient of $H^*(M;\mathbb{Q})$ by the ideal generated by $[\omega]$ and all elements of dimension $> m$. Since multiplication by ω is injective, the dimensions

$$\dim R^k = h_k - h_{k-1} = g_k$$

form just the g-vector of P. By a theorem of Macaulay [28], a vector $(g_1, \ldots, g_m)$ is the vector of dimensions of some commutative graded $\mathbb{Q}$-algebra generated by its degree 1 part if and only if it is a Macaulay vector. This yields the last inequality, so we have proved the necessity of the McMullen conditions.

Sufficiency of the McMullen conditions was proved by Billera and Lee [5]. □

Example 16.13 ($n = 4$). For $n = 4$ the h-vector is

$$h_4 = 1, \quad h_3 = f_0 - 4, \quad h_2 = f_1 - 3f_0 + 6,$$
$$h_1 = f_2 - 2f_1 + 3f_0 - 4, \quad h_0 = f_3 - f_2 + f_1 - f_0 + 1.$$

The Dehn-Sommerville equations are equivalent to

$$f_0 - f_1 + f_2 - f_3 = 0, \quad f_2 = 2f_3,$$

which express the Euler characteristic and the simplicity condition. Positivity of the g-vector yields

$$g_1 = h_3 - h_4 = f_0 - 5 \ge 0, \quad g_2 = h_2 - h_3 = f_1 - 4f_0 + 10 \ge 0.$$

The equation $f_0 \ge 5$ is clear, but $f_1 \ge 4f_0 - 10$ is not obvious (at least not to me). Finally, the last inequality $g_2 \le \binom{g_1+1}{2}$ becomes the quadratic inequality mentioned before,

$$f_1 \le \frac{f_0(f_0-1)}{2}.$$

Problem 16.3. Prove Macaulay's theorem for $m = 2$.

Problem 16.4. From every simplicial convex polytope in $\mathbb{R}^n$ we get a new one by pulling out an interior point of an $(n-1)$-dimensional face F and connecting it to all proper faces of F. This increases f_k by $\binom{n}{k}$ for $k < n-1$, and f_{n-1} by $n-1$.

(i) Show that the new polytope satisfies the McMullen conditions if the old one does.

(ii) Show that the pull-out construction corresponds to blow-up of the associated toric orbifold.

(iii) Show that for $n = 3$, all combinatorial types of simplicial polytopes are obtained by this construction, starting from the tetrahedron. So all types are realized as moment polytopes of some blow-up of $\mathbb{C}P^3$.

(iv) Show that for $n = 4$, all polytopes obtained from the standard simplex by the pull-out construction satisfy $f_1 = 4f_0 - 10$, i.e. the fourth inequality is sharp. Conclude that not all simplicial polytopes are obtained in this way.

(v) Construct a simplicial 4-dimensional polytope with f-vector $(6, 15, 18, 9)$.

Chapter 17

The Duistermaat-Heckmann theorems

17.1 Variation in cohomology of the reduced symplectic form

We have seen that the symplectic form on a toric manifold varies linearly with the parameter λ describing the moment polytope $P = \{\xi \mid \langle \xi, u_i \rangle \geq \lambda_i\}$. Equivalently, the toric manifold is the symplectic quotient $\mathbb{C}^d//H(-i^*\lambda)$, and the reduced symplectic form depends linearly on the moment value $-i^*\lambda$. The first version of the Duistermaat-Heckmann Theorem generalizes this result to general quotients by torus actions. The discussion follows [13].

Let (M^{2n}, ω, μ) be a Hamiltonian T-space, where $T = T^k$ is a k-dimensional torus with $k \leq n$. For $\xi \in \mathfrak{t}^*$ set

$$Z_\xi := \mu^{-1}(\xi), \quad M_\xi := \mu^{-1}(\xi)/T = M//H(\xi).$$

Suppose that 0 is a regular value of μ and $\mu^{-1}(0)$ is compact.

Lemma 17.1. *There exists T-invariant a 1-form $A \in \Omega^1(Z_0, \mathfrak{t})$ satisfying $A(\underline{X}) = X$ for all $X \in \mathfrak{t}$. A is called a* connection 1-form.

Proof. If T acts freely on Z_0 then A is just a connection 1-form on the principal bundle $Z_0 \to M_0$. In general, pick a T-invariant Riemanian metric $\langle\ ,\ \rangle$ on Z_0. For $x \in Z_0$ define an inner product on $\mathfrak{t}$ by $\langle X, Y \rangle_x := \langle \underline{X}_x, \underline{Y}_x \rangle$. These products satisfy $\langle\ ,\ \rangle_{gx} = \langle\ ,\ \rangle_x$ for $g \in T$. Now

$$\langle A_x(v), Y \rangle_x := \langle v, \underline{Y}_x \rangle, \quad Y \in \mathfrak{t},$$

is a 1-form with the desired properties.

Note by the way that an invariant compatible almost complex structure defines an invariant metric and therefore a connection A, which played a role in Kähler reduction in Chapter 15. □

For a small neighbourhood U of 0 in $\mathfrak{t}^*$, $\mu^{-1}(U)$ is equivariantly diffeomorphic to $Z_0 \times U$ with coordinates (x, ξ) and moment map ξ. A connection A induces a closed 2-form on $Z_0 \times U$,

$$\omega_A := \omega|_{Z_0} - d\langle \xi, A \rangle.$$

In coordinates X_j on $\mathfrak{t}$ and dual coordinates ξ_j on $\mathfrak{t}^*$,

$$\omega_A = \omega|_{Z_0} - d(\sum \xi_j A_j).$$

At points of $Z_0 \times 0$, $\omega_A = \omega|_{Z_0} - \sum d\xi_j \wedge A_j$, so $\omega_A(\frac{\partial}{\partial X_j}, \frac{\partial}{\partial \xi_j}) = 1$. This shows that ω_A is symplectic and agrees with ω on the coisotropic submanifold $Z_0 \times 0$. Moreover, ω_A is T-invariant, and its moment map is ξ in view of

$$i_{\underline{X}} \omega_A = \langle d\xi, A(\underline{X}) \rangle = d\langle \xi, X \rangle.$$

Therefore an equivariant version of the Coisotropic Neighbourhood Theorem implies

Lemma 17.2. *For U sufficiently small, $(\mu^{-1}(U), \omega, \mu)$ is isomorphic to $(Z_0 \times U, \omega_A, \xi)$ as a Hamiltonian T-space.*

Suppose now that T acts freely on $\mu^{-1}(U)$. Denote by ω_ξ the reduced symplectic form on the quotient M_ξ. Let $F_A \in \Omega^2(M_0, \mathfrak{t})$ be the curvature of A defined by $dA = \pi^* F_A$, $\pi : Z_0 \to M_0$. Then by the preceding lemma, ω_ξ is isomorphic to

$$\omega_0 - \langle \xi, F_A \rangle.$$

In particular, ω_ξ varies linearly with $\xi \in U$. However, this statement depends on the particular diffeomorphisms between the reduced spaces, which are not canonical. Therefore, let us pass to the level of cohomology.

Let $P^0 \subset P$ be a *free chamber* of the moment polytope, i.e. a maximal connected open subset such that T acts freely on $\mu^{-1}(P^0)$. Let

$$\phi : Z_0 \times P^0 \to \mu^{-1}(P^0)$$

be a T-equivariant trivialization of the bundle $\mu^{-1}(P^0) \to P^0$. It induces diffeomorphisms $M_0 \to M_\xi$ and bundle isomorphisms between $Z_0 \to M_0$ and $Z_\xi \to M_\xi$, both unique up to isotopy. Thus we get canonical isomorphisms

$$H^*(M_0; R) \to H^*(M_\xi; R)$$

(with any coefficient ring R) sending the Chern class of the bundle $Z_0 \to M_0$ to that of $Z_\xi \to M_\xi$.

The *Chern class* $c \in H^2(M_0; \mathbb{R}^k)$ is defined as follows: Write the torus as $T = \mathbb{R}^k/(2\pi\mathbb{Z})^k$, and let $A \in \Omega^1(Z_0, \mathbb{R}^k)$ be a connection 1-form with curvature $F_A \in \Omega^2(M_0, \mathbb{R}^k)$. Then

$$c := [-\frac{1}{2\pi} F_A].$$

The Chern class c has the following properties:

(i) It is integral, i.e. $\langle c, \sigma\rangle \in \mathbb{Z}^k$ for $\sigma \in H_2(M_0;\mathbb{Z})$.

(ii) If $\chi \in (\mathbb{Z}^k)^*$ is a character, then $\langle \chi, c\rangle \in H^2(M_0;\mathbb{Z})$ is the first Chern class of the line bundle $L_\chi = Z_0 \times_T \mathbb{C}$ associated to χ. This property uniquely characterizes c.

Theorem 17.3 (Duistermaat-Heckmann I). *For ξ, ξ_0 in the same free chamber $P^0 \subset P$,*

$$[\omega_\xi] = [\omega_{\xi_0}] + 2\pi\langle \xi - \xi_0, c\rangle \in H^2(M_{\xi_0}; \mathbb{R}).$$

Proof. After a translation we may assume $\xi_0 = 0$. Then the formula is a direct consequence of Lemma 17.2. It follows that $[\omega_\xi]$ is a linear function of ξ in a neighbourhood of every point of P^0, so it must be the same linear function throughout P^0. □

Corollary 17.4. *The symplectic volume of the reduced spaces is a polynomial of degree $\leq n-k$ in $\xi \in P^0$,*

$$\int_{M_\xi} \frac{\omega_\xi^{n-k}}{(n-k)!} = \int_{M_{\xi_0}} \frac{(\omega_{\xi_0} + 2\pi\langle \xi - \xi_0, c\rangle)^{n-k}}{(n-k)!}.$$

17.2 The Duistermaat-Heckmann measure

As before let (M, ω, μ) be a Hamiltonian T-space. Let $\mathfrak{m}_L$ be the Lebesgue on $\mathfrak{t}^* \cong \mathbb{R}^k$, normalized such that $\mathfrak{t}^*/\mathfrak{t}^*_{\mathbb{Z}}$ has volume 1.

Definition. The *Duistermaat-Heckmann measure* on $\mathfrak{t}^*$ is the push-forward of the symplectic volume under μ,

$$\mathfrak{m}_{DH}(B) := \int_{\mu^{-1}(B)} \frac{\omega^n}{n!}, \quad B \subset \mathfrak{t}^* \text{Borel measurable.}$$

Theorem 17.5 (Duistermaat-Heckmann II). *The Duistermaat-Heckmann measure equals $\mathfrak{m}_{DH} = f\mathfrak{m}_L$ with a continuous function $f : \mathfrak{t}^* \to \mathbb{R}_{\geq 0}$ which is zero outside P and a polynomial of degree $\leq n-k$ on each regular chamber P^0 of P:*

$$f(\xi) = \int_{Z_{\xi_0}} \frac{1}{(n-k)!} (\omega - \langle \xi, dA\rangle)^{n-k} \wedge A_1 \wedge \cdots \wedge A_k.$$

Corollary 17.6. *On a free chamber P^0,*

$$f(\xi) = (2\pi)^k \int_{M_{\xi_0}} \frac{(\omega_{\xi_0} + 2\pi\langle \xi, c\rangle)^{n-k}}{(n-k)!}.$$

Proof. Clearly $\mathfrak{m}_{DH} = f\mathfrak{m}_L$ with a continuous function f supported on P. So it only remains to show that f is polynomial in a neighbourhood of every regular point $\xi_0 \in P$. After translation, assume that $\xi_0 = 0$. By Lemma 17.2, for a sufficiently small neighbourhood U of 0, $\mu^{-1}(U) \cong Z_0 \times U$ with the symplectic form $\omega_A = \omega|_{Z_0} - d\langle \xi, A\rangle$. For any open set $B \subset U$,

$$\begin{aligned}
\mathfrak{m}_{DH}(B) &= \int_{Z_0 \times B} \frac{1}{n!}(\omega|_{Z_0} - \langle \xi, dA\rangle + \sum A_j \wedge d\xi_j)^n \\
&= \int_{Z_0 \times B} \frac{1}{(n-k)!}(\omega|_{Z_0} - \langle \xi, dA\rangle)^{n-k} \wedge A_1 \wedge d\xi_1 \wedge \cdots \wedge A_k \wedge d\xi_k \\
&= \int_{Z_0} \frac{1}{(n-k)!}(\omega|_{Z_0} - \langle \xi, dA\rangle)^{n-k} \wedge A_1 \wedge \cdots \wedge A_k \ \mathfrak{m}_L(B).
\end{aligned}$$

If T acts freely on $\mu^{-1}(U)$,

$$\begin{aligned}
f(\xi) &= \int_{M_0 \times T} \frac{1}{(n-k)!}(\omega|_{Z_0} - \langle \xi, dA\rangle)^{n-k} \wedge A_1 \wedge \cdots \wedge A_k \\
&= (2\pi)^k \int_{M_0} \frac{1}{(n-k)!}(\omega_0 - \langle \xi, F_A\rangle)^{n-k}.
\end{aligned}$$

□

Example 17.7. For a toric manifold ($k = n$),

$$\mathfrak{m}_{DH} = (2\pi)^n \mathfrak{m}_L.$$

For $M = S^2$ with moment map the height function, this recovers a result known to Archimedes around 230 B.C.: The area of the spherical segment between heights a and b equals $2\pi(b-a)$. In particular, the area depends only on the height difference!

In the previous example we applied the Duistermaat-Heckmann Theorem to a toric manifold viewed as a Hamiltonian T^n-space. We can also apply it to $\mathbb{C}^d$ with the action of the torus H whose quotient at zero is the toric manifold

$$\Big(M_0 = (i^*\mu)^{-1}(0)/H, \omega_0\Big).$$

For $h \in \mathbb{R}^d$ close to zero, consider the toric manifolds

$$\Big(M_h = (i^*\mu)^{-1}(i^*h)/H, \omega_h\Big)$$

corresponding to the moment polytopes

$$P_h = \{\xi \in (\mathbb{R}^n)^* \mid \langle \xi, u_j\rangle \geq \lambda_j - h_j,\ 1 \leq j \leq d\}.$$

In order to apply Duistermaat-Heckmann, we need to know the Chern class c of the H-principal bundle $Z_0 = \mu^{-1}(0) \to M_0$.

Recall that the standard basis vectors e_j^* of $\mathbb{R}^d$ yield characters $i^* e_j^* \in \mathfrak{h}_{\mathbb{Z}}^*$, and thus line bundles $L_j \to M_0$ with first Chern classes $c_j \in H^2(M_0;\mathbb{Z})$. These combine to an $\mathbb{R}^d$-valued class

$$(c_1, \ldots, c_d) = \sum c_j e_j \in H^2(M_0;\mathbb{R}^d).$$

By Lemma 16.6,

$$\pi \sum c_j e_j = \sum c_j u_j = 0,$$

so $\sum c_j e_j = ic$ for a unique $\mathfrak{h}$-valued class $c \in H^2(M_0;\mathfrak{h})$.

Lemma 17.8. *The unique class $c \in H^2(M_0;\mathfrak{h})$ with $ic = \sum c_j e_j$ is the Chern class of the H-principal bundle $Z_0 \to M_0$.*

Proof. It suffices to check that for every character $\chi \in \mathfrak{t}_{\mathbb{Z}}^*$, $\langle \chi, c\rangle$ is the first Chern class of the associated line bundle $L_\chi = Z_0 \times_H \mathbb{C}$. Write $\chi = \sum \chi_j i^* e_j^*$ with integers χ_j. Then $L_\chi = \sum \chi_j L_j$ and

$$\begin{aligned} c_1(L_\chi) &= \sum \chi_j c_j = \sum \chi_j \langle e_j^*, ic\rangle \\ &= \langle \sum \chi_j i^* e_j^*, c\rangle = \langle \chi, c\rangle. \end{aligned}$$

□

Now we apply the Duistermaat-Heckmann Theorem:

Proposition 17.9. *The toric symplectic forms corresponding to the moment polytopes P_h vary linearly with h by*

$$[\omega_h/2\pi] = [\omega_0/2\pi] + \sum h_j c_j,$$

where c_j are the first Chern classes of the line bundles corresponding to the codimension 1 faces of P_0.

Combining this with $\mathfrak{m}_{DH} = (2\pi)^n \mathfrak{m}_L$ we obtain a formula which encodes all the intersection numbers of the faces of P_0!

Corollary 17.10. *The Euclidean volume of P_h is the following polynomial of degree $\leq n$:*

$$\mathrm{vol}(P_h) = \int_{M_0} \frac{(\omega_0/2\pi + \sum h_j c_j)^n}{n!}.$$

Example 17.11. Let $P_0 \subset (\mathbb{R}^2)^*$ be the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ corresponding to $(\mathbb{C}P^2, 2\omega_{FS})$. The faces have normal vectors $u_1 = (1,0)$, $u_2 = (0,1)$ and $u_3 = (-1,-1)$. By elementary geometry,

$$\mathrm{vol}(P_h) = \frac{1}{2}(1 + h_1 + h_2 + h_3)^2.$$

On the other hand, this equals

$$\mathrm{vol}(P_h) = \frac{1}{2}([\omega_{FS}/\pi] + h_1 c_1 + h_2 c_2 + h_3 c_3)^2.$$

Comparing these expressions we read off all cup products

$$[\omega_{FS}/\pi]^2 = [\omega_{FS}/\pi] c_j = c_i c_j = 1.$$

By nondegeneracy of the cup product this implies

$$[\omega_{FS}/\pi] = c_1 = c_2 = c_3,$$

agreeing with our previous description of the cohomology ring.

17.3 The method of stationary phase

This section provides the background for the Duistermaat-Heckmann Formula in the next section. The discussion follows [16].

Various problems in physics lead to oscillatory integrals of the type

$$\int_M a(x) e^{itf(x)} \mathrm{dvol}$$

over a manifold M. For example, in wave optics, the complex amplitude at a point y of light reflected from a surface S is given by

$$\int_S \frac{a(x)}{|x-y|} e^{\frac{2\pi i}{\lambda}|x-y|} \mathrm{darea}.$$

Here $a(x)$ is the complex amplitude at a point on the mirror, and λ is the wavelength. In the semi-classical approximation, one is interested in the asymptotic behaviour of such integrals for small wavelength λ, i.e. for large t. The term "semi-classical approximation" comes from quantum mechanics, where such oscillatory integrals arise as path integrals.

Intuitively, the rapidly oscillating term $e^{itf(x)}$ should lead to large cancellations in regions where f varies. So one expects contributions to the integral to be concentrated near the critical points of f. This is indeed the case, and leads to asymptotic expansions in terms of the Taylor series at the critical points. The simplest case arises if all critical points are nondegenerate:

Theorem 17.12 (Stationary Phase Lemma). *Let M^n be a manifold with volume form* dvol*, $f : M \to \mathbb{R}$ a Morse function and $a : M \to \mathbb{R}$ a function with compact support. Then for large t,*

$$\left(\frac{t}{2\pi}\right)^{\frac{n}{2}} \int_M a(x) e^{itf(x)} \mathrm{dvol} = \sum_{p \in \mathrm{Crit}(f)} \frac{e^{\frac{i\pi}{4} \mathrm{sgn}\,\mathrm{Hess}_f(p)}}{\sqrt{|\det \mathrm{Hess}_f(p)|}} a(p) e^{itf(p)} + O\Big(\frac{1}{t}\Big),$$

where $\mathrm{Hess}_f(p)$ *is the Hessian of f at p, and* sgn *is its signature.*

Proof. Step 1. Since both sides are linear in a, using a partition of unity, it suffices to prove the statement for a supported in an arbitrarily small coordinate neighbourhood. So we may assume $M = \mathbb{R}^n$ and $d\text{vol} = d^n x$.

Suppose that $\operatorname{supp}(a)$ contains no critical point of f. Then $\nabla f = \sum \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j}$ does not vanish on $\operatorname{supp}(a)$ and we get

$$\nabla f \cdot e^{itf} = it|\nabla f|^2 e^{itf},$$

$$\begin{aligned}\int a e^{itf} &= \int \frac{a}{it|\nabla f|^2} \nabla f \cdot e^{itf} \\ &= \frac{i}{t} \int \sum \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_j} \Big(\frac{a}{|\nabla f|^2} \Big) e^{itf} \\ &= \frac{1}{t} \int b e^{itf}\end{aligned}$$

for a smooth function b with $\operatorname{supp}(b) \subset \operatorname{supp}(a)$. Iterating this procedure we find that $\int a e^{itf} = O(1/t^N)$ for every $N \in \mathbb{N}$. So the highest order term comes only from neighbourhoods of the critical points.

Step 2. Let p be a critical point. By the Morse Lemma, there exist coordinates z_j near p in which

$$f(z) = f(p) + \frac{1}{2} Q(z), \quad Q(z) = -z_1^2 - \cdots - z_l^2 + z_{l+1}^2 + \cdots + z_n^2.$$

Change of variables yields

$$\int a e^{itf} d^n x = e^{itf(p)} \int a e^{\frac{itQ(z)}{2}} |\det \frac{\partial x}{\partial z}| d^n z.$$

The matrix Q of $Q(z)$ is related to the Hessian $\operatorname{Hess}_f(p) = \Big(\frac{\partial^2 f}{\partial x_j \partial x_k} \Big)$ by

$$Q = \Big(\frac{\partial x}{\partial z} \Big)^T \Big(\frac{\partial^2 f}{\partial x_j \partial x_k} \Big) \Big(\frac{\partial x}{\partial z} \Big),$$

so

$$|\det \frac{\partial x}{\partial z}(p)| = |\det \operatorname{Hess}_f(p)|^{-\frac{1}{2}}.$$

We have thus reduced the proof to showing

$$\Big(\frac{t}{2\pi} \Big)^{\frac{n}{2}} \int a(z) e^{\frac{itQ(z)}{2}} d^n z = a(0) e^{\frac{i\pi}{4}(n-2l)} + O\Big(\frac{1}{t} \Big)$$

for $a(z)$ compactly supported and $Q(z)$ the quadratic form above.

Step 3. Write

$$a(z) - a(0) = \int_0^1 \frac{d}{dt} a(tz) dt = \sum z_j \int_0^1 \frac{\partial a}{\partial z_j}(tz) dt = \sum z_j a_j(z).$$

Here the functions a_j are smooth but no longer compactly supported. But the a_j and all their derivatives are bounded on $\mathbb{R}^n$. The lemma below shows that this is sufficient for the integrals $\int a_j(z)e^{\frac{itQ(z)}{2}}d^nz$ to exist and be uniformly bounded for $|t| \geq \varepsilon$. Using this we can write

$$\int z_j a_j(z) e^{\frac{itQ(z)}{2}} d^nz = \pm\frac{1}{it}\int a_j \frac{\partial}{\partial z_j}\left(e^{\frac{itQ(z)}{2}}\right) d^nz = \mp\frac{1}{it}\int \frac{\partial a_j}{\partial z_j} e^{\frac{itQ(z)}{2}} d^nz.$$

Since the last integral is again uniformly bounded for $|t| \geq \varepsilon$, this shows that the highest order term comes from the constant term $a(0)$.

It remains to prove the existence of the integrals:

Lemma 17.13. *Let $h : \mathbb{R} \to \mathbb{R}$ be a C^2-function which is bounded and with bounded first two derivatives. Then the integral*

$$\int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2}} h(x)dx$$

converges uniformly for $\Re\lambda \geq 0$, $|\lambda| \geq \varepsilon$.

Proof. For $0 < R < S$ we have

$$\begin{aligned}
\int_R^S e^{-\frac{\lambda x^2}{2}} h(x)dx &= -\frac{1}{\lambda}\int_R^S \left(e^{-\frac{\lambda x^2}{2}}\right)' \frac{h(x)}{x}dx \\
&= \left[-\frac{1}{\lambda}e^{-\frac{\lambda x^2}{2}}\frac{h(x)}{x}\right]_R^S + \frac{1}{\lambda}\int_R^S e^{-\frac{\lambda x^2}{2}}\left(\frac{h(x)}{x}\right)' dx \\
&= \left[e^{-\frac{\lambda x^2}{2}}\left\{-\frac{1}{\lambda}\frac{h(x)}{x} + \frac{1}{\lambda^2}\frac{1}{x}\left(\frac{h(x)}{x}\right)'\right\}\right]_R^S \\
&\quad + \frac{1}{\lambda^2}\int_R^S e^{-\frac{\lambda x^2}{2}}\left\{\frac{1}{x}\left(\frac{h(x)}{x}\right)'\right\}' dx.
\end{aligned}$$

The term evaluated at S tends to zero as $S \to \infty$, and the term evaluated at R is converges for fixed R. The integral on the right-hand side is absolutely convergent as $S \to \infty$ because each term of the integrand contains a factor $1/x^2$. Since the integrals from $-R$ to R converge for fixed R, this proves the lemma. ◻

Step 4. By Step 3, it only remains to show

$$\left(\frac{t}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{itQ(z)}{2}} d^nz = e^{\frac{i\pi}{4}(n-2l)}.$$

The integral factorizes into one-dimensional integrals, and the statement reduces to

$$\int_{-\infty}^{\infty} e^{\pm\frac{itx^2}{2}}dx = \left(\frac{2\pi}{t}\right)^{\frac{1}{2}} e^{\pm\frac{i\pi}{4}}.$$

By the lemma above, the integral

$$\int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2}} dx$$

defines a holomorphic function for $\Re\lambda > 0$ which is continuous for $\Re\lambda \geq 0$, $\lambda \neq 0$. For λ real, $\int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi}$ and change of variables implies

$$\int_{-\infty}^{\infty} e^{-\frac{\lambda x^2}{2}} dx = \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}}, \quad 0 \neq \lambda \in \mathbb{R}.$$

By analytic continuation, the same formula holds for $\Re\lambda > 0$, where the square root is continued from the real axis. Continuing the this for $\lambda \to \pm it$ yields the desired formula and concludes the proof of the Stationary Phase Lemma. □

17.4 The Duistermaat-Heckmann Formula

In this section we apply the method of stationary phase to the oscillatory integrals

$$\int_M e^{it\langle \mu, X\rangle} \frac{\omega^n}{n!},$$

where μ is the moment map of a compact Hamiltonian T^k-space (M, ω, μ).

Let $X \in \mathfrak{t}$ be such that $\mathrm{Crit}(\langle \mu, X\rangle) = \mathrm{Fix}(T)$. Assume that the action has only isolated fixed points so that $f := \langle \mu, X\rangle$ is a Morse function. Consider the local normal form near a fixed point p,

$$f(z) = f(p) + \frac{1}{2} \sum \langle w_j(p), X\rangle\, |z_j|^2,$$

where $w_1(p), \ldots, w_n(p) \in \mathfrak{t}^*_{\mathbb{Z}}$ are the weights at p. We read off:

$$\begin{aligned} \sqrt{|\det \mathrm{Hess}_f(p)|} &= |\prod \langle w_j(p), X\rangle|, \\ \mathrm{sgn}\, \mathrm{Hess}_f(p) &= 2n - 4\#\{j \mid \langle w_j(p), X\rangle < 0\}, \\ \frac{e^{\frac{i\pi}{4} \mathrm{sgn}\, \mathrm{Hess}_f(p)}}{\sqrt{|\det \mathrm{Hess}_f(p)|}} &= \frac{i^n}{\prod \langle w_j(p), X\rangle}. \end{aligned}$$

So the Stationary Phase Lemma yields the asymptotics as $t \to \infty$,

$$\left(\frac{t}{2\pi i}\right)^n \int_M e^{it\langle \mu, X\rangle} \frac{\omega^n}{n!} = \sum_{p \in \mathrm{Fix}(T)} \frac{e^{it\langle \mu(p), X\rangle}}{\prod \langle w_j(p), X\rangle} + O\left(\frac{1}{t}\right).$$

It turns out that in this case the stationary phase approximation is in fact exact! This is referred to as the *Duistermaat-Heckmann Formula*:

Theorem 17.14 (Duistermaat-Heckmann III). *Let (M, ω, μ) be a compact Hamiltonian T^k-space with isolated fixed points, and let $X \in \mathfrak{t}$ be such that $\langle w_j(p), X\rangle \neq 0$ for all weights at fixed points. Then for $t \neq 0$,*

$$\int_M e^{it\langle \mu, X\rangle} \frac{\omega^n}{n!} = \Big(\frac{2\pi i}{t}\Big)^n \sum_{p \in \mathrm{Fix}(T)} \frac{e^{it\langle \mu(p), X\rangle}}{\prod \langle w_j(p), X\rangle}.$$

Proof. Let $X \in \mathfrak{t}$ be such that $\mathrm{Crit}(\langle \mu, X\rangle) = \mathrm{Fix}(T)$ and X generates a circle in T. Since there is only a finite number of isotropy groups, such X are dense among the vectors with $\langle w_j(p), X\rangle \neq 0$ for all weights. Since both sides of the Duistermaat-Heckmann Formula depend continuously on X, it suffices to prove it for such X.

Let $\mathfrak{m}_{DH}$ be the Duistermaat-Heckmann measure of the S^1-action generated by X. Then by the Duistermaat-Heckmann Theorem,

$$\begin{aligned}\int_M e^{it\langle \mu, X\rangle} \frac{\omega^n}{n!} &= \int_{\mathbb{R}} e^{it\xi} d\mathfrak{m}_{DH}(\xi) \\ &= \int_{\mathbb{R}} f(\xi) e^{it\xi} d\xi\end{aligned}$$

for a continuous compactly supported function f which is piecewise polynomial of degree $\leq n-1$. Note that the right-hand side is just the inverse Fourier transform $\mathcal{F}^{-1} f$ of f. Multiplying by $(t/2\pi i)^n$ yields

$$\Big(\frac{t}{2\pi i}\Big)^n \int_M e^{it\langle \mu, X\rangle} \frac{\omega^n}{n!} = \Big(\frac{t}{2\pi i}\Big)^n \mathcal{F}^{-1} f(t) = \Big(\frac{i}{2\pi}\Big)^n \mathcal{F}^{-1}\Big(\frac{d^n f}{d\xi^n}\Big)(t).$$

The right-hand side is the inverse Fourier transform of a linear combination of derivatives of Dirac distributions, so it is a linear combination of terms $t^l e^{iat}$. On the other hand, by the Stationary Phase Lemma, the left-hand side equals

$$\sum_{p \in \mathrm{Fix}(T)} \frac{e^{it\langle \mu(p), X\rangle}}{\prod \langle w_j(p), X\rangle} + O\Big(\frac{1}{t}\Big).$$

The first term is a linear combination of terms e^{iat}, and the second (error) term is an L^2-function. Since neither of the $t^l e^{iat}$ is L^2, this is only possible if the L^2-term $O(1/t)$ equals zero. □

Note that the left-hand side extends smoothly over $t = 0$. This forces cancellations on the right-hand side and leads to relations between the fixed point data Moreover, putting $t = 0$ expresses the volume of M in terms of fixed point data.

Example 17.15 ([4]). Suppose $T = S^1$ acts with moment map H, with isolated fixed points, and *without finite stabilizers*, so all weights equal ± 1. Then $\prod w_j(p) = (-1)^{\mathrm{ind}(p)/2}$, and the Duistermaat-Heckmann Formula reads

$$\int_M e^{itH} \frac{\omega^n}{n!} = \Big(\frac{2\pi i}{t}\Big)^n \sum_p (-1)^{\mathrm{ind}(p)/2} e^{itH(p)}.$$

For $n = 1$ this gives us back Archimedes' formula. For $n = 2$, let h_k be the number of critical points of index $2k$, and expand the right-hand side:

$$\int_M e^{itH}\frac{\omega^2}{2} = \Big(\frac{2\pi i}{t}\Big)^2\Big\{(h_0 - h_1 + h_2) + it\Big(\sum_{\operatorname{ind}(p)=0,4} H(p) - \sum_{\operatorname{ind}(p)=2} H(p)\Big)$$

$$= \frac{(it)^2}{2}\Big(\sum_{\operatorname{ind}(p)=0,4} H(p)^2 - \sum_{\operatorname{ind}(p)=2} H(p)^2\Big)\Big\}.$$

As $t \to 0$ we first infer

$$h_0 - h_1 + h_2 = 0.$$

Since $h_0 = h_2 = 1$, there are precisely 2 critical points a, b of index 2. Vanishing of the second term yields

$$H_{\min} + H_{\max} - H(a) - H(b) = 0,$$

so $H(a) - H_{\min} = H_{\max} - H(b)$. Finally, we get a formula for the volume of M,

$$\operatorname{vol}(M) = \frac{2\pi)^2}{2}\Big(H_{\min}^2 + H_{\max}^2 - H(a)^2 - H(b)^2\Big).$$

Note that the right-hand side is $(2\pi)^2$ times the area of the trapezoid with vertices $(H_{\min}, 0)$, $\big((H(a), H(a) - H_{\min}\big)$, $\big((H(b), H(a) - H_{\min}\big)$ and $(H_{\max}, 0)$.

Problem 17.1. (i) Show that the graph of the Duistermaat-Heckmann measure is the trapezoid described in the preceding example.

(ii) Find S^1-actions with isolated fixed points and without finite stabilizers on $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, and verify that the Duistermaat-Heckmann measure has the required shape. Hint: Both are toric manifolds.

(iii) Show that no such S^1-action exists on $\mathbb{C}P^2$.

Example 17.16 ([31]). Consider the S^1-action on $\mathbb{C}P^n$ given by

$$e^{i\theta}[z_0 : \cdots : z_n] = [z_0 : e^{-i\theta}z_1 : \cdots : e^{-in\theta}z_n]$$

with moment map

$$H([z]) = \frac{1}{2|z|^2}\sum_{k=1}^{n} k|z_k|^2.$$

It has isolated fixed points $p_k = [0 : \cdots : 1 : \ldots 0]$ with

$$H(p_k) = \frac{k}{2}, \quad \prod w_j(p_k) = (-1)^k k!(n-k)!.$$

Putting $t = 0$ in the Duistermaat-Heckmann formula yields

$$\operatorname{vol}(\mathbb{C}P^n) = \frac{(-1)^n(2\pi)^n}{n!}\sum_{k=1}^{n}\frac{(-1)^k k^n}{2^n k!(n-k)!}.$$

Since $\mathrm{vol}(\mathbb{C}P^n) = \pi^n/n!$, this is equivalent to the combinatorial formula

$$(-1)^n n! = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^n.$$

Note. Here is a combinatorial proof of this identity due to Eric Katz: Notice that the right-hand side equals

$$\left(x \frac{d}{dx}\right)^n (1-x)^n \Big|_{x=1}.$$

But since all terms involving $(1-x)$ vanish for $x = 1$, this equals

$$x^n \left(\frac{d}{dx}\right)^n (1-x)^n \Big|_{x=1} = (-1)^n n!.$$

Chapter 18

Equivariant cohomology

18.1 An alternative proof of the Duistermaat-Heckmann formula

Since the work of Atiyah-Bott [3] and Kirwan [26], equivariant cohomology has become the main tool to study the cohomology of symplectic quotients. As a motivation, I will show in this section how the simplest version of equivariant cohomology leads to a simple and elegant proof of the Duistermaat-Heckmann formula. This observation is due to Atiyah-Bott [3], but I follow the presentation in [21].

Let M^n be a compact manifold, and X be a vector field that preserves some Riemannian metric on M. Let $\Omega^k_X(M)$ be the X-invariant k-forms and

$$\Omega^*_X(M) := \bigoplus_{k=0}^{n} \Omega^k_X(M).$$

Define a linear operator $d_X : \Omega^*_X(M) \to \Omega^*_X(M)$ by

$$d_X\alpha := d\alpha + i_X\alpha.$$

It satisfies $d_X^2 = 0$ because

$$d_X^2\alpha = i_X d\alpha + d\, i_X\alpha = L_X\alpha = 0.$$

Lemma 18.1 (Localization Lemma). *Let $\alpha = \alpha_n + \alpha_{n-2} + \dots$ be d_X-closed. Then $\alpha_n = d\beta_{n-1}$ on $M^* := \{x \in M \mid X(x) \neq 0\}$.*

Proof. Define a 1-form θ on M^* by

$$\theta(v) := \frac{\langle X, v\rangle}{\|X\|^2},$$

where $\langle\ ,\ \rangle$ is an X-invariant metric. This form satisfies

$$L_X\theta = 0, \quad i_X\theta \equiv 1, \quad i_X d\theta = L_X\theta - d\,i_X\theta = 0.$$

Set

$$\beta := \theta \wedge (1 + d\theta)^{-1} \wedge \alpha \in \Omega^*_X(M^*),$$

where $(1 + d\theta)^{-1} = 1 - d\theta + d\theta \wedge d\theta \mp \dots$ is a finite sum.

I claim that $i_X d\beta = i_X\alpha$ on M^*.

Indeed, using $d\alpha + i_X\alpha = 0$,

$$\begin{aligned} d\beta &= d\theta \wedge (1 + d\theta)^{-1} \wedge \alpha - \theta \wedge (1 + d\theta)^{-1} \wedge d\alpha \\ &= d\theta \wedge (1 + d\theta)^{-1} \wedge \alpha + \theta \wedge (1 + d\theta)^{-1} \wedge i_X\alpha, \\ i_X d\beta &= d\theta \wedge (1 + d\theta)^{-1} \wedge i_X\alpha + (1 + d\theta)^{-1} \wedge i_X\alpha \\ &= i_X\alpha. \end{aligned}$$

The claim implies $i_X d\beta_{n-1} = i_X\alpha_n$ on M^*, and therefore $d\beta_{n-1} = \alpha_n$ because $X \neq 0$ on M^*. □

Proof of the Duistermaat-Heckmann Formula. As in the proof of Theorem 17.14, it suffices to prove it for $T = S^1$. Let X be the vector field generating the circle action. It preserves any S^1-invariant metric, and thus defines a differential d_X as above. Now the main observation is:

$\mu - \omega \in \Omega^*_X(M)$ *is* d_X*-closed.*

This follows right from the definition of the moment map $\mu : M \to \mathbb{R}$:

$$d_X(\mu - \omega) = d\mu - i_X\omega = 0.$$

It follows that

$$\alpha := e^{it(\mu-\omega)} := e^{it\mu}\Big\{1 - it\omega + \frac{(it\omega)^2}{2!} \mp \dots\Big\}$$

is d_X-closed. (Check this directly for the finite sum on the right-hand side!). The highest order term on α is $\alpha_{2n} = e^{it\mu}(-it)^n\mathrm{vol}$. So by the Localization Lemma,

$$e^{it\mu}\mathrm{vol} = d\beta_{2n-1}$$

on $M^* = M \setminus Fix(S^1)$.

Now assume that the action has only isolated fixed points, and pick small balls $B(p)$ around them. Then by Stokes' Theorem,

$$\begin{aligned} \int_M e^{it\mu}\mathrm{vol} &= \sum_{p\in Fix(S^1)} \int_{B(p)} e^{it\mu}\mathrm{vol} + \int_{M\setminus\cup B(p)} d\beta_{2n-1} \\ &= \sum_{p\in Fix(S^1)} \Big(\int_{B(p)} e^{it\mu}\mathrm{vol} - \int_{\partial B(p)} d\beta_{2n-1}\Big). \end{aligned}$$

Now in suitable coordinates on $B(p)$, vol $= d^{2n}z$ and

$$\mu(z) = \mu(p) + \frac{1}{2}\sum w_j(p)|z_j|^2$$

with weights $w_1(p), \dots, w_n(p) \in \mathbb{Z}$. Another application of Stokes' Theorem yields

$$\int_{B(p)} e^{it\mu}\text{vol} - \int_{\partial B(p)} d\beta_{2n-1} = \int_{\mathbb{C}^n} e^{it\left(\mu(p)+\frac{1}{2}\sum w_j(p)|z_j|^2\right)} d^{2n}z.$$

(This step requires justification. We need to replace β by another primitive that is defined on all of $\mathbb{C}^n$ and whose integral over large spheres becomes small.)

The integral on the right-hand side is a product of 1-dimensional integrals which have been evaluated in Step 4 of the proof of the Stationary Phase Lemma:

$$\int_{-\infty}^{\infty} e^{\pm\frac{itx^2}{2}} dx = \left(\frac{2\pi}{t}\right)^{\frac{1}{2}} e^{\pm\frac{i\pi}{4}}.$$

So the right-hand side becomes

$$\left(\frac{2\pi i}{t}\right)^n \frac{e^{it\mu(p)}}{\prod w_j(p)},$$

proving the Duistermaat-Heckmann Formula. □

18.2 The Cartan model of equivariant cohomology

In this section we develop more systematically the Cartan model of equivariant cohomology. For simplicity, we restrict ourselves to torus actions.

Let $T = T^k$ be a torus and M^n a manifold. Denote by $\Omega_T^*(M)$ the polynomial maps from $\mathfrak{t} = \mathbb{R}^k$ to the T-invariant differential forms on M. Define a differential $d_T : \Omega_T^*(M) \to \Omega_T^*(M)$ by

$$d_T\alpha(X) := d\big(\alpha(X)\big) + i_{\underline{X}}\big(\alpha(X)\big), \quad X \in \mathfrak{t}.$$

More explicitly, pick a basis $e_1, \dots, e_k$ and write α as a polynomial

$$\alpha = \sum_I \alpha_I u^U, \quad u^I = u_1^{i_1} \dots u_k^{i_k}.$$

Then

$$d_T\alpha = \sum_I d\alpha_I u^I + \sum_{I,j} i_{\underline{e_j}}\alpha_I u^{I+j}.$$

If we give $\alpha_I u^I$ the grading $\deg \alpha_I + 2|I|$, then $d_T : \Omega_T^k(M) \to \Omega_T^{k+1}(M)$ has degree $+1$. It satisfies $d_T^2 = 0$, and we define the *equivariant cohomology* (with real coefficients)

$$H_T^k(M) := \ker d_T \big/ \mathrm{im} d_T.$$

For $T = S^1$ and forgetting the grading we recover the model of the first section.

Note that $H_T^*(\mathrm{pt}) = \mathbb{R}[u_1, \ldots, u_k]$ and $H_T^*(M)$ is a module over this ring. Equivariant cohomology has the usual functorial proerties.

Integration over the fibre

Let $\pi : E \to M$ be a smooth fibration with compact base. In ordinary cohomology we have *integration over the fibre*

$$\pi_* : H^*(E)_c \to H^{*-k}(M),$$

where H_c^* is cohomology with compact support and k is the fibre dimension. If T acts on E and M compatible with π, integration over the fibre carries over to equivariant cohomology,

$$\pi_* : H_T^*(E)_c \to H_T^{*-k}(M).$$

Thom and Euler class

Let $\pi : E \to M$ be a rank k (real) vector bundle with compact base. In ordinary cohomology there exists a unique class $\tau \in H^k(E)_c$, called the *Thom class*, such that

$$\pi_* \tau = 1.$$

The *Euler class* is the restriction of τ to the zero section M,

$$e := \tau_M.$$

Now suppose that T acts linearly on E with fixed points set M. We wish to construct an equivariant Thom class $\tau_T \in H_T^k(E)_c$ with $\pi_* \tau_T = 1$.

Let E first be a complex line bundle. Pick a T-invariant metric on E and a T-invariant connection form $\theta \in \Omega^1(E \setminus M)$ with curvature F_θ. Let r be the norm in the fibre and $f(r)$ a decreasing function with $f \equiv 1$ near $r = 0$ and $f \equiv 0$ for $r \geq r_0$. Then the ordinary Thom and Euler class are represented by

$$\tau = \Big[d\Big(-\frac{f(r)\theta}{2\pi} \Big) \Big],$$
$$e = \tau|_M = \Big[-\frac{F_\theta}{2\pi} \Big].$$

The class τ is not equivariantly closed. Indeed $\theta(\underline{X}) = \langle -w, X\rangle$ where $w = \sum w^{(i)}u^i \in \mathfrak{t}^*$ is the weight of the T-action at M. Therefore

$$i_{\underline{X}}\tau = -\frac{f'\langle w, X\rangle}{2\pi}dr = d\Big(-\frac{f\langle w, X\rangle}{2\pi}\Big).$$

So the form

$$\tau_T = \frac{1}{2\pi}\Big\{d(-f\theta) + fw\Big\} = \tau + \frac{fw}{2\pi}$$

is d_T-closed and represents the equivariant Thom class. The equivariant Euler class is

$$e_T := \tau_T|_M = e + \frac{w}{2\pi}.$$

If $E = L_1 \oplus \cdots \oplus L_l$ is a sum of complex line bundles,

$$\begin{aligned}
\tau_T(E) &= \prod \tau_T(L_j),\\
e_T(E) &= \prod \Big(e(L_j) + \frac{w_j}{2\pi}\Big)\\
&= e(E) + \cdots + \frac{\prod w_j}{(2\pi)^l}.
\end{aligned}$$

Note that since all the weights w_j are nonzero, the term $\prod w_j$ is a product of nontrivial linear polynomials and therefore invertible in the field of rational functions. Since all other terms in $e_T(E)$ are nilpotent (because they contain differential forms of positive degrees), the Euler class $e_T(E)$ is invertible in the "localized" ring $H_T^*(M)_{loc}$ in which polynomials are enlarged to rational functions.

These considerations generalize to the following result, the proof of which can be found in [22]:

Proposition 18.2. *Let $\pi : E \to M$ be a rank $2l$ oriented (real) vector bundle, and let T act linearly on E with fixed points set M. Then there exists a unique class $\tau_T \in H_T^{2l}(E)_c$, called the* equivariant Thom class*, satisfying*

$$\pi_*\tau = 1.$$

The map $\wedge\tau_T : H_T^{*-2l}(M) \to H_T^*(E)_c$ is an isomorphism. The equivariant Euler class $e_T = \tau_T|_M$ *is of the form*

$$e_T = e + \cdots + \frac{\prod w_j}{(2\pi)^l},$$

where $w_1, \ldots, w_l \in \mathfrak{t}^*$ are the weights at M, and e_T is invertible in the localized ring $H_T^*(M)_{loc}$.

The invertibility of the Euler class can be rephrased in a more useful way. Let E_1 be the unit disk bundle of E. Then in the long exact sequence of the pair

$(E_1, \partial E_1)$ we have the isomorphisms

$$H_T^*(M) \cong H_T^*(E_1),$$
$$H_T^{*-2l}(M) \overset{\wedge \tau_T}{\cong} H_T^*(E)_c \cong H_T^*(E_1, \partial E_1).$$

Under these isomorphisms, the map $H_T^*(E_1, \partial E_1)$ corresponds to the map

$$\wedge e_T : H_T^{*-2l}(M) \to H_T^*(M)$$

which is injective. So the long exact sequence splits into short exact sequences that fit in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_T^{*-2l}(M) & \xrightarrow{\wedge e_T} & H_T^*(M) & & & & \\
 & & \cong \big\downarrow \wedge \tau_T & & \big\downarrow \cong & & & & \\
0 & \longrightarrow & H_T^*(E_1, \partial E_1) & \longrightarrow & H_T^*(E_1) & \longrightarrow & H_T^*(\partial E_1) & \longrightarrow & 0
\end{array}$$

18.3 Localization

Lemma 18.3. *Let T act on the manifold M^n with compact fixed point set $Fix(T)$. Then the map in top dimension*

$$H_T^n(M) \to H_T^n\Big(Fix(T)\Big)$$

induced by the inclusion $Fix(T) \subset M$ is injective.

Proof. Suppose $\alpha \in \Omega_T^*(M)$ is d_T-exact on $Fix(T)$. Let us first make some simplifications. After subtracting a primitive we may assume $\alpha = 0$ on $Fix(T)$. Pick a map $\phi : M \to M$, homotopic to the identity, which maps a tubular neighbourhood of $Fix(T)$ onto $Fix(T)$. Then $\phi^*\alpha - \alpha$ is d_T-exact and $\phi^*\alpha = 0$ on a neighbourhood of $Fix(T)$. Replace α by $\phi^*\alpha$.

Finally, pick a circle $S^1 \subset T$ with $Fix(S^1) = Fix(T)$. Then α defines a class in $H_{S^1}^*(M)$, and it suffices to show that this class vanishes (see [22]).

After all simplifications we may assume that $T = S^1$ and $\alpha = 0$ on a neighbourhood of $Fix(S^1)$. Now the proof is similar to Lemma 18.1 (cf. [31]). Let X be the vector field generating the action. Pick $\theta \in \Omega^1(M^*)$, $M^* = M \setminus Fix(S^1)$, with $\theta(X) \equiv 1$, $i_X d\theta = 0$. If $n = 2m$ is even,

$$\alpha = \alpha_0 u^m + \alpha_2 u^{m-1} + \cdots + \alpha_{2m},$$

define $\beta \in \Omega_{S^1}^{2m-1}(M)$ inductively by $\beta_1 := \alpha_0 \theta$ and

$$\beta_{2k+1} := \alpha_{2k} \wedge \theta - \beta_{2k-1} \wedge d\theta.$$

One easily checks inductively that $i_X d\beta_{2k+1} + d\beta_{2k-1} = \alpha_{2k}$ for $k < m$. For $k = m$,

$$\begin{aligned} d\beta_{2m-1} &= d\alpha_{2m-1} \wedge \theta + \alpha_{2m-2} \wedge d\theta - d\beta_{2m-3} \wedge d\theta \\ &= -i_X \alpha_{2m} \wedge \theta + i_X \beta_{2m-1} \wedge d\theta. \end{aligned}$$

So $i_X d\beta_{2m-1} = i_X \alpha_{2m}$, which implies $d\beta_{2m-1} = \alpha_{2m}$ because $2m$ is the top dimension and $X \neq 0$. This proves $d_{S^1}\beta = \alpha$.

If $n = 2m+1$ is odd,

$$\alpha = \alpha_1 u^m + \alpha_3 u^{m-1} + \cdots + \alpha_{2m+1},$$

define $\beta \in \Omega^{2m}_{S^1}(M)$ inductively by $\beta_0 := 0$ and

$$\beta_{2k+2} := -\alpha_{2k+1} \wedge \theta - \beta_{2k} \wedge d\theta.$$

□

Theorem 18.4 (Localization Theorem). *Let T act on the manifold M with compact fixed point set $Fix(T)$. Then for every equivariant cohomology class $\alpha \in H^*_T(M)$,*

$$\alpha = \sum_F \frac{\alpha|_F}{e_T(F)} \wedge \tau_T(F),$$

where the sum runs over the connected components of the fixed point set and $\tau_T(F)$, $e_T(F)$ are the equivariant Thom and Euler classes of the normal bundle.

Remark. Here the equality is to be understood in the localized ring $H^*_T(M)_{loc}$. Note, however, that α lies in the non-localized ring, so the equality forces the fractional terms on the right-hand side to cancel.

Proof. The restriction of $\alpha - \sum_F \frac{\alpha|_F}{e_T(F)} \wedge \tau_T(F)$ to a fixed point component F equals

$$\alpha|_F - \frac{\alpha|_F}{e_T(F)} e_T(F) = 0,$$

so the result follows from the previous lemma. □

Integrating both sides over M we obtain

Corollary 18.5. *For every $\alpha \in H^*_T(M)$,*

$$\int_M \alpha = \sum_F \int_F \frac{\alpha}{e_T(F)}.$$

Duietermaat-Heckmann revisited

Now let (M^{2n}, ω, μ) be a Hamiltonian T-space. The form $\mu - \omega$ is d_T-closed. Apply the preceding corollary to $\alpha := e^{\mu-\omega}$ to obtain

$$\int_M (-1)^n e^{\mu} \frac{\omega^n}{n!} = \sum_F \int_F \frac{\alpha}{e_T(F)}.$$

(Since e^{μ} is a formal power series, the equality should be checked for each degree separately). This expresses the integral in terms of fixed point data.

To get a more explicit formula, suppose all fixed points p are isolated. Then the equivariant Euler class is determined by the weights $w_j(p)$,

$$e_T(p) = \frac{\prod w_j(p)}{(2\pi)^n},$$

and the formula becomes

$$\int_M e^{\mu} \frac{\omega^n}{n!} = (-2\pi)^n \sum_p \frac{e^{\mu(p)}}{\prod w_j(p)}.$$

Here $\mu(p)$ and $w_j(p)$ are linear polynomials in $u_1, \ldots, u_k$, and the equation relates formal power series. Evaluating it at a point λX, $\lambda \in \mathbb{R}$, $X \in \mathfrak{t}$, yields an equation between power series in λ,

$$\int_M e^{\lambda} \langle \mu, X \rangle \frac{\omega^n}{n!} = (\frac{-2\pi}{\lambda})^n \sum_p \frac{e^{\lambda \langle \mu(p), X \rangle}}{\prod \langle w_j(p), X \rangle}.$$

Since both sides converge for $\lambda \in \mathbb{C} \setminus 0$, this is an equality between analytic functions of λ. For $\lambda = it$ we recover the Duistermaat-Heckmann formula.

18.4 Kirwan surjectivity

This section follows [38]. Let T act on a manifold M, and let F be a compact component of the fixed point set. Suppose that there exist a T-invariant function $f : M \to \mathbb{R}$ which is Morse-Bott nondegenerate along F, $f(F) = c$, and F is the only critical set in $f^{-1}(-\varepsilon, \varepsilon)$. Set

$$M^+ := f^{-1}(c - \varepsilon, \infty), \quad M^- := f^{-1}(-\infty, c + \varepsilon).$$

By Morse-Bott theory, the pair $\big(M^+ \setminus M^-, \partial(M^+ \setminus M^-)\big)$ deformation retracts onto $(E_1^-, \partial E_1^-)$, where E^- is the negative normal bundle of F. So the exact sequence following Proposition 18.2 yields:

Lemma 18.6. *The long exact sequence of the pair (M^+, M^-) splits into short exact sequences that fit in the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_T^{*-2l}(F) & \xrightarrow{\wedge e_T} & H_T^*(F) & & & & \\
 & & \cong \big\downarrow \wedge\tau_T & & \big\uparrow & & & & \\
0 & \longrightarrow & H_T^*(M^+, \partial M^-) & \longrightarrow & H_T^*(M^+) & \longrightarrow & H_T^*(M^-) & \longrightarrow & 0,
\end{array}$$

where τ_T, e_T are the equivariant Thom and Euler class of the negative normal bundle of F.

Problem 18.1 ([38]). Derive Lemma 18.3 from the preceding lemma by induction over the critical levels, under the assumption that there exists a T-equivariant Morse-Bott function f with $Crit(f) = Fix(T)$. This applies, e.g., to Hamiltonian T-actions.

Now let (M, ω, μ) be a Hamitonian T-space with finitely many fixed components. Suppose that $\langle \mu, Y \rangle$ is proper and bounded below for some $Y \in \mathfrak{t}$, and that 0 is a regular value.

The inclusion $\mu^{-1}(0) \subset M$ composed with the isomorphism $H_T^*\big(\mu^{-1}(0)\big) \cong H^*\big(M//T(0)\big)$ induces the *Kirwan map*

$$\kappa : H_T^*(M) \to H^*\Big(M//T(0)\Big).$$

Theorem 18.7 (Kirwan surjectivity). *The Kirwan map*

$$\kappa : H_T^*(M) \to H^*\Big(M//T(0)\Big)$$

is surjective.

Proof. (for $T = S^1$). Suppose $T = S^1$. Then the square of the moment map, $f = \mu^2 : M \to \mathbb{R}_{\geq 0}$, is Morse-Bott on every critical level except zero. We want to prove the following statement by induction over the critical levels c: The map

$$H_T^*\Big(f^{-1}[0, c + \varepsilon)\Big) \to H^*\Big(M//T(0)\Big)$$

is surjective.

For $c = 0$ this is obvious. For $c > 0$ it follows by simple diagram chasing from Lemma 18.6.

If T is a higher dimensional torus the function μ^2 is no longer Morse-Bott on positive levels. But Kirwan has shown that it still has the homological properties of a Morse-Bott function that are needed for the argument to go through ([26], cf. [38]). □

Kirwan's theorem provides a tool to compute the cohomology of a symplectic quotient in terms of the equivariant cohomology of the total space. For this, we have to identify the kernel of the Kirwan map. This is done in the following theorem by Tolman and Weitsman, which we will prove in the case $T = S^1$:

Let (M, ω, μ) be as above. For $X \in \mathfrak{t}$ define

$$\begin{aligned} M_X &:= \{x \in M \mid \langle \mu(x), X\rangle \le 0\}, \\ K_X &:= \{\alpha \in H_T^*(M) \mid \alpha|_{M_X} = 0\}. \end{aligned}$$

Theorem 18.8 (Tolman and Weitsman [38]). *The kernel of the Kirwan map equals $K := \sum_{X\in\mathfrak{t}} K_X$.*

Proof. (for $T = S^1$). For $T = S^1$ we have to show that the kernel equals $K = K_+ + K_-$ where

$$\begin{aligned} M_+ := \mu^{-1}[0,\infty), \quad M_- &:= \mu^{-1}(-\infty, 0], \\ K_\pm &:= \{\alpha \in H_T^*(M) \mid \alpha|_{M_\pm} = 0\}. \end{aligned}$$

$K \subset \ker\kappa$ is obvious because the Kirwan map factors through the restrictions $H^*_{S^1}(M) \to H^*_{S^1}(M_\pm)$.

Now let $\alpha \in \ker\kappa$. Order the fixed components $F_0 = \mu^{-1}(0), F_1, \ldots, F_N$ by increasing value of μ^2. By the Localization Theorem it suffices to prove by induction over l:

For every l there exists a $\beta \in K$ such that $\alpha|_{F_i} = \beta|_{F_i}$ for all $i \le l$.

For $l = 0$ this is just the hypothesis $\alpha \in \ker\kappa$. For the induction step, we may assume that $\alpha|_{F_i} = 0$ for $i < l$. By symmetry, assume that $c := \mu(F_l) > 0$. Apply Lemma 18.6 to $f = \mu^2$ at the critical set F_l. Since $\alpha|_{f^{-1}(-\infty,c-\varepsilon)} = 0$, it follows that $\alpha|_{F_l}$ is a multiple of the Euler class $e_T(F_l)$. But F_l is also a critical set for $f = \mu$ with the same negative normal bundle and Euler class. Applying Lemma 18.6 to $f = \mu$, we see that $\alpha_{\mu^{-1}(-\infty,c+\varepsilon)}$ lies in the image of the map

$$H^*_{S^1}\Big(\mu^{-1}(-\infty, c+\varepsilon), \mu^{-1}(-\infty, c-\varepsilon)\Big) \to H^*_{S^1}\Big(\mu^{-1}(-\infty, c+\varepsilon)\Big).$$

This means that there exists a $\beta \in H^*_{S^1}(M)$ such that $\beta|_{F_l} = \alpha|_{F_l}$ and $\beta = 0$ on $\mu^{-1}(-\infty, c-\varepsilon)$. Since $c > 0$, for ε small this implies $\beta \in K$, and $\beta|_{F_i} = \alpha|_{F_i}$ for all $i \le l$.

For the proof for higher dimensional torus actions see [38]. □

Example 18.9. Consider the standard circle action on $\mathbb{C}^n$ with moment map

$$\mu(z) = \frac{1}{2}(|z|^2 - 1).$$

Since M_- contains the origin and the restriction to the origin is injective in equivariant cohomology, $K_- = \{0\}$. M_+ retracts onto S^{2n-1}. Let θ be a connection 1-form on S^{2n-1}. From

$$d_{S^1}\theta = u + d\theta$$

and $(d\theta)^n = 0$ it follows that K_+ is the ideal generated by u^n. Thus

$$H^*_{S^1}(\mathbb{C}^n)\Big/K = \mathbb{R}[u]\Big/ < u^n > \cong H^*(\mathbb{C}P^{n-1}),$$

and the isomorphism is given by $u \mapsto [-d\theta] \in H^2(\mathbb{C}P^{n-1})$.

Example 18.10. Consider the rotation around the z-axis on $S^2 = \{(x,y,z) \mid x^2 + y^2 + z^2 = 1\}$ with moment map

$$\mu(x,y,z) = z.$$

The equivariant cohomology equals

$$H^*_{S^1}(S^2) = \mathbb{R}[u, \mu u - \omega].$$

$K_\pm$ are determined by the restriction to $(0,0,\pm 1)$, thus

$$K_\pm = < u \mp (\mu u - om) >, \quad K = < u, \mu u - \omega > .$$

It follows that

$$H^*_{S^1}(S^2)\Big/K = \mathbb{R}[u, \mu u - \omega]\Big\langle u, \mu u - \omega >= \mathbb{R} \cong H^*(\mathrm{pt}).$$

18.5 Last visit to toric varieties

We keep the notations from Chapter 14. So T^d acts on $\mathbb{C}^d$ in the standard way with moment map

$$\mu(z) = \frac{1}{2}(|z_1|^2, \ldots, |z_d|^2) + \lambda,$$

and $i^*\mu$ is the moment map of the action of H induced by

$$0 \to H \xrightarrow{i} T^d \xrightarrow{\pi} T^n \to 0.$$

The inclusion i induces a natural homomorphism $H^*_{T^d}(\mathbb{C}^d) \to H^*_H(\mathbb{C}^d)$ whose kernel is the ideal

$$I = < \sum a_j u_j \mid a \in \ker i^* = \mathrm{im}\pi^* > .$$

This is precisely the ideal I of linear relations in Theorem 16.7, and we have

$$H^*_H(\mathbb{C}^d) = \mathbb{R}[u_1, \ldots, u_d]\Big/I.$$

For an irrational $X \in \mathfrak{h}$,

$$M_X = \{z \in \mathbb{C}^d \mid \langle i^*\mu(z), X\rangle = \frac{1}{2}\sum X_j|z_j|^2 + c \le 0\},$$

where $X_j = \langle i^* i^*_j, X\rangle$ and $c = \langle i^*\lambda, X\rangle$.

If ≤ 0 then $0 \in M_X$, so $K_X = \{0\}$.

Suppose $c > 0$. Then $z \in M_X$ iff

$$\sum_{i \in I} (-X_i)|z_i|^2 \geq 2c + \sum_{j \notin I} X_j |z_j|^2 \geq 2c,$$

where

$$I = \Big\{ i \in \{1, \ldots, d\} \mid X_i < 0 \Big\}.$$

So M_X is equivalent to

$$\{z \in \mathbb{C}^d \mid \sum_{i \in I} |z_i|^2 > 0\}.$$

Problem 18.2. Prove that the ideal of elements in $H^*_{T^l}(\mathbb{C}^l)$ for the standard action which vanish on $\mathbb{C}^l \setminus 0$ is generated by $u_1 \ldots u_l$.

By the preceding problem,

$$K_X = < u^I > .$$

Thus the ideal K is generated by the monomials u^I for which there exists an $X \in \mathfrak{h}$ such that $\langle i^* \lambda, X \rangle > 0$ and $\langle i^* e_j^*, X \rangle > 0$ for all $j \notin I$. This is the case iff $-i^* \lambda$ does not lie in the cone generated by $i^* e_j$, $j \notin I$.

Now $-i^* \lambda$ lies in the cone generated by $i^* e_j$, $j \notin I$, iff there exists a $z \in \mathbb{C}^d$ with $i^* \mu(z) = 0$ and $z_i = 0$ for $i \in I$, or equivalently, if the intersection of the codimension 1 faces F_i, $i \in I$, is a codimension $|I|$ face.

Thus the ideal K is generated by the monomials u^I for which the intersection of the faces F_i, $i \in I$, is not a codimension $|I|$ face. But this is precisely the ideal J in Theorem 16.7, so we have reproved Theorem 16.7 (for real cohomology): The real cohomology ring of the toric variety equals

$$\mathbb{R}[u_1, \ldots, u_d] \Big/ (I + J).$$

Appendix A

Normal forms near group orbits

The material in this section is not needed in the proofs in this chapter. I have just included it for possible later reference.

Consider a point x in a Hamiltonian G-space with $\mu(x) = 0$. Since $\omega(\underline{X}_x, \underline{Y}_x) = \langle \mu(x), [X, Y] \rangle = 0$ for all $X, Y \in \mathfrak{g}$, the orbit $G \cdot x$ is ω-isotropic. Let

$$E_x := (T_x G \cdot x)^{\perp_\omega} \Big/ T_x G \cdot x$$

be the symplectic normal complement to $T_x G \cdot x$. The stabilizer $H := G_x$ acts linearly on (E_x, ω_x). This representation is called the *isotropy representation.* It can be made unitary (hence Hamiltonian) by choosing an invariant almost complex structure on M. It induces the symplectic normal bundle over $G \cdot x \cong G/H$,

$$G \times_H E_x \to G/H.$$

The form ω_x induces a closed 2-form on the total space of this bundle, still denoted by ω_x, which makes each fibre isomorphic to (E_x, ω_x). By the Isotropic Neighbourhood Theorem, a neighbourhood of $G \cdot x$ is symplectomorphic to

$$T^*(G \cdot x) \times (G \times_H E_x)$$

with the symplectic form $\omega_{\mathrm{st}} \oplus \omega_x$, where $\omega_{\mathrm{st}} = dq \wedge dp$ on $T^*(G \cdot x)$. The proof shows that this symplectomorphism can be made G-equivariant, where G acts on $T^*(G \cdot x)$ by the lift of left multiplication and on $G \times_H E_x$ as in the Slice Theorem.

Finally note that by subtracting a constant from μ the hypothesis $\mu(x) = 0$ can be weakened to $\mu(x) \in center(\mathfrak{g}^*)$. Let us record this result:

Lemma A.1. *A neighbourhood of an orbit $G \cdot x$ with $\mu(x) \in center(\mathfrak{g}^*)$ is* *isomorphic, as a Hamiltonian G-space, to*

$$T^*(G \cdot x) \times (G \times_H E_x)$$

with the symplectic form and G-action as described above and the moment map which agrees with μ at the point x.

Remark. This is a special case of the result in [19] that the neighbourhood of any orbit (even if $\mu(x)$ is not central) is determined uniquely by $\mu(x)$, G_x, and the isotropy representation of G_x on E_x. However, the normal form is not so easy to describe if $\mu(x)$ is not central.

Remark. The first factor in the local model, $T^*(G/H)$, can also be described as follows: Consider a free group action $H \times Q \to Q$. The cotangent bundle of the quotient Q/H is naturally identified with the quotient of the "transverse subbundle" by the lifted action,

$$T^*(Q/H) \cong \{(q,p) \in T^*Q \mid p \text{ vanishes on } T_q(H \cdot q)\}/H.$$

Recall that the moment map of the lifted action on T^*Q is $\langle \mu(q,p), X \rangle = \langle p, \underline{X}^Q_q \rangle$. So the transverse subbundle equals $\mu^{-1}(0)$, and the above identification is just the symplectic quotient

$$T^*(Q/H) \cong T^*Q//H(0).$$

Not surprisingly, this isomorphism also identifies the canonical symplectic form on $T^*(Q/H)$ with the reduced symplectic form. To see this, observe that the retriction of the canonical 1-form $p\,dq$ to $\mu^{-1}(0)$ is H-invariant and horizontal and thus descends to a 1-form on the quotient, and check that this 1-form is the canonical 1-form on $T^*(Q/H)$. This is a special case of the cotangent bundle reduction discussed in [29].

Now apply the preceding discussion to $Q = G$ and $H < G$ acting by right multiplication. The moment map $\mu : T^*G \to \mathfrak{h}^*$ of the lifted action equals $\mu(g,p) = -i^* L_g^* p$, where $i^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is induced by the inclusion $i : \mathfrak{h} \to \mathfrak{g}$. Under the left trivialization

$$G \times \mathfrak{g}^* \to T^*G, \quad (g, \xi) \mapsto (g, L^*_{g^{-1}} \xi),$$

the H-action and moment map become

$$h \cdot (g, \xi) = (gh^{-1}, \mathrm{Ad}^*_{h^{-1}} \xi), \quad \mu(g, \xi) = i^* \xi.$$

So with $\mathfrak{h}^\perp := \ker i^* \subset \mathfrak{g}^*$ the symplectic quotient becomes

$$T^*(G/H) \cong T^*G//H(0) \cong G \times_H \mathfrak{h}^\perp,$$

where H acts on $\mathfrak{h}^\perp$ by the coadjoint action. The G-action on $G \times_H \mathfrak{h}^\perp$ is given by $\tilde{g} \cdot [g, \xi] = [\tilde{g} g, \xi]$. So the local model in the lemma can be written as an associated bundle

$$(G \times \mathfrak{h}^\perp \times E_x)/H.$$

Now specialize to the case that $G = T$ is a torus. Let H be the isotropy group at $x \in M^{2n}$. Since the coadjoint action on $\mathfrak{h}^\perp$ is trivial, the local model is

$$(T \times_H E_x) \times \mathfrak{h}^\perp.$$

If the abelian group H is disconnected the bundle $T \times_H E_x$ can be nontrivial. For H connected this does not happen: In this case we have a splitting $T \cong T^k \times T^l$ with $H = T^l$. The bundle

$$T \times_H E_x \cong T^k \times (H \times_H E_x) \cong T^k \times E_x$$

is naturally trivialized via

$$(H \times_H E_x) \to E_x, \quad [h, v] \mapsto hv,$$

with the residual action of H on E_x given by the isotropy action $(h, v) \mapsto hv$. Let $w_1, \dots, w_{n-k} \in \mathbb{Z}^l$ be the isotropy weights on $E_x \cong \mathbb{C}^{n-k}$, and write elements in T as (τ, σ) with $\tau_i, \sigma_j \in \mathbb{R}/2\pi\mathbb{Z}$. Putting everything together, we have:

Lemma A.2. *In the notation above, the neighbourhood of an orbit $T \cdot x$ with stabilizer $H = T^l$ is isomorphic, as a Hamiltonian T-space, to*

$$T^k \times \mathbb{R}^k \times \mathbb{C}^{n-k}.$$

In coordinates $(\theta, \eta, z = x + iy)$ the symplectic form, T-action and moment map are given by

$$\omega = \sum d\theta_i \wedge d\eta + \sum dx_j \wedge dy_j,$$
$$(\tau, \sigma) \cdot (\theta, \eta, z) = (\theta + \tau, \eta, e^{-i\langle w_j, \sigma\rangle} z_j),$$
$$\mu(\theta, \eta, z) = (\eta, \frac{1}{2} \sum |z_j|^2 |w_j).$$

In particular, on a toric manifold ($l = n - k$) we can pick coordinates on T^{n-k} such that the weights become the standard basis, so

$$(\tau, \sigma) \cdot (\theta, \eta, z) = (\theta + \tau, \eta, e^{-i\sigma_j} z_j),$$
$$\mu(\theta, \eta, z) = (\eta, \frac{1}{2}|z_j|^2).$$

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