Seminar: Gauge theory

Summer term 2018

Prof. Bernd Ammann, Dr. habil. Raphael Zentner

Monday 16-18
Special dates:

- 21.5. Pfingstmontag

1 Aim of the seminar

The main aims of the seminar are to establish a proof of the following two theorems and an understanding of the gauge theoretic methods used for proving these results:

Theorem 1 (Donaldson, 1982) *Let X be a simply-connected closed smooth four-manifold with definite intersection form. Then the intersection form q_X is the standard diagonal intersection form.*

Here the intersection form is the pairing

$$
\begin{aligned}
q_X : H^2(X) \times H^2(X) &\to \mathbb{Z} \\
(a, b) &\mapsto \langle a \cup b, [X] \rangle.
\end{aligned}
$$

Donaldson's original proof of this result uses instanton gauge theory. Meanwhile simpler proofs have appeared, but Donaldson's original one is probably the conceptionally clearest one (and also most beautiful one), and it also illustrates many important phenomena in instanton gauge theory. We will follow Freed and Uhlenbeck's book [2].
Freedman's famous theorem states that for any unimodular bilinear form q over $\mathbb{Z}$ there is a simply connected topological four-manifold which has q as intersection form. There are many non-diagonal definite unimodular forms which are not diagonal. By Freedman's theorem they can be realised as intersection forms of topological four-manifolds, but by Donaldson's theorem they cannot admit a smooth structure.

Theorem 2 (Donaldson, Friedman-Morgan, Okonek-van-de-Ven) *There are infinitely many simply connected closed smooth four-manifolds X_i for $i = 1, 2, \dots$ which are all homeomorphic, but no two are diffeomorphic to each other.*

We will only prove a slightly weaker version of this result, namely that there are two homeomorphic four-manifolds X_1, X_2 which are not diffeomorphic. One will be $X_1 = K3\#\overline{CP}^2$, the blow-up of the K3-surface. The other will be $X_2 = 3\mathbb{CP}^2\#20\overline{\mathbb{CP}^2}$. Our proof will be based on Seiberg-Witten gauge theory, and in particular the Seiberg-Witten invariants.

2 Instanton gauge theory

This part of the seminar addresses instanton gauge theory in particular, and in general analytic concepts valid both in Seiberg-Witten and instanton gauge theory.

A word of warning regarding the early source [2]: The convention in this book is that one studies self-dual connections. People later shifted to study anti self-dual connections instead, because of some natural identifications in complex geometry. The two approaches are entirely equivalent, but one has to bare this in mind if one goes back and forth between this sources and later sources such as Donaldson-Kronheimer's book [1].

Talk no. 1: Bundles and connections, Chern-Weil theory, elliptic complexes, the Yang-Mills equation, Donaldson's theorem. *9.4.* GERRIT HERRMANN.

This is some foundational material used further in the theory. It also covers Chern-Weil theory and elliptic complexes. Donaldson's theorem and its strategy of proof is formulated [2, Chap. 2]

Talk no. 2: Sobolev spaces, the slice theorem, and local structure of the moduli space for generic metrics. *16.4* FELIX EBERHART.

Here the study of the moduli space of instantons starts in earnest. A slice theorem for the action of the gauge group, the group of bundle automorphisms acting on the space of connections, is proved. Furthermore, reducible connections need particular care. Then Freed-Uhlenbeck's metric perturbation theorem is proved (Donaldson's original approach used other kind of perturbations): For a "generic metric" the part of the instanton moduli space consisting of irreducible connections is a smooth manifold, and of dimension 5 for the bundles we consider. [2, Chap. 3]

Talk no. 3: The local structure around the reducibles, and orientability. *23.4.* FELIX EBERHART.

Here the structure of the moduli space around the reducible connections is considered. They will have neighbourhoods consisting of cones over complex projective spaces. We will black-box the orientability of the moduli space or give only a brief sketch of it. [2, Chap. 4 and 5]

Talk no. 4: Instantons on S^4, introduction to Taubes's drafting procedure. *30.4.* BERND AMMANN.

The instanton moduli space of charge 1 over S^4 admit an explicit description due to Atiyah, Hitchin and Singer. To show the existence of non-reducible instantons over negative definite manifold, Taubes used these instantons over S^4 to glue them, in some sense via a connected sum, to a general negative definite four-manifold X, in order to obtain first almost-instantons on X which then are perturbed to yield instantons on X. [2, Chap. 6]

Talk no. 5: Taubes's Theorem. *7.5.* RAPHAEL ZENTNER.

This is the technical heart of Taubes's theorem. It uses tools from analysis and the continuation method to produce instantons via the grafting procedure. [2, Chap. 7]

Talk no. 6: Compactness. *14.5.* Nobuhiko Otaba.
This talk analyses compactness properties of the instanton moduli space, and in particular, it gives a quite explicit understanding in which way compactness can fail, yielding what is now known as the Uhlenbeck compactification of the moduli space. [2, Chap. 8]

Talk no. 7: The collar neighbourhood theorem and Donaldson's proof of his theorem. *28.5.* Raphael Zentner.

Here the compactness properties together with Taubes's theorem give a description of the moduli space "near the ends", and this is then used to give Donaldson's proof of his theorem. [2, Chap. 9]

3 Seiberg-Witten gauge theory

This part of the seminar addresses Seiberg-Witten gauge theory. We will define $Spin^c$ structures, the Dirac operator, and then we shall write down the Seiberg-Witten equations. These are equations involving a section of some spinor bundle and an abelian connection in some line bundle. We will prove compactness of the moduli space, discuss generic perturbations, and define the Seiberg-Witten invariants.

Talk no. 8: $Spin^c$-structures and Dirac operators. *4.6.* Roman Schiessl.
Here the basic spin-geometric objects are introduced necessary to define a Clifford multiplication, a $Spin^c$ structure, and a Dirac operator. [3, Chap. 2 and 3, ignoring 3.4]

Talk no. 9: The Seiberg-Witten equations, compactness of the moduli space. *11.6.* Julian Seipel.
The Seiberg-Witten equations are formulated. A Weitzenböck formula yields an a priori estimate on the spinor section of a solution to the Seiberg-Witten equation. This yields a compactness theorem for the moduli space of the Seiberg-Witten equations. This compactness makes the study of Seiberg-Witten gauge theory significantly simpler compared to instanton gauge theory. A lot of the material is already familiar from the analogous situation in the instanton setting – action of a gauge group, elliptic complexes, reducibles... [3, Chap. 4 and 5]

Talk no. 10: Avoiding reducibles, perturbations of the equations, and the Seiberg-Witten invariants. *18.6.* .
Here we will see how the reducible solutions can be avoided if $b_2^+(X) > 0$, and along one-dimensional families if $b_2^+(X) > 1$. In these cases the moduli space is a smooth manifold cut out transversally by the equations, for perturbation with a generic self-dual 2-form (this is much simpler than in the analogous instanton situation). The "parametrized moduli space" – this is the space of solutions together with the perturbation parameter – yields cobordisms between

moduli spaces associated to fixed parameters. A suitable count of these moduli spaces which only depends on this cobordism class hence yield invariants of the underlying smooth four-manifold. The Seiberg-Witten invariant is such a count, and it is introduced in this talk. [3, Chap. 6, ignoring 6.8 and 6.9]

Talk no. 11: The non-vanishing result for Kähler surfaces (or more generally symplectic 4-manifolds). *25.6.* .
Spin^c-structures for complex manifolds are introduced, and the Dirac operator takes a particularly simple form. Kähler surfaces have non-vanishing Seiberg-Witten invariants, as can be seen by complex geometric methods (solutions have interpretations as divisors). Alternatively one may go straight to Taubes's more general non-vanishing result for symplectic four-manifolds, according to the taste and motivation of the orator.
The a priori estimate from a previous talk imply that a four-manifold with positive scalar curvature metric has vanishing Seiberg-Witten invariants. Hence Kähler surfaces cannot admit positive scalar curvature metrics. Furthermore, we will prove the theorem about pairwise homeomorphic, but not diffeomorphic four-manifolds mentioned in the introduction. [3, Chap 3.4 and Chap. 7]

Talk no. 12: The adjunction inequality, and lower bounds for the genus of 2-dimensional homology classes (The Thom conjecture). *2.7.* .
If time permits we may prove the adjunction inequality and show how this implies the Thom conjecture (in fact, it is easier to prove it for Kähler surfaces of $b_2^+(X) > 1$, and hence not for the original case formulated for $\mathbb{CP}^2$.

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/gauge`

Literatur

[1] S. Donaldson, P. Kronheimer, The geometry of four-manifolds, Oxford University Press

[2] D. Freed, K. Uhlenbeck, Instanton gauge theory and four-manifolds, MSRI 1991

[3] J. Morgan, Seiberg-Witten theory, Princeton University Press