

Seminar: Positive scalar curvature and bordism, Part II

Summer term 2018

Prof. Bernd Ammann, Prof. Niko Naumann, Dr. Nobuhiko Otoba

Tuesday 14.15–16.00, SFB seminar room

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Distribution of talks: At the end of the first session on April 10th.

Goal:

We want to prove the following theorem by Stolz: Assume that M is a closed simply connected spin manifold of dimension n at least 5. Then M carries a metric of positive scalar curvature if and only if $\text{ind}(M) \in \text{KO}^{-n}(pt) = \text{KO}_n(pt) = \text{KO}_n$ vanishes.

A main ingredient in the proof is the map

$$\Psi : \Omega_{n-8}^{\text{Spin}}(\text{BPSp}(3)) \rightarrow \Omega_n^{\text{Spin}}$$

which maps a spin bordism class of a G -principal bundle $P \rightarrow M$ with $G = \text{PSp}(3) = \text{Isom}(\mathbb{H}P^2)$ to the total space of the associated bundle $P \times_{\text{PSp}(3)} \mathbb{H}P^2$. The key step in the proof is to show that the image of Ψ coincides with the kernel of the index map $\Omega_n^{\text{Spin}} \rightarrow \text{KO}_n$.

Talks

The dates after the talks are still preliminary, and will be adapted to the evolution of the seminar.

Talk no. 1: Main result in Stolz' article and collapsing fiber bundles.

April 10+April 17.

State the main result of our seminar (see above). We discuss to which extent this problem has now been reduced to statements about bordism groups which we will investigate in more detail during this semester. In particular we will study the geometry of fiber bundles $F \rightarrow M \rightarrow B$ where F carries a metric g_F of positive scalar curvature such that the structure group of the bundle is within the isometry group of (F, g_F) . Show that this implies that the total space carries a metric of positive scalar curvature as well, whose restriction to the fibers is homothetic to g_F . We also discuss the projective symplectic groups and their action on the quaternionic projective spaces.

Sources: Stolz' article [2], Besse [1] and others.

Talk no. 2: Spectral sequences. *April 24+May 8th.*

[9, Chap. 2] You discuss spectral sequences from scratch. Try to concentrate on the subtle aspects and don't hesitate to tell people to read about the details for

themselves. There is no way to understand anything here without seeing many examples for which you can consult "A user's guide to spectral sequences". (When done properly, this will take at least 2, rather 3 sessions).

Talk no. 3: The Adams spectral sequence. *May 15th.*

[9, Chap. 4] We need to cover 4.1, 4.2 and 4.6. The convergence theorem 4.29 is a bit subtle but unfortunately absolutely indispensable for our application, so it should be treated with some care. (This will fit into 1,5 or rather 2 talks)

Talk no. 4: The dual Steenrod algebra. *May 29th.*

[9, Chap. 7] We need 7.1 through 7.5. This can be done with almost no prerequisites as it is only linear algebra. It is however important to internalize a little the basic structure floating around, including an easy switch between modules and comodules, and a little homological algebra. A very helpful addendum might be the relevant appendix of "Ravenel's green book" [8]. We will need to know about the subalgebras $A(n)$, try to give examples involving $A(1)$ at the prime $p = 2$, as those will be relevant later on.

Talk no. 5: The Homotopy groups of MO. *June 12th.*

You can follow this blog by Akhil Mathew:

<https://amathew.wordpress.com/2012/05/23/the-unoriented-cobordism-ring>

Talk no. 6: The Construction of the map $MSpin \rightarrow ko$. *June 19th.*

The goal is to describe models for the spaces $MSpin$ and KO in terms of orthogonal spectra. Then construct the map $D: MSpin \rightarrow KO$ and discuss its lift to ko . [4, Sec. 3 and following]

Talk no. 7: Construction of the Thom multiplication T . *July 3rd.*

Construct the map $T: MSpin \wedge \Sigma^8 BG_+ \rightarrow MSpin$ following [2][Sec. 2]. This map is important for our seminar, as the induced map on homotopy groups $T_*: \pi_n(MSpin \wedge \Sigma^8 BG_+) \rightarrow \pi_n(MSpin)$ coincides with the map Ψ defined above. The speaker should then prove that the composition

$$MSpin \wedge \Sigma^8 BG_+ \xrightarrow{T} MSpin \xrightarrow{D} ko$$

is null homotopic (Prop. 1.1, proven in Sec. 2 in [2]).

Talk no. 8: Cohomology of $BPSp(3)$. *June 19th.*

We study the $\mathbb{Z}/2$ -cohomology of $BPSp(3)$. We cite Kono's theorem which tells us that in the sense of algebras over $\mathbb{Z}/2$ we have

$$H^*BG \cong \mathbb{Z}/2[t_2, t_3, t_8, t_{12}]$$

where t_i is of degree i . The main subject of the talk is the action of the Steenrod squares on this algebra. [2][Sec. 3]

Talk no. 9: Homology of $MSpin$. *June 26th.*

Discuss the homology of $MSpin$ and the induced map $D_*: H_* MSpin \rightarrow H_* ko$ [2][Sec. 4]

Talk no. 10: Homology of $M\text{Spin}$ -module spectra. *July 3rd.*

The Thom multiplication map T has turned $\Sigma^8 BG_+$ into a $M\text{Spin}$ -module spectrum. We thus need a better understanding of the homology of $M\text{Spin}$ -module spectra which is the subject of this talk. The speaker should follow [2][Sec. 5].

Talk no. 11: The Adams spectral sequence of $M\text{Spin} \wedge BG_+$ collapses. *July 10th.*

Study the Adams spectral sequence of $M\text{Spin} \wedge BG_+$, see [2][Sec. 6].

Talk no. 12: \widehat{T} induces a split surjection. .

We already know from previous talks that the map T lifts to a map $\widehat{T} : M\text{Spin} \wedge BG_+ \rightarrow \widehat{M}\text{Spin}$ where $\widehat{M}\text{Spin}$ is the homotopy fiber of $D : M\text{Spin} \rightarrow \text{ko}$. In this talk one should now prove that \widehat{T} induces a split surjective map of comodules over the dual Steenrod algebra, see [2][Sec. 7].

Talk no. 13: The primitives in $H^* M\text{Spin}$ and $H^*(H^* M\text{Spin}; Q_0)$. *July 10.*

The last piece in the proof of the main result is to understand the structure of $H^* M\text{Spin}$ and $H^*(H^* M\text{Spin}; Q_0)$. This is the subject of this talk, following [2][Sec. 8].

Seminar-Homepage

<http://www.mathematik.uni-regensburg.de/ammann/psc>

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