Seminar: Positive scalar curvature and bordism, Part II

Summer term 2018

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Tuesday 14.15–16.00, SFB seminar room
Distribution of talks:

At the end of the first session on April 10th.
Goal:
We want to prove the following theorem by Stolz: Assume that M is a closed simply connected spin manifold of dimension n at least 5. Then M carries a metric of positive scalar curvature if and only if $\mathrm{ind}(M) \in \mathrm{KO}^{-n}(pt) = \mathrm{KO}_n(pt) = \mathrm{KO}_n$ vanishes.
A main ingredient in the proof is the map

$$\Psi : \Omega^{\mathrm{Spin}}_{n-8}(\mathrm{B}\,\mathrm{PSp}(3)) \to \Omega^{\mathrm{Spin}}_n$$

which maps a spin bordism class of a G-principal bundle $P \to M$ with $G = \mathrm{PSp}(3) = \mathrm{Isom}(\mathbb{H}P^2)$ to the total space of the associated bundle $P \times_{\mathrm{PSp}(3)} \mathbb{H}P^2$. The key step in the proof is to show that the image of Ψ coincides with the kernel of the index map $\Omega^{\mathrm{Spin}}_n \to \mathrm{KO}_n$.

Talks

The dates after the talks are still preliminary, and will be adapted to the evolution of the seminar.

Talk no. 1: Main result in Stolz' article and collapsing fiber bundles. *April 10.* Bernd Ammann.
State the main result of our seminar (see above). We discuss to which extend this problem has now been reduced to statements about bordism groups which we will investigate in more detail during this semester. In particular we will study the geometry of fiber bundles $F \to M \to B$ where F carries a metric g_F of positive scalar curvature such that the structure group of the bundle is within the isometry group of (F, g_F). Show that this implies that the total space carries a metric of positive scalar curvature as well, whose restriction to the fibers is homothetic to g_F. We also discuss the projective symplectic groups and their action on the quaternionic projective spaces.
Sources: Stolz' article [9], Besse [1] and others.

Talk no. 2: Spectral sequences. *April 17.* ??.
[8, Chap. 2] You discuss spectral sequences from scratch. Try to concentrate on the subtle aspects and don't hesitate to tell people to read about the details for

themselves. There is no way to understand anything here without seeing many examples for which you can consult "A user's guide to spectral sequences". (When done properly, this will take at least 2, rather 3 sessions).

Talk no. 3: The Adams spectral sequence. *April 24.* ??.
[8, Chap. 4] We need to cover 4.1, 4.2 and 4.6. The convergence theorem 4.29 is a bit subtle but unfortunately absolutely indispensable for our application, so it should be treated with some care. (This will fit into 1,5 or rather 2 talks)

Talk no. 4: The dual Steenrod algebra. *May 8th.* ??.
[8, Chap. 7] We need 7.1 through 7.5. This can be done with almost no prerequisites as it is only linear algebra. It is however important to internalize a little the basic structure floating around, including an easy switch between modules and comodules, and a little homological algebra. A very helpful addendum might be the relevant appendix of "Ravenel's green book" [7]. We will need to know about the subalgbras $A(n)$, try to give examples involving $A(1)$ at the prime $p = 2$, as those wil be relevant later on.

Talk no. 5: The Homotopy groups of MO**.** *May 15th.* ??.
You can follow this blog by Akhil Mathew:

https://amathew.wordpress.com/2012/05/23/the-unoriented-cobordism-ring

Talk no. 6: Construction of the Thom multiplication T. *May 29.* ??.
Construct the map $T : \mathrm{MSpin} \wedge \Sigma^8 BG_+ \to \mathrm{MSpin}$ following [9][Sec. 2]. This map is important for our seminar, as the induced map on homotopy groups $T_* : \pi_n(\mathrm{MSpin} \wedge \Sigma^8 BG_+) \to \pi_n(\mathrm{MSpin})$ coincides with the map Ψ defined above. The speaker should then prove that the composition

$$\mathrm{MSpin} \wedge \Sigma^8 BG_+ \xrightarrow{T} \mathrm{MSpin} \xrightarrow{D} \mathrm{ko}$$

is null homotopic (Prop. 1.1, proven in Sec. 2 in [9]).

Talk no. 7: Cohomology of $BPSp(3)$. *June 5.* ??.
We study the $\mathbb{Z}/2$-cohomology of $B\mathrm{PSp}(3)$. We cite Kono's theorem which tells us that in the sense of algebras over $\mathbb{Z}/2$ we have

$$H^* BG \cong \mathbb{Z}/2[t_2, t_3, t_8, t_{12}]$$

where t_i is of degree i. The main subject of the talk is the action of the Steenrod squares on this algebra. [9][Sec. 3]

Talk no. 8: Homology of MSpin**.** *June 12.* ??.
Discuss the homology of MSpin and the induced map $D_* : H_* \mathrm{MSpin} \to H_* \mathrm{ko}$ [9][Sec. 4]

Talk no. 9: Homology of MSpin**-module spectra.** *June 19.* ??.
The Thom multiplication map T has turned $\Sigma^8 BG_+$ into a MSpin-module spectrum. We thus need a better understanding of the homology of MSpin-module spectra which is the subject of this talk. The speaker should follow [9][Sec. 5].

Talk no. 10: The Adams spectral sequence of $\mathrm{MSpin} \wedge BG_+$ **collapses.** *June 26.* ??.
Study the Adams spectral sequence of $\mathrm{MSpin} \wedge BG_+$, see [9][Sec. 6].

Talk no. 11: $\widehat{T}$ **induces a split surjection.** *July 3.* ??.
We already know from previous talks that the map T lifts to a map $\widehat{T} : \mathrm{MSpin} \wedge BG_+ \to \widehat{\mathrm{MSpin}}$ where $\widehat{\mathrm{MSpin}}$ is the homotopy fiber of $D : \mathrm{MSpin} \to \mathrm{ko}$. In this talk one should now prove that $\widehat{T}$ induces a split surjective map of comodules over the dual Steenrod algebra, see [9][Sec. 7].

Talk no. 12: The primitives in $H^* \mathrm{MSpin}$ **and** $H^*(H^* \mathrm{MSpin}; Q_0)$**.** *July 10.* ??.
The last piece in the proof of the main result is to understand the structure of $H^* \mathrm{MSpin}$ and $H^*(H^* \mathrm{MSpin}; Q_0)$. This is the subject of this talk, folllowing [9][Sec. 8].

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/psc`

References

[1] Besse, A. L. Einstein Manifolds. No. 10 in Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. Springer-Verlag, 1987.

[2] Epstein, D. Cohomology Operations: Lectures by N.E. Steenrod. Princeton University Press, 1962.

[3] Gromov, M., and Lawson, H. B. The classification of simply connected manifolds of positive scalar curvature. Ann. of Math. (2) 111, 3 (1980), 423–434.

[4] Kosinski, A. A. Differential manifolds, vol. 138 of Pure and Applied Mathematics. Academic Press Inc., Boston, MA, 1993.

[5] Lawson, H. B., and Michelsohn, M.-L. Spin Geometry. Princeton University Press, Princeton, 1989.

[6] Mosher, R. E., and Tangora, M. C. Cohomology operations and applications in homotopy theory. Harper & Row, Publishers, New York-London, 1968.

[7] Ravenel, D.C. Complex cobordism and stable homotopy groups of spheres

[8] Rognes, J. The Adams Spectral Sequence. https://folk.uio.no/rognes/papers/notes.050612.pdf

[9] Stolz, S. Simply connected manifolds of positive scalar curvature. Ann. of Math. (2) 136, 3 (1992), 511–540.

[10] Stolz, S. Manifolds of positive scalar curvature. In Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), vol. 9 of ICTP Lect. Notes. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 661–709.

[11] Switzer, R. M. Algebraic topology—homotopy and homology. Springer-Verlag, New York-Heidelberg, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

[12] Walsh, M. Metrics of positive scalar curvature and generalised Morse functions, Part I. Mem. Amer. Math. Soc. 209, 983 (2011).

[13] Walsh, M. Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics. Proc. Amer. Math. Soc. 141, 7 (2013), 2475–2484.

[14] Walsh, M. Metrics of positive scalar curvature and generalised Morse functions, Part II. Trans. Amer. Math. Soc. 366, 1 (2014), 1–50.