Seminar: Analytic K-homology

Winter term 2018/19

Prof. Bernd Ammann

Monday 16-18

1 Aim of the seminar

In the seminar we develop an analytic approach for K-homology. This establishes links between Riemannian geometry, algebraic topology and functional analysis. The origins of the area lie on the one hand side in the theory of linear operators on Hilbert spaces, and on the other side on the Atiyah-Singer index theorem which already provides a method to calculate the Fredholm index of an elliptic operator in terms of characteristic classes from algebraic topology. One goal of the seminar is an analytic definition of Kasparov's K-homology and the associated index pairing. K-homology can also be defined abstractly as the homology theory dual to the generalized cohomology theory given by Atiyah-Hirzebruch K-theory. Thus our seminar provides a concrete and geometric picture of this homology theory, establishing a helpful link to operator theory.

An important result in the seminar will be the Brown-Douglas-Fillmore theorem which states that two essentially normal operators T_1 and T_2 with the same essential spectrum X are essentially unitarily equivalent if and only if $\mathrm{Index}(T_1 - \lambda\,\mathrm{id}) = \mathrm{Index}(T_2 - \lambda\,\mathrm{id})$ for every $\lambda \in \mathbb{C} \setminus X$.

The seminar might be continued in the summer term, discussing the Kasparov product, KK-theory, further links to the classical Atiyah-Singer index theorem, index theory for hypersurfaces, higher index theory and obstructions against metrics of positive scalar curvature.

2 Talks

We will mainly follow a textbook written by Nigel Higson and John Roe [4]. Almost all chapters have about the same size and probably contain a bit more information than what can be presented in one session. Additionally each chapter contains some very helpful exercise. We thus reserve two sessions per chapter, but the speaker should leave some time to discuss the exercises (and choose some exercises as homework for the audience).

Talk no. 1: C^*-algebras and operator theory. *15.10. + 22.10.* N.N..
This chapter is summarizes important facts about operator algebras and C^*-algebras [4, Chap. 1]. *Remark: The structures and results of this chapters are also important in quantum mechanics (in particular the GNS construction) and in spectral theory.*

Talk no. 2: Index theory and extensions. *29.10. + 5.11.* GUADALUPE CASTILLO.
In this talk we discuss Fredholm operators, the essential spectrum, Toeplitz operators and essentially normal operators. This then leads to various extensions and the Calkin algebra. [4, Chap. 2]

Talk no. 3: Completely positive maps. *12.11. + 19.11.* N.N..
Part I: Completely positive maps, quasicentral approximate units, nuclearity, statement of Voiculescu's theorem and block-diagonal matrices [4, Sec. 3.1 to 3.5]
Part II: Proof of Voiculescu's theorem. Property T and Ext, Kasparov's Technical Theorem [4, Sec. 3.6 to 3.10]

Talk no. 4: K-Theory. *26.11. + 3.12.* N.N..
The talk should follow [4, Sec. 4]. Additional information can be obtained in [5] and [3].

Talk no. 5: Duality Theory. *10.12.* N.N..
Discuss the dual C^*-algebra associated to a representation on a Hilbert space. This allows us to see extension groups as K-groups (Prop 5.1.6). We now can define (relative) K-homology and discuss its basic properties [4, Sec. 5]. As this chapter is shorter than the other ones, it would be good to do it in one single session.

Talk no. 6: Coarse Geometry and K-Homology. *17.12. + 7.1.* SEBASTIAN RUPPRECHT.
Coarse structures are a tool to discuss the global structure of non-compact spaces, in a way which is robust under local (e.g. compact) changes of the space. In this talk we discuss coarse structures, their relation to metric structures and the associated K-theory [4, Sec. 6].

Talk no. 7: The Brown-Douglas-Fillmore theorem. *14.1. + 21.1.* N.N..
In this talk we proof the Brown-Douglas-Fillmore theorem (stated above) which is one of the mot important results of the seminar. [4, Sec. 7]. It is a consequence of an index map

$$\text{Index} : K^p(X(X)) \to \text{Hom}(K_p(C(X)), \mathbb{Z})$$

which is an isomorphism for all p.

Talk no. 8: Kasparov's K-homology. *28.1. + 4.2.* N.N..
The previous definition of K-homology depends of a choice of representation for each C^*-algebra. The goal of this talk is to avoid this dependence by using Fredholm modules over C^*-algebras. This leads to Kasparov's approach to K-homology [4, Sec. 8].

Seminar-Homepage

`http://www.mathematik.uni-regensburg.de/ammann/k-homology`

Literatur

[1] Bruce Blackadar; K-Theory for Operator Algebras; MSRI Research Publications

[2] Joachim Cuntz, Ralf Meyer, Jonathan Rosenberg; Topological and Bivariant K-Theory; Oberwolfach Seminars **36**; Birkhäuser

[3] Thomas Friedrich; Vorlesungen über K-Theorie; B. G. Teubner-Verlagsgesellschaft, Leipzig; 1978

[4] Nigel Higson, John Roe; Analytic K-Homology; Oxford Mathematical Monographs; Oxford University Press

[5] M. Rørdam, F. Larsen, N. J. Laustsen; An Introduction to K-Theory for C^*-Algebras; London Mathematical Society Student Texts **49**