# Seminar: The $h$-cobordism theorem

## Winter term 2019/20

## Prof. Bernd Ammann, Dr. habil Raphael Zentner

Monday 16.15-18.00
**Special dates:**

- Nov 18: no session
- Tuesday, Dec 3rd, 4-6 pm, instead of Monday Dec 2nd
- What to do on Dec 9th? Intersection with Windberg meeting
- Christmas break already on Dec 23rd

# Summary

In the main part of the seminar we want to treat the classical h-cobordism theorem, following the classical reference by Milnor. This should take between 6 and 8 talks of 90 minutes. We should have time for several extension: we preent several alternatives: e.g. the s-cobordism theorem and the obstruction coming from Whitehead torsion, and considerations which lead to Morse homology.

# I. The $h$-cobordism theorem

A (co)bordism is a (differentiable) compact manifold $W$ with boundary $\partial W$, together with a decomposition $\partial W = M_1 \amalg M_2$ of the boundary into two open and closed subset $M_1$ and $M_2$.
Suppose that $W$ is simply connected and $M_1 \hookrightarrow W$ and $M_2 \hookrightarrow W$ are homotpy equivalences. The $h$-cobordism theorem states that under these conditions $W$ is diffeomorphic to a cylinder $M_1 \times [0,1]$ ist.
Our goal is to prove this theorem.
As an application we will the Poincaré conjecture in higher dimension: every closed simply connected differentiable manifold of dimension $n \geq 5$ with the integral homology of a sphere $S^n$ is also homeomorphic to $S^n$.
In the case $n = 5$ and $n = 6$ we even obtain "diffeomorphic to a sphere". However in dimensions above 6 there are closed manifolds homeomorphic to a sphere, but not diffeomorphic. Such manifolds are called exotic spheres. The $h$-cobordism also tells us that any such exotic $n$-dimensional sphere is obtained by gluing two $n$-dimensional closed balls $B_1$ and $B_2$ together at the boundary by a diffeomorphism $\partial B_1 \cong S^{n-1} \to \partial B_2 \cong S^{n-1}$.
Further applications can be found in [11, Chap. 9]. A crucial step in the proof is to simplify Morse functions on a bordism.
For an experienced audience the first three talks could be condensed to one talk.

**Talk no. 1: Summary of Morse theory.** *14.10.* ALEXANDER KÖNIG.
A quick summary about Morse functions on compact manifolds. The idea of the talk is that someone who has attended the seminar in the summer term about Morse theory summarizes the main facts and some applications of Morse theory.
The talk should cover the definitions and results from [11, Sec. 2]. More material may be taken e.g. from [12] according to a choice of the speaker. However, the talk should not exceed one session.

**Talk no. 2: Elementary Cobordisms.** *21.10.* JONATHAN GLÖCKLE.
Introduction to elementary cobordisms. These are cobordisms which carry a Morse function with exactly one (non-degenerate) critical point. The index $\lambda$ of the Hessian of the Morse function in this point is called the index of the elementary cobordism. Relations to handle attachments (Theorem 3.14), relative homology groups $H_*(W, M_1)$.
*Literature: [11, Sec. 3].*

**Talk no. 3: Rearrangements of Cobordisms.** *28.10.* GUADALUPE CASTILLO SOLANO.
Any cobordism can be decomposed into elementary cobordisms. Suppose that we have a cobordism $W$ that can be decomposed in two elementary cobordisms. A rearrangement of $W$ is by definition a way to decompose $W$ in another way into two elementary cobordisms.
*Literature: [11, Sec. 4].*

**Talk no. 4: A cancellation theorem.** *4.11.* + *11.11.* (First half) BERND AMMANN.
We present a theorem which can be used that under suitable conditions the composition of a cobordism of index $\lambda$ with a cobordism of index $\lambda + 1$ is diffeomorphic to a cylinder. So in some sense the two bordism cancel each other. *Literature: [11, Sec. 5].*

**Talk no. 5: A stronger cancellation theorem.** *11.11.* (Second half) + *25.11.* BERND AMMANN.
The same conclusion using weaker assumptions, but additionally simply connectedness of the bordism and the boundaries, plus a dimesion condition.
*Literature: [11, Sec. 6].*
**No Talk:** *18.11.* (Several participants absent)

**Talk no. 6: Cancelation in the middle dimensions.** ***3.12.*** + *16.12.* (First half) FELIX EBERHART.
*Literature: [11, Sec. 7].*
**No Talk:** *9.12.* (Several participants absent)

**Talk no. 7: Cancelation in dimension** 0 **and** 1. *16.12.* (Second half) + **7.1.** ROMAN SCHIESSL.
*Literature: [11, Sec. 8].*

**Talk no. 8: $h$-cobordism theorem and applications.** *13.1.* JULIAN SEIPEL.
*Literature: [11, Sec. 9].*

**Supplementary Talk no. 1: Morse homology** *.1 overview talk, or 2 talks* As we will have seen at this point ([11, Sec. 7, page 98]), the cancellation in the middle dimension relies on studying a sequence of free abelian groups

$$C_{n-2} \to C_{n-3} \to C_{n-4} \to \ldots \to C_2,$$

where $n$ is the dimension of the bordism, and where $C_\lambda$ is the free abelian group generated by the critical points of index $\lambda$. In fact, this leads to a definition of Morse homology, which puts it into a more conceptual framework [2, Chap. 2 to 4]. It turns out that Morse homology is isomorphic to the standard homology theories for smooth manifolds with boundary (e.g. cellular), so drom a purely topological point of view, nothing new is achieved. The advantage is that it prepares the ground for an introduction of Floer Homology for symplectic manifolds. Thus this talk is particularly interesting if followed by the following talk. *Literature: [2, Chap. 2 to 4].*

**Supplementary Talk no. 2: Floer homology** *.Can range from an overview talk to a subject for the summer term* Symplectic manifolds, Arnold conjecture, Floer Homology. *Literature: [2, Chap. 5 and following].*

# II. Extensions

Remaining sessions: *20.1.* , *27.1.* , *3.2.* , *next term* , *next term* , *next term* , *next term* . We can either choose supplementary talks as above, or one/several one of the following topics.

## 1. Exotic spheres

**Talk no. 9: The group of homotopy spheres.** *1 or 2 sessions* José Q.. *Literature: an important original article is [9], a good overview article is [13], a lecture book where exotic spheres are considered is [8].* More literature can be found on `https://www.maths.ed.ac.uk/~v1ranick/exotic.htm`, in particular the (English) Wikipedia page gives a good overview: `https://en.wikipedia.org/wiki/Exotic_sphere`.
To prepare the talk a good strategy can be to read [13] first and then to add various material from other sources.

**Talk no. 10: Geometric aspects of exotic spheres.** After this one could discuss curvature properties for such spheres [7], [4]

## 2. The $s$-cobordism theorem

**Talk no. 11: The $s$-cobordism theorem and Whitehead torsion.**
The $s$-cobordism theorem generalizes the $h$-cobordism theorem to closed manifolds with arbitrary fundamental group $\pi_1$. An new obstruction arises: the Whitehead torsion of the bordism is an element in the Whitehead group $Wh(\pi_1)$ which is a modification of the group $K_1(\mathbb{Z}\pi_1)$. The bordism is diffeomorphic to

a cylinder if and only if – additionally – the Witehead torsion vanishes.
*Literature: [3, Section 1 and 2]*
*This talk can be also a first step to enter into surgery theory [3].*

### 3. Morse homology and Floer Homology

*As presented in the supplementary talks sketched above.*

## Seminar Homepage

`http://www.mathematik.uni-regensburg.de/ammann/h-cobord`

## References


[1] T. Bröcker, K. Jänich, Einführung in die Differentialtopologie, Springer, 1990.

[2] M. Audin, M. Damian, Morse Theory and Floer homology, Springer Universitext, 2014.

[3] D. Crowley, W. Lück, T. Macko, Surgery theory: Foundations, book project, downloadable at `http://www.mat.savba.sk/~macko/surgery-book.html`

[4] S. Goette, M. Kerin, K. Shankar Highly connected 7-manifolds and non-negative curvature, `https://arxiv.org/abs/1705.05895`

[5] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002
available at `http://www.math.cornell.edu/~hatcher`.

[6] M.W. Hirsch, Differential topology, Graduate Texts in Mathematics **33**, Springer-Verlag, New York, 1994.

[7] M. Joachim, D. Wraith, Exotic spheres and curvature, Bull. AMS **45** (2008), 595–616.

[8] A. Kosinski, Differential manifolds, Pure and Applied Mathematics **138**, Academic Press Inc., Boston (Mass. USA), 1993.

[9] M. Kervaire, J. Milnor, Groups of homotopy spheres I, Ann. of Math. (2) **77** (1963), 504–537.

[10] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math. (2) **64** (1956), 399–405.

[11] J. Milnor, Lectures on the $h$-cobordism theorem, Princeton University Press, Princeton, 1965.

[12] *J. Milnor, Morse theory, Princeton University Press, Princeton, 1963.*

[13] *J. Milnor, Differental topology: forty-six years later, Notices AMS,* **58** *(2001), 804–809.* `https://www.maths.ed.ac.uk/~v1ranick/papers/hedrick2.pdf`