Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
Prof. Bernd Ammann, Jonathan Glöckle, Roman Schießl
Exercises during the week April 12 to 16

Online exercise sheet

1. Exercise.
We assume that (M, g) is a semi-Riemannian manifold.

(a) Let g_{ij} be the coefficients of g with respect to a given chart. Let γ^j be the coefficient of a geodesic. Argue why the following local formulas that we already know in the Riemannian setting also hold for semi-Riemannian manifolds:

$$\Gamma_{ij}{}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$$
$$\ddot{\gamma}^k(t) = -\Gamma_{ij}{}^k \dot{\gamma}^i(t)\dot{\gamma}^j(t)$$
$$R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \Gamma_{jk}{}^m \Gamma_{im}{}^l - \Gamma_{ik}{}^m \Gamma_{jm}{}^l$$

Note that we are using the Einstein summation convention here.

(b) Let $e_1, \ldots, e_n$ be a generalized orthonormal basis of T_pM, and let $e_1^\flat, \ldots, e_n^\flat$ be the basis of T_p^*M consisting of the metric duals $e_i^\flat = g(e_i, -)$. In the following, let $R \in \Gamma(T^{1,3}M)$, ric $\in \Gamma(T^*M \otimes T^*M)$, Ric $\in \Gamma\big(\mathrm{End}(TM)\big)$ and scal $\in C^\infty(M)$ be the Riemann curvature tensor, the Ricci tensor, the Ricci endomorphism and the scalar curvature function, respectively. Please add the correct signs $\epsilon_i := g(e_i, e_i)$ in the following Riemannian formulas in order to adapt them to the semi-Riemannian situation.

$$\mathrm{ric}(X,Y) = \sum_{i=1}^n \; e_i^\flat\big(R(e_i, X)Y\big) \qquad \mathrm{Ric}(X) = \sum_{i=1}^n \; R(X, e_i)e_i$$
$$\mathrm{scal} = \sum_{i=1}^n \; \mathrm{ric}(e_i, e_i) \qquad \mathrm{scal} = \sum_{i=1}^n \; e_i^\flat\big(\mathrm{Ric}(e_i)\big)$$

(c) Will the formulae change, if we replace the metric dual $e_j^\flat$ by the dual basis in the sense of finite-dimensional vector spaces, i.e. by $e^j \in T^*M$ with $e^j(e_k) = \delta_k^j$?

2. Exercise.
Let $n \geq 1$ and g be a non-degenerate symmetric bilinear form on a real $n+1$-dimensional vector space V of signature $(n, 1)$. In this case, a vector $v \in V \smallsetminus \{0\}$ is called *causal* if $g(v,v) \leq 0$. Show the following:

(a) For all causal $v, w \in V$ the *inverse Cauchy-Schwarz inequality* holds:

$$g(v,w)^2 \geq g(v,v)g(w,w) \tag{1}$$

(b) The set of causal vectors in $V \smallsetminus \{0\}$ has precisely two components. If two causal vectors $v, w \in V$ belong to the same component, then $g(v,w) \leq 0$.

(c) If $v, w \in V$ belong to the same component of causal vectors, then the *inverse triangle equality* holds:

$$\sqrt{-g(v+w, v+w)} \geq \sqrt{-g(v,v)} + \sqrt{-g(w,w)} \tag{2}$$

When does equality hold in (1) and (2)?

3. Exercise.
Consider the most common curvature notions:

- Riemannian curvature tensor: R
- Sectional curvature: K
- Ricci tensor: ric
- Ricci endomorphism: Ric
- Ricci curvature map: $TM \ni v \mapsto \mathrm{ric}(v, v)$
- Scalar curvature: scal

(a) Which of these determines which others?

(b) How does the answer to part (a) change, when the manifold has low dimension?

(c) Do your findings also hold for semi-Riemannian manifolds?