Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, April 20**

Exercises Sheet no. 1

1. Exercise (4 points).
Let g be a symmetric bilinear form on a finite-dimensional vector space V, and let n_+, n_0 and n_- be the numbers associated to a basis as in Sylvester's law of inertia. Calculate

$$\max\left\{\dim W \,\middle|\, W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive definite}\right\}$$
$$\max\left\{\dim W \,\middle|\, W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative definite}\right\}$$
$$\max\left\{\dim W \,\middle|\, W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive semi-definite}\right\}$$
$$\max\left\{\dim W \,\middle|\, W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative semi-definite}\right\}$$
$$\max\left\{\dim W \,\middle|\, W \text{ is a linear subspace of } V \text{ with } g|_{W\times W} = 0\right\}$$

in terms of n_+, n_0 and n_-. Conclude that n_+, n_0 and n_- do not depend on the chosen basis.

2. Exercise (4 points).
Let $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ be the standard scalar product of $\mathbb{R}^{n,k}$. Show that the pseudosphere

$$\mathbb{S}^{n-1,k} := \left\{x \in \mathbb{R}^{n,k} \,\middle|\, \langle\!\langle x, x\rangle\!\rangle = 1\right\}$$

is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}^k$, and show that the pseudohyperbolic space

$$\mathbb{H}^{n,k-1} := \left\{x \in \mathbb{R}^{n,k} \,\middle|\, \langle\!\langle x, x\rangle\!\rangle = -1\right\}$$

is diffeomorphic to $\mathbb{R}^n \times \mathbb{S}^{k-1}$.

3. Exercise (4 points).
We define $\mathcal{L}(c)$ as in Definition 1.2.7 of the script.

a) For any nonspacelike C^1-curve c in $\mathbb{R}^{n,1}$ starting from 0 the endpoint x is nonspacelike with
$$\mathcal{L}(c) \leq \sqrt{-\langle\!\langle x, x\rangle\!\rangle}. \tag{1}$$

b) Let $x \in \mathbb{R}^{n,1}$ be nonspacelike. Determine all nonspacelike C^1-curves c from 0 to x for which we have equality in (1).

c) Does (1) also hold for nonspacelike piecewise C^1-curves c?

d) Now let x be lightlike. Determine all nonspacelike C^1-curves c from 0 to x.

4. Exercise (4 points).
Consider $M = (2m, \infty) \times S^2 \ni (r, x)$ with the Riemannian metric

$$g(r,x) = \frac{1}{1 - \frac{2m}{r}} \mathrm{d}r^2 + r^2 g_{S^2}(x),$$

where g_{S^2} denotes the standard metric on the 2-dimensional sphere.

a) Determine a function $h\colon (2m, \infty) \to \mathbb{R}$ such that $M \to \mathbb{R}^4$, $(r, x) \mapsto (rx, h(r))$ is an isometric embedding ($\mathbb{R}^4$ carries the standard flat metric).

We fix such an embedding and identify M with its image.

b) Determine a unit normal vector field $\nu\colon M \to \mathbb{R}^4$ on M. Calculate the Weingarten map $W \in \Gamma(\mathrm{End}(TM))$ of the embedding and its eigenvalues, the *principal curvatures* of M in $\mathbb{R}^4$.
Hint: Recall that a unit normal is characterized by $\langle \nu(p), \nu(p) \rangle = 1$ and $\langle \nu(p), X \rangle = 0$ for all $p \in M$ and $X \in T_pM$, and the Weingarten map is given by $W(X) = -D\nu(X)$.

c) Calculate the Riemann curvature tensor of g.
Hint: This is most easily done in a g-orthonormal basis of eigenvectors of W using the Gauß formula (for the special case of hypersurfaces in Euclidean $\mathbb{R}^n$)

$$R(X,Y)Z = g(W(Y), Z)W(X) - g(W(X), Z)W(Y).$$

d) Verify that (M, g) is scalar-flat but not Ricci-flat.