Differential Geometry II: Exercises

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Please hand in the exercises until **Tuesday, April 27**

Exercises Sheet no. 2

1. Exercise (4 points).
Let $m \in \mathbb{N}_{>0}$ and $x \in \mathbb{R}^{m,1}$, $x \neq 0$ spacelike. In the following, we write the coefficients of $y \in \mathbb{R}^{m,1}$ always as y^0, y^1, $\dots y^m$.

a) Show that there are matrices $A, B, C \in \mathrm{SO}_\uparrow(m,1)$ with $(Ax)^0 < 0$, $(Bx)^0 = 0$, and $(Cx)^0 > 0$.

b) Determine a Poincaré transformation $\mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ that preserves time-orientation and that maps $(1,0,0)^\intercal$ to $(1+\sqrt{2},1,1)^\intercal$, $(1,2,0)^\intercal$ to $(3+\sqrt{2},1+2\sqrt{2},1)^\intercal$, and $(1,0,2)^\intercal$ to $(1+\sqrt{2},1,3)$

c) We define $L_i := \mathbb{R} \times \{i\} \subset \mathbb{R}^{1,1}$ for $i \in \{0,1\}$. Calculate $\ell \in \mathbb{R}$ such that

$$\begin{pmatrix} 0 \\ \ell \end{pmatrix} + B_\alpha \cdot L_0 = B_\alpha \cdot L_1$$

where $B_\alpha = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}$.

2. Exercise (2+0.5+1+0.5 points).

a) Let g and g' be symmetric forms of signature $(m,1)$ on V. Assume that

$$\forall v \in V : g(v,v) = 0 \Leftrightarrow g'(v,v) = 0.$$

Show that there is a constant $\lambda \in \mathbb{R}_{\neq 0}$ with $g' = \lambda g$.

b) Show that $\lambda > 0$ if $m \geq 2$.

c) Let $f : \mathbb{R}^{m,1} \to \mathbb{R}^{m,1}$, $m \geq 2$ be an isomorphism of vector spaces, that maps lightlike vectors to lightlike vectors. Show that there is a constant $\mu \in \mathbb{R}_{>0}$ such that $\mu \cdot f \in \mathrm{O}(m,1)$.

d) Show by providing a counterexample that c) does not hold for $m = 1$.

3. Exercise (4 points).
For an open subset $\Omega \ni 0$ of $\mathbb{R}^{n,1}$ define

$$I_+^\Omega(0) := \{x \in \Omega \mid \exists \text{ future-directed timelike piecewise } C^1 \text{ curve in } \Omega \text{ from } 0 \text{ to } x\}$$

and

$$J_+^\Omega(0) := \{x \in \Omega \mid \exists \text{ future-directed causal piecewise } C^1 \text{ curve in } \Omega \text{ from } 0 \text{ to } x\} \cup \{0\}.$$

Here we only allow piecewise C^1 curves $c\colon [a,b] \to \Omega$ with $b > a$. Such a curve is called future-directed, if for any subinterval $[a_{j-1}, a_j]$ on which c is C^1, we have

$$\forall t \in [a_{j-1}, a_j] : \left(\frac{d}{dt}\left(c|_{[a_{j-1},a_j]}\right)\right)^0 > 0.$$

In the following $\overline{A}$ is the closure of A in Ω.

a) Compute $I_+^\Omega(0)$, $J_+^\Omega(0)$ and $\overline{I_+^\Omega(0)}$ for $\Omega := \mathbb{R}^{n+1}\backslash(1,1,0,\dots,0)^\intercal$.

b) Compute $I_+^\Omega(0)$, $J_+^\Omega(0)$ and $\overline{I_+^\Omega(0)}$ for $\Omega := \mathbb{R}^2\backslash\{(1,t)^\intercal | t \in [0,\infty)\}$.

c) Prove that $I_+^\Omega(0)$ is dense in $J_+^\Omega(0)$ for any Ω as above.

4. Exercise (4 points).
Let X be a smooth vector field on a manifold M, and let $p \in M$. From ODE theory it follows that there is an open neighbourhood U of p, a non-empty interval $(-\varepsilon,\varepsilon)$ and a smooth map $\phi^X : U \times (-\varepsilon,\varepsilon) \to M$, $(x,t) \mapsto \phi_t^X(x)$ such that ϕ_t^X is a diffeomorphism to its image for all $t \in (-\varepsilon,\varepsilon)$ and that for all $x \in U$ the curve $\gamma(t) := \phi_t^X(x)$ satisfies

$$\gamma(0) = x \text{ and } \dot\gamma(t) = X|_{\gamma(t)}.$$

Such a ϕ^X is called a local flow of X. Let further ω be an (r,s)–tensor and recall that the diffeomorphism ϕ_t^X induces isomorphisms

$$\mathrm{d}\phi_t^X : T_pM \to T_{\phi_t^X(p)}M \text{ and } (\mathrm{d}\phi_t^X)^* : T^*_{\phi_t^X(p)}M \to T_p^*M,\ \alpha \mapsto \alpha \circ \mathrm{d}\phi_t^X$$

and

$$I := \left((\mathrm{d}\phi_t^X)^{-1}\right)^{\otimes r} \otimes \left((\mathrm{d}\phi_t^X)^*\right)^{\otimes s} : T^{(r,s)}_{\phi_t^X(p)}M \to T_p^{(r,s)}M,$$

which allows to pull back (r,s)-tensors: $((\phi_t^X)^*\omega)|_p = I \circ \omega|_{\phi_t^X(p)}$. The *Lie derivative* of ω by X at p is then defined by

$$(\mathcal{L}_X(\omega))|_p := \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\left((\phi_t^X)^*\omega\right)|_p,$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes derivation in ordinary vector spaces.

Prove the following equalities:

- $\mathcal{L}_X(f) = X(f)$ for functions f,
- $\mathcal{L}_X(\omega\otimes\sigma) = (\mathcal{L}_X(\omega))\otimes\sigma + \omega\otimes\mathcal{L}_X(\sigma)$ for two tensors ω and σ,
- $\mathcal{L}_X(\omega(Y_1,\dots,Y_s)) = \mathcal{L}_X(\omega)(Y_1,\dots,Y_s) + \sum_{i=1}^s \omega(Y_1,\dots,\mathcal{L}_X(Y_i),\dots,Y_s)$ for an (r,s)–tensor ω and vector fields $Y_1,\dots,Y_s$.

Prove that for a vector field Y it holds that $\mathcal{L}_X(Y) = [X,Y]$. *Hint: Work in a chart described by the flow.*