Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, May 25**

Exercises Sheet no. 6

1. Exercise (4 points).
Continuing exercise 1 of sheet 5 let (M, g) be a semi-Riemannian manifold and let $\gamma : (a, b) \to M$ be a geodesic. Consider the manifold $M = (a, b) \times S^{n-1}$ equipped with the warped product metric

$$g = dt \otimes dt + w(t)^2 g_{\text{sph}},$$

where g_{sph} is the standard metric on S^{n-1} and $w > 0$ is smooth. We write $\gamma(\tau) = (\gamma^1(\tau), \sigma(\tau))$ with $\sigma(\tau) \in S^{n-1}$.

a) Show that if $F \subset N$ is a submanifold lying in the fixed point set of an isometry φ of a semi-Riemannian manifold N and some geodesic $c : (a, b) \to N$ satisfies $\dot{c}(t_0) \in T_{c(t_0)}F, t_0 \in (a, b)$, then $\text{im}(c)$ is contained in the fixed point set of φ.

b) Show that the image of σ lies on a great circle (= image of a geodesic) of S^{n-1}.

c) Show that there are constants $c_1, c_2 \geq 0$ with

$$\left(\dot{\gamma}^1(\tau)\right)^2 = c_1 - \frac{c_2}{w\left(\gamma^1(\tau)\right)^2}.$$

Hint: Choose the skew-symmetric matrix of exercise 1, sheet 5 properly.

d) Assume additionally $-\infty < a < b < \infty$ and $\lim_{t\to a} w(t) = \lim_{t\to b} w(t) = 0$. Determine all geodesics on (M, g) that cannot be extended to geodesics defined on the whole of $\mathbb{R}$.

2. Exercise (4 points).
Let (M, g) be a Lorentzian manifold. Show that there is a time-orientable Lorentzian manifold $(\hat{M}, \hat{g})$ and an isometric local diffeomorphism $f : \hat{M} \to M$ that is surjective and such that each point in M has two preimages. (This Lorentzian manifold $(\hat{M}, \hat{g})$ is called the time-orientation covering of (M, g).)
Hint: Define a suitable topology, an atlas and a Lorentzian metric $\hat{g}$ on

$$\hat{M} := \{(p, \mathcal{O}_p) \mid p \in M,\ \mathcal{O}_p \text{ a time-orientation on } (T_pM, \hat{g}_p)\}.$$

3. Exercise (4 points).
Recall from sheet 5 the Schwarzschild space (M, g), where

$$g(t,r,x) = -\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} \mathrm{d}r^2 + r^2 g_{S^{n-1}}(x) \text{ and } M = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$$

and the Schwarzschild halfplane is the manifold (P, g'), where

$$g'(t,r) = -\left(1 - \frac{2m}{r^{n-2}}\right) \mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} \mathrm{d}r^2 \text{ and } P = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty)$$

with the smooth projection $\pi : M \to P, (t,r,x) \mapsto (t,r)$. Let $\gamma : I \to M$ be a causal curve. Prove the following:

a) The curve $\pi \circ \gamma : I \to P$ is causal.

b) If $\pi \circ \gamma(\tau) = (t(\tau), r(\tau))$ runs towards $r(\tau) \to (2m)^{\frac{1}{n-2}}$, then $t(\tau) \to \pm\infty$.

4. Exercise (4 points).
Let M be a time-oriented Lorentzian manifold. A causal future-directed piecewise C^1-curve $c : I \to M$ is called C^0-future (resp. past) inextendible if there is no smooth orientation preserving diffeomorphism $\phi : I \to \tilde{I}$ with $\tilde{I}$ a bounded interval, such that $c \circ \phi^{-1}$ can be extended to a continuous map $\hat{c} : J \to M$ such that $\{t \in J \mid t > s \, \forall s \in \tilde{I}\} \neq \emptyset$ (resp. $\{t \in J \mid t < s \, \forall s \in \tilde{I}\} \neq \emptyset$).

a) Prove that if $c : I \to M$ is C^0-future inextendible, then $\sup I \notin I$.

For a subset A of a Lorentzian manifold M one defines the future causal domain of dependence

$$D^+(A) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{any } C^0\text{-past inextendible future-directed} \\ \text{causal curve running through } p \text{ meets } A, \end{array} \right\}$$

and the past causal domain of dependence

$$D^-(A) := \left\{ p \in M \;\middle|\; \begin{array}{l} \text{any } C^0\text{-future inextendible future-directed} \\ \text{causal curve running through } p \text{ meets } A. \end{array} \right\}$$

Define also $D(A) := D^+(A) \cup D^-(A)$.

b) Consider $M_1 := (\mathbb{R} \times (-2,2)) \setminus \overline{B_1(0)} \subset \mathbb{R}^{1,1}$ and $M_2 := \mathbb{R}^{1,1} / \mathbb{Z}e_0$, $e_0 := (1,0)$. Prove that the following curves are C^0-future and C^0-past inextendible:

$$c_1 : (1, \infty) \to M_1, \; t \mapsto (t, 0), \text{ and } c_2 : \mathbb{R} \to M_2, \; t \mapsto (t \bmod \mathbb{Z}, 0).$$

c) Let $M := \mathbb{R}^{1,1} / \mathbb{Z}e_0$, $e_0 := (1,0)$. Determine $D(A), A = B \times \{0\}$, for the cases $B = [0, \frac{1}{2}]$, $B = [0,3]$ and $B = \mathbb{R}$.

d) Prove that if $A \subset M$ is acausal (i.e. no pair $p, q \in A$ satisfies $p < q$), then $D(A)$ does not contain any closed causal curve.