# Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Roman Schießl
Please hand in the exercises until **Tuesday, June 1**

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## Exercises Sheet no. 7

**1. Exercise** (4 points).
Let $(M, g)$ be a time-oriented Lorentzian manifold. Show that $M$ satisfies the strong causality condition if and only if for any $p \in M$ the *causal diamonds* $J(q, q') \coloneqq J_+(q) \cap J_-(q')$ with $q \ll p \ll q'$ form a neighborhood basis of $p$, i. e. for every open set $U$ containing $p$ there exist $q, q'$ as above with $J(q, q') \subseteq U$.

**2. Exercise** (4 points).
Let $(M, g)$ be Lorentzian manifold that is a generalized warped product with basis $(\mathbb{R}, -\mathrm{d}t^2)$. Show the following:

a) $(M, g)$ is time-orientable.

b) $(M, g)$ satisfies the causality condition.

*Remark:* One can even show that $(M, g)$ satisfies the strong causality condition, but this is not demanded here.

**3. Exercise** (4 points).
Consider again the Schwarzschild space $(M, g)$ with $M = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$ and

$$g(t, r, x) = -\left(1 - \frac{2m}{r^{n-2}}\right)\mathrm{d}t^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}}\mathrm{d}r^2 + r^2 g_{S^{n-1}}(x).$$

Let $F\colon ((2m)^{\frac{1}{n-2}}, \infty) \to \mathbb{R}$ be a function whose derivative is $r \mapsto \left(1 - \frac{2m}{r^{n-2}}\right)^{-1}$.

a) Show that $\Phi\colon \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1} \to M$, $(v, r, x) \mapsto (v - F(r), r, x)$ defines a diffeomorphism.

b) Calculate $\Phi^* g$ and determine the maximal subset $\hat{M}$ of $\mathbb{R}^2 \times S^{n-1}$ to which it analytically extends as a Lorentzian metric.
*Hint:* Analyticity here means that the coefficients of the metric are real-analytic in every chart of a real-analytic atlas. The standard analytic structure on $S^{n-1}$ is generated e. g. by the atlas consisting of the stereographic projections.

c) Show that $(\hat{M}, \Phi^* g)$ is a vacuum solution of the Einstein equations.
*Hint:* Show that the Ricci tensor is analytic as well.