Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, June 8**

Exercises Sheet no. 8

1. Exercise (4 points).
Let M be a time-oriented Lorentzian manifold. Prove that the set of points in M at which the chronology (resp. causality) condition fails is a (possibly empty) disjoint union of sets of the form $I^+(p) \cap I^-(p)$ (resp. $J^+(p) \cap J^-(p)$).

2. Exercise (0,5+1+3+0,5+1 points).
Let X and Y be Hausdorff spaces. The compact-open topology on the set of continuous maps $C(X,Y)$ is defined as follows: For a compact $K \subset X$ and an open $V \subset Y$ set

$$U(K,V) := \{f \in C(X,Y) \mid f(K) \subset V\}$$

and let $\mathcal{B} \subset \mathcal{P}(C(X,Y))$ be the set containing these, with K running through all compact and V through all open subsets. The compact-open togology is the (set-theoretically) smallest topology $\mathcal{O} \subset \mathcal{P}(C(X,Y))$ that contains $\mathcal{B}$.

a) Describe the open sets of this topology in terms of the $U(K,V)$.

b) Prove that $C(X,Y)$ with the compact-open topology is Hausdorff.

From now on, assume that X is locally compact. Here, we say that X is locally compact if any neighborhood of any point p contains a compact neighborhood of p.

c) Let (Y,d) be a metric space and define $d_\infty(f,g) := \sup_{x \in X}\{d(f(x),g(x))\}$. If X is compact, then we already know that d_∞ is a metric. Show that d_∞ induces the compact-open topology on such $C(X,Y)$.

d) Now let X be compact and Y a manifold. If g and $\tilde{g}$ are Riemannian metrics on Y, prove that uniform convergence of a seqence of functions w.r.t. g is equivalent to uniform convergence w.r.t. $\tilde{g}$.

e) Give an example of a non-compact topological space X and show that the conclusion of c) does not hold for it.

3. Exercise (1+1+2,5+0,5+1 points).
The aim of this exercise is to describe the *conformal compactification* of Minkowski space $(M,g) \coloneqq (\mathbb{R}^{n,1}, \langle\!\langle \bullet, \bullet \rangle\!\rangle)$. Up to the line $\mathbb{R}\times\{0\}$, spherical coordinates $(t,r,x) \in \mathbb{R}\times(0,\infty)\times S^{n-1}$ describe M via (t,rx). We define $v = t-r$, $w = t+r$ and $V = \arctan(v)$, $W = \arctan(w)$ as well as $T = V + W$, $R = W - V$.

a) Show that $(t,rx) \mapsto (T,R,x)$ defines a diffeomorphism of $M \setminus (\mathbb{R}\times\{0\})$ onto its image $U \subseteq \mathbb{R}^2 \times S^{n-1}$ and determine U.

b) Determine the pulled-back metric $\tilde{g}$ on U of the conformally transformed metric $\frac{4}{(1+v^2)(1+w^2)} g \eqqcolon \Omega^2 g$ of M.

c) Show that $(M\setminus(\mathbb{R}\times\{0\}), \Omega^2 g) \cong (U,\tilde{g})$ isometrically embeds into $(\mathbb{R}\times S^n, -\mathrm{d}t^2 + g_{S^n})$ via $\mathbb{R}^2\times S^{n-1} \to \mathbb{R}\times S^n$, $(T,R,x) \mapsto (T, \sin(R)x, \cos(R))$ and that this extends to an isometric embedding of $(M,\Omega^2 g)$.

d) Show that the closure of the image of M in $\mathbb{R}\times S^n$ is compact.

e) Determine the following subsets of the closure of the image:

$$\mathcal{I}^0 = \left\{ \lim_{t\to\infty} \gamma(t) \mid \gamma\colon \mathbb{R} \to M \text{ is a spacelike geodesic} \right\}$$
$$\mathcal{I}^\pm = \left\{ \lim_{t\to\pm\infty} \gamma(t) \mid \gamma\colon \mathbb{R} \to M \text{ is a future timelike geodesic} \right\}$$
$$\mathcal{J}^\pm = \left\{ \lim_{t\to\pm\infty} \gamma(t) \mid \gamma\colon \mathbb{R} \to M \text{ is a future lightlike geodesic} \right\}.$$

You may use without proof the following fact from elementary multivariable analysis (1 bonus point if you add a proof):
If $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with $\varphi\big((t,r)\big) = -\varphi\big((t,-r)\big)$, then there is a smooth map

$$\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad \Phi(t, r\sigma) = \varphi(t,r)\sigma \quad \forall t \in \mathbb{R}, r \in [0,\infty), \sigma \in S^{n-1}.$$