Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Roman Schießl
Please hand in the exercises until **Tuesday, July 6**

Exercises Sheet no. 12

1. Exercise (1+1+1+1 points).
We assume that $M = (a,b) \times N$ carries the warped product metric

$$g = -dt \otimes dt + w(t)^2 h^N$$

for some Riemannian metric h^N on N and some positive function $w \in C^\infty\big((a,b)\big)$. Show

a) If g is Ricci-flat, then W is affine-linear, i.e. of the form $\alpha t + \beta$. Furthermore h^N is an Einstein metric with non-positive scalar curvature.
Hint: You may use the formulas derived in section 2.5 of this lecture's partial lecture notes.

b) Assume $\dim M = 4$ and (N, h^N) simply-connected and complete. Show that (M,g) is isometric to $\big((a,b) \times \mathbb{R}^3, -dt \otimes dt + g_{\mathrm{eucl}}\big)$ or to $\big((\tilde{a},\tilde{b}) \times \mathbb{H}^3, -dt \otimes dt + t^2 g_{\mathrm{hyp}}\big)$ for some constants $\tilde{a} > 0$ and $\tilde{b}$, where $(\mathbb{H}^n, g_{\mathrm{hyp}})$ denotes the n-dimensional hyperbolic space.

c) Show that the following 3 semi-Riemannian manifolds are isometric:

$$\big(\mathbb{R}_{>0} \times \mathbb{H}^n, -dt \otimes dt + t^2 g_{\mathrm{hyp}}\big)$$
$$\big(\mathbb{R} \times \mathbb{H}^n, e^{2r}\left(-dr \otimes dr + g_{\mathrm{hyp}}\right)\big)$$
$$I_+(0) \subset \mathbb{R}^{n,1}$$

2. Exercise (4 points).

a) Show that $\mathbb{S}^{3,1}$ is isometric to a Robertson-Walker spacetime.

b) Calculate the sectional curvature of $\mathbb{M} := \big((0,\pi) \times \mathbb{H}^3, -dt \otimes dt + \sin(t)^2 g_{\mathrm{hyp}}\big)$ by applying the formulas for warped products in Section 2.5.

c) Construct two isometric embeddings $i_1, i_2 : \mathbb{M} \to \mathbb{H}^{3,1}$ with disjoint image such that $i_1(\mathbb{M}) \cup i_2(\mathbb{M})$ is dense in $\mathbb{H}^{3,1}$.
Hint: You may consider the map $(0,\pi) \times \mathbb{H}^3 \to \mathbb{H}^{3,1}$ *that maps* (t,x), $x \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ *to* $\big(\sin(t)x, -\cos(t)\big)$.

Note (not part of the exercise): $\mathbb{H}^{3,1}$ is not isometric to a Robertson-Walker spacetime, as it violates the causality condition, see next exercise.

3. Exercise (4 points).
Recall from exercise 2 of sheet 1 that there is a diffeomorphism $\mathbb{R}^n \times S^1 \to \mathbb{H}^{n,1} = \{z \in \mathbb{R}^{n,2} \,|\, \langle\!\langle z,z \rangle\!\rangle = -1\}, (x,y) \mapsto (x, \sqrt{\|x\|^2_{\mathrm{eucl}} + 1}\, y)$, $x \in \mathbb{R}^n$, $y \in S^1$. Pulling back the metric of the pseudohyperbolic space along this diffeomorphism and further along the covering map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times S^1$, $(x,t) \mapsto (x, \exp(2\pi i t))$, we obtain a model $(\mathbb{R}^n \times \mathbb{R}, g_{\mathrm{AdS}})$ of the *Anti-de Sitter spacetime*. We choose the time-orientation such that for any $x \in \mathbb{R}^n$ the curve $t \mapsto (x,t)$ (or $t \mapsto (x, \exp(2\pi i t))$) is future-directed. Show the following:

a) Anti-de Sitter spacetime satisfies the causality condition whereas it is violated by the pseudohyperbolic space $\mathbb{H}^{n,1}$.

b) The causal diamond $J\big((0,-\frac{1}{4}),(0,\frac{1}{4})\big)$ in $(\mathbb{R}^n \times \mathbb{R}, g_{\mathrm{AdS}})$ maps diffeomorphically to a subset $J_0 \subseteq \mathbb{H}^{n,1}$.

c) The curve $\gamma\colon [0,\infty) \to \mathbb{H}^{n,1}$, $t \mapsto (t,0,\dots,0,t,-1)$ is a lightlike pregeodesic in $\mathbb{H}^{n,1}$.
Hint: Show that the image of γ lies in the intersection of $\mathbb{H}^{n,1}$ with the plane $P \subset \mathbb{R}^{n,2}$ defined by $x_1 = x_{n+1}$, $x_2 = \dots = x_n = 0$ and use suitable reflections to show that this intersection contains the image of a geodesic.

d) The image of γ is contained in J_0.

e) Anti-de Sitter spacetime is not globally hyperbolic.