# Differential Geometry II: Exercises

University of Regensburg, Summer term 2021
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Please hand in the exercises until **Tuesday, July 13**

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## Exercises Sheet no. 13

**1. Exercise** (4 bonus points).
Let $(M,g)$ be the *interior Schwarzschild spacetime* with $M = \mathbb{R} \times (0,(2m)^{\frac{1}{n-2}}) \times S^{n-1}$ and

$$g(t,r,x) = -\left(1 - \frac{2m}{r^{n-2}}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{2m}{r^{n-2}}}\mathrm{d}r^2 + r^2 g_{S^{n-1}}(x).$$

We choose the time-orientation such that $-\frac{\partial}{\partial r}$ is future-pointing.

a) Show that $S = \mathbb{R} \times \{r_0\} \times S^{n-1}$ for $r_0 \in (0,(2m)^{\frac{1}{n-2}})$ is a Cauchy hypersurface.

b) Show that there is a bound for the lengths of future curves emanating from $S$.

**2. Exercise** (3+1+1 bonus points).
Let $(M,g)$ be a globally hyperbolic time-oriented Lorentzian of dimension $n+1$, and $N$ a compact spacelike hypersurface with future directed unit normal field $\nu$. For any $x \in N$ there is a $a_x < 0 < b_x$ such that for all $t \in (a_x, b_x)$ the expression $\phi(x,t) := \exp_x(t\nu|_x)$ is well-defined. We set $U := \{(x,t) \mid a_x < t < b_x\}$. You may assume that $U$ is open in $N \times \mathbb{R}$ and that $\phi : U \to M$ is a diffeomorphism onto its image (which is the case if $U$ is a sufficiently small neighborhood of $N \times \{0\}$).

a) Show that $\phi^* g = -dt^2 + h_t$ for some suitable family $h_t$ of Riemannian metrics on $N$. *Attention: the most subtle part is that the hypersurfaces* $N_{t_0} := \phi(N \times \{t_0\})$ *are orthogonal to the vector field* $\frac{\partial}{\partial t} = \frac{\partial \phi}{\partial t}$*. This can be proven in analogy to the Gauß lemma using Jacobi fields. If this is too hard, you may skip this part and continue the exercise.*

b) Let $\vec{\mathrm{II}}$ be the vector-valued second fundamental form of $N_{t_0}$ in $M$. Then

$$g(\vec{\mathrm{II}}(\cdot,\cdot),\nu) = -\frac{1}{2}\frac{d}{dt}\Big|_{t=t_0} h_t.$$

*Hint: this is almost immediate from formulas in the lecture's partial notes.*

c) Let $\mathrm{dvol}^{h_h}$ denote integration with respect to the measure defined by $h_t$. Prove that

$$\frac{d}{dt}\mathrm{vol}(N_t, h_t) = -n\int_{N_t}\left\langle \vec{H}, \frac{\partial}{\partial t}\right\rangle \mathrm{dvol}^{h_h}.$$

*Hint: "Variationsformel" from Analysis IV. Argue why you can use this!*

**3. Exercise** (1+2+2 bonus points).

We assume the setting of the previous exercise. In the following we write $H_t(x) := \left\langle \vec{H}, \frac{\partial}{\partial t} \right\rangle$ for the mean curvature of $N_t$ at $x \in N_t$. You may use (without proof)

$$n \cdot \frac{d}{dt} H_t = \mathrm{ric}\left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) + \|\mathrm{II}\|^2$$

where $\|\mathrm{II}\|^2 = \sum_{i,j=1}^{n} \langle \vec{\mathrm{II}}(e_i, e_j), \nu \rangle^2$ for any orthonormal basis $(e_1, \ldots, e_n)$ of $T_pN$ for $p \in N$.

a) Show $\|\mathrm{II}(x,t)\|^2 \geq n H_t(x)^2$ *Hint: Cauchy-Schwarz inequality*

b) Now we assume $\mathrm{ric}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \geq 0$ and $H_0 \geq \beta$ along $N = N_0$ for some $\beta > 0$. (In particular this holds assuming the conditions of Hawking's singularity theorem in case of a compact Cauchy hypersurfaces.) Show that for $0 \leq t \leq \inf_{x \in M} b_x$:

$$\frac{1}{H_t} \leq \frac{1}{\beta} - t$$

c) Conlude if $b_x \geq 1/\beta$ for all $x \in N$, then for $t \geq 0$

$$\mathrm{vol}(N_t, h_t) \leq (1 - \beta t)^n \,\mathrm{vol}(N, h_0).$$

*Note: It might be interesting to compare this exercise to Theorem 55B in Section 14 of O'Neill's book, which shows under the same conditions that $M$ is not future timelike complete. The above exercise reflects the contracting nature of this setting.*