Geometric Non-Commutative Geometry

Joint work with Moulay-Tahar Benameur

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Invariants in Foliation Index Theory.

Denote by F a foliation of a compact manifold M.

Associated to any leafwise elliptic (pseudo)differential operator D on (M, F) are many index invariants:

- in the K-theory of C^* algebras or smooth subalgebras;
- in the cyclic homology of smooth subalgebras of C^* algebras;
- in the Haefliger cohomology $\mathrm{H}^*_c(M/F)$ of the foliation.

$$\vdots$$

In K Theory

- Given D acting on E over F, there is an operator Q such that

$$S_0 = I - QD \text{ and } S_1 = I - DQ \quad \in \quad C_c^\infty(\mathcal{G}; \mathrm{End}(E)).$$

  $\mathcal{G}$ is the holonomy groupoid of F.

- The class of the idempotent $\begin{pmatrix} S_0^2 & -Q \circ (S_1 + S_1^2) \\ -S_1 \circ D & I - S_1^2 \end{pmatrix}$

  in $K_0(C_c^\infty(\mathcal{G}))$ is an index class denoted $\mathrm{Ind}_c^\infty(D)$.

- Using the natural map $K_0(C_c^\infty(\mathcal{G})) \to K_0(C^*(M,F))$, one gets the Connes C^* index class $\mathrm{Ind}(D)$, which is a deep invariant. ($C^*(M,F)$ is Connes' C^* algebra of the "space of leaves of F".)

  For the Atiyah-Hirzebruch (signature) operator, this is a (leafwise) homotopy invariant.

In Cohomology

- Using a Chern character in Haefliger cohomology, Benameur-H get a cohomology class $\mathrm{ch}_a(\mathrm{Ind}_c^\infty(D)) \in \mathrm{H}_c^*(M/F)$.

  For fibrations $M \to B$, this is the usual families index Chern character in $\mathrm{H}^*(B; \mathbb{R}) = \mathrm{H}_c^*(M/F)$.

- Using Connes' Chern character yields a cyclic homology class.

- When the index bundle $\mathrm{Ker}(D)$ exists, its Chern character in Haefliger cohomology $\mathrm{ch}(\mathrm{Ker}(D))$ was defined by JLH.

- Using the heat equation, one defines "other" index classes:
  - Extending the Bismut superconnection construction, JLH defined a cohomology class $\mathrm{Tr}(e^{-\mathbb{B}_s^2}) \in \mathrm{H}_c^*(M/F)$.
  - Nistor and Gorokhovsky-Lott extended JLH superconnection approach to cyclic homology.
  - The so-called Wassermann idempotent.

Natural Questions

- Compare and relate all these invariants.
- Compute them in terms of topological constructions and characteristic classes.
- Use the different definitions to investigate properties of the foliation:
  - produce homotopy invariants;
  - develop obstruction theory;
  - prove rigidity results;
  - give interesting examples;

  ⋮

Some results... (somewhat paraphrased)

Theorem (Connes-Skandalis)

P any leafwise elliptic ΨDO. Then in $K_0(C^*(M,F))$, $\mathrm{Ind}(P) = \mathrm{Ind}_t(P)$, where $\mathrm{Ind}_t(P)$ is constructed using shrieck maps in Kasparov's theory of C^*-algebras.

Theorem (H-Lazarov)

For a Dirac type operator D and under suitable spectral assumptions,

$$\mathrm{ch}_a(\mathrm{Ker}(D)) = \int_F AS(D; M, F).$$

Theorem (Benameur-H)

Under much weaker spectral assumptions,

$$\mathrm{ch}_a(\mathrm{Ind}(D)) = \mathrm{ch}_a(\mathrm{Ker}(D)) \quad (= \int_F AS(D; M, F)).$$

Theorem (Benameur-H)

(Relation between Connes-Skandalis and H-L and B-H)
For any leafwise elliptic pseudodifferential operator P,

- *The Connes-Skandalis topological index class can be factorised through a class $\mathrm{Ind}_t(P) \in \widehat{K}_0(C_c^\infty(\mathcal{G}))$.*
- *In Haefliger cohomology, we have $\mathrm{ch}_a(\mathrm{Ind}_t(P)) = \int_F AS(P; M, F)$.*

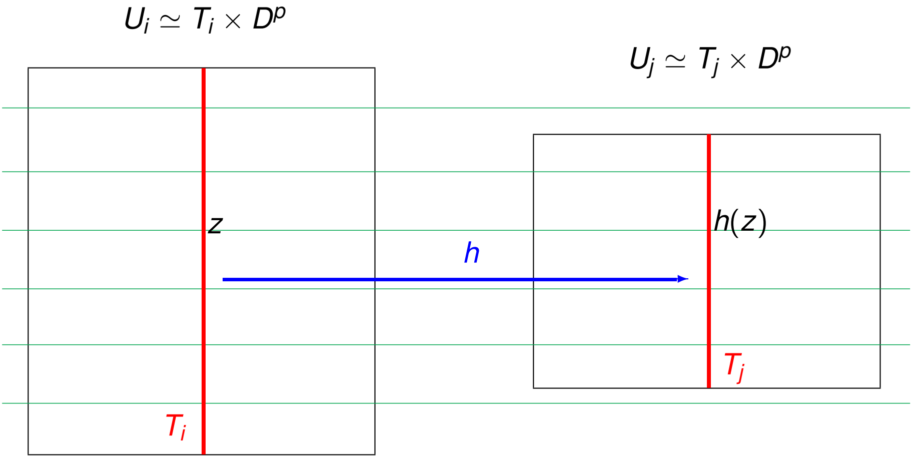
$\widehat{K}_0(C_c^\infty(\mathcal{G})) = \varinjlim_k K_0(C_c^\infty(\mathcal{G} \times \mathbb{R}^k))$, the so-called Bott system.

See also the thesis of Carillo Rouse.

Theorem (Benameur-H)

If D^+ is the leafwise signature operator and $\mathrm{Ker}(D^+)$ is a smooth (usually infinite dimensional) bundle, then $\mathrm{ch}_a(\mathrm{Ker}(D^+))$ is an oriented foliated homotopy invariant (which coincides with the higher signature in Haefliger cohomology).

Holonomy and Local Integration for F of dim p



- $h^* : \Omega_c^k(T_j) \to \Omega_c^k(T_i)$.
- If $\omega_i \in \Omega_c^{p+k}(U_i)$, get $\int \omega_i \in \Omega_c^k(T_i)$.

Haefliger Cohomology

- Write $\quad M = \bigcup U_i \qquad U_i$ foliation charts for F.
- Choose transversals $T_i \subset U_i$ so that $T = \bigcup T_i$ is disjoint union.
- In C^∞ k forms with compact support $= \Omega_c^k(T)$, consider
  $L^k = \overline{\text{span}\{\alpha - h^*\alpha\}}, \quad h \in$ holonomy pseudogroup.
- Set $\qquad \Omega_c^k(M/F) = \Omega_c^k(T)/L^k.$
- $d : \Omega_c^k(T) \to \Omega_c^{k+1}(T)$ induces $d_H : \Omega_c^k(M/F) \to \Omega_c^{k+1}(M/F)$.
- $\mathrm{H}_c^*(M/F) =$ cohomology of this complex.
- If F given by a fibration $M \to B$, then $H_c^*(M/F) = H^*(B;\mathbb{R})$.
- Independent of all choices.

Integration over the fiber of F

- $\int_F : \Omega^{p+k}(M) \to \Omega_c^k(M/F)$.
- Given $\omega \in \Omega^{p+k}(M)$, write $\omega = \sum_i \omega_i$, where $\omega_i \in \Omega_c^{p+k}(U_i)$.
- Integrate ω_i along the fibers of $U_i \to T_i$. Get $\int \omega_i \in \Omega_c^k(T_i)$.
- $\int_F \omega \equiv$ class of $\sum_i \int \omega_i$. $\int_F \omega \in \Omega_c^k(M/F)$ well defined.
- $d_H \circ \int_F = \int_F \circ d$ so get $\int_F : H^{p+k}(M) \to H_c^k(M/F)$.

Holonomy groupoid $\mathcal{G}$ of F

- $\mathcal{G}$ = equivalence classes of leafwise paths in M.
- Paths equivalent if start at same point, end at same point, and have same holonomy germ.
- $s,r : \mathcal{G} \to M$: $\quad s\big([\gamma]\big) = \gamma(0), \quad r\big([\gamma]\big) = \gamma(1)$.
- F_s foliation of $\mathcal{G}$, leaves are $\widetilde{L}_x = s^{-1}(x)$.
- $r : \widetilde{L}_x \to L_x$ is the holonomy cover of L_x.
- $x \in M$ gives $\overline{x} \in \mathcal{G}$, $\overline{x}$ = class of constant path at x.
- So, $x \to \overline{x}$ $\quad$ gives $\quad M \simeq \mathcal{G}_0 \subset \mathcal{G}$, and $M \subset \mathcal{G}$.

Recall that the Families Index Theorem is almost immediate from:

Theorem

The following diagram commutes.

$$\begin{array}{ccc} \mathrm{K}^0_c(N) & \xrightarrow{f_!} & \mathrm{K}^0(M) \\ \mathrm{ch}(\cdot)\wedge \mathrm{Td}(f) \downarrow & & \downarrow \mathrm{ch} \\ \mathrm{H}^*_c(N;\mathbb{R}) & \xrightarrow{f_{**}} & \mathrm{H}^*(M;\mathbb{R}). \end{array}$$

- $f : N \to M$ a K-oriented map, i.e. $TN \oplus f^*TM$ has Spin^c structure.
- $\mathrm{Td}(f) = \mathrm{Td}(TN \otimes \mathbb{C}) / \mathrm{Td}(f^*(TM \otimes \mathbb{C}))$.
- $f_{**} = PD \circ f_* \circ PD$, $\quad f_* : H_*(N;\mathbb{R}) \to H_*(M;\mathbb{R})$.
- If f a submersion, $f_{**} = \int$ over the fibers of f.

We extend this to foliations.

Theorem (Benameur-H)

M compact manifold, F oriented foliation.
$f : N \to M/F$ a K-oriented map. (M/F = "space of leaves of F")
For k large, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{K}^0_c(N) & \xrightarrow{\quad f_! \quad} & \mathrm{K}_0(C^\infty_c(\mathcal{G} \times \mathbb{R}^{2k})) \\ \mathrm{ch}(\cdot) \wedge \mathrm{Td}(f) \ \downarrow & & \downarrow \ \mathrm{ch}_a \\ \mathrm{H}^*_c(N; \mathbb{R}) & \xrightarrow{\quad f_{**} \quad} & \mathrm{H}^*_c(M/F). \end{array}$$

$f_!$ the Connes-Skandalis push forward map.

The maps ch_a and f_{**} to be defined.

In simplest terms,

"the push forward maps in K-theory and Haefliger H^* are compatible."

Definition of f_{**} (simplest case):

- Assume $f : M \times K \to M$.
- Have foliation $\widehat{F} = F \times K$ on $M \times K$. Leaves are $L \times K$.
- Each $T_i \subset U_i$ gives $\widehat{T}_i = T_i \times \{pt\} \subset U_i \times K$.
- Can compute $H_c^*(M \times K/F \times K)$ using the $\widehat{T}_i$.
- Note that $f : \widehat{T}_i \to T_i$ is a diffeomorphism.

Given $\omega \in \Omega_c^*(\widehat{T}_i)$, set $f_*(\omega) = (f^{-1})^*(\omega) \in \Omega_c^*(T_i)$.

$f_* : \Omega_c^*(\widehat{T}) \to \Omega_c^*(T)$ induces $f_* : H_c^*(M \times K/F \times K) \to H_c^*(M/F)$.

$$f_{**} : H_c^*(M \times K; \mathbb{R}) \xrightarrow{\int_{\widehat{F}}} H_c^*(M \times K/F \times K) \xrightarrow{f_*} H_c^*(M/F)$$

Definition of f_{**} (submersion case):

- $f : N \to M/F$ is a $\mathcal{G}$ valued cocycle $(V_\alpha, f_{\alpha\beta})$.
- $\{V_\alpha\}$ locally finite open cover of N.
- $f_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathcal{G}$
- Cocycle condition $\quad f_{\alpha\beta}(x) f_{\beta\gamma}(x) = f_{\alpha\gamma}(x) \quad \Longrightarrow$
  - $f_{\alpha\alpha} : V_\alpha \to \mathcal{G}_0 = M$
  - $f^*(\nu)$ well defined, $\nu \subset TM$ normal bundle of F.
- f is K-oriented if $TN \oplus f^*(\nu)$ is Spin^c.
- f is a "submersion" if each $f_{\alpha\alpha}$ is transverse to F, i.e. if f is a submersion to the "space of leaves" of F.

- Assume f is a K-oriented submersion. (Use K-oriented later).
- Have oriented foliation F_N of N. Leaves are f^{-1} of leaves of F.
- May assume:
  - each V_α of the cocycle $(V_\alpha, f_{\alpha\beta})$ is a chart for F_N;
  - For each α, there is an $i(\alpha)$ with $\overline{f_{\alpha\alpha}(V_\alpha)} \subset U_{i(\alpha)}$.

Denote by S_α and $T_{i(\alpha)}$ transversals of V_α and $U_{i(\alpha)}$. Then

- $f_{\alpha\alpha} : S_\alpha \to T_{i(\alpha)}$ is a diffeomorphism onto its image.
- Have $\quad f_{\alpha *} : \Omega^*_c(S_\alpha) \to \Omega^*_c(T_{i(\alpha)})$, so get $\quad f_* : \Omega^*_c(S) \to \Omega^*_c(T)$.
- For $[\omega] \in H^*_c(N; \mathbb{R})$, set

$$f_{**}([\omega]) = [f_*(\int_{F_N} \omega)].$$

Definition of f_{**} (general case):

- Construct a manifold W, and K-oriented maps

$$i : N \to W \quad \text{and} \quad g : W \to M/F,$$

  so that g is a submersion, and $f = g \circ i$.

- $g_{**} : H^*_c(W;\mathbb{R}) \to H^*_c(M/F)$ as above.

- $i_{**} \equiv PD \circ i_* \circ PD : H^*_c(N;\mathbb{R}) \to H^*_c(W;\mathbb{R})$.

  $PD =$ Poincaré Duality.

Then

$$f_{**} : H^*_c(N;\mathbb{R}) \xrightarrow{i_{**}} H^*_c(W;\mathbb{R}) \xrightarrow{g_{**}} H^*_c(M/F).$$

The Connes-Skandalis push forward map

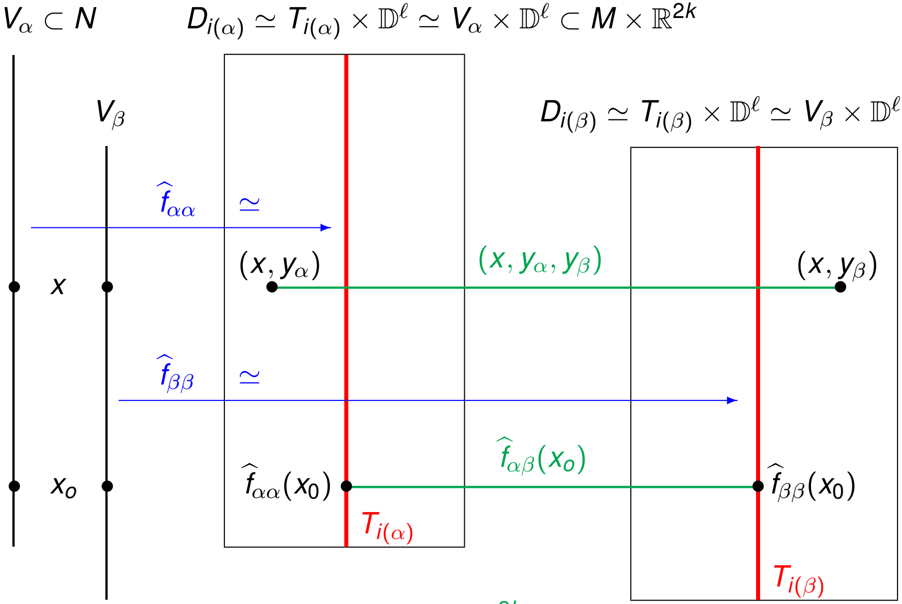
- $i : N \to W$ is K-oriented, so have $i_! : K^0_c(N) \to K^0_c(W)$. Replacing N by W, may assume that f is a submersion.
- Want $f : N \to M/F$ to be **étale**, i.e. $f_{\alpha\alpha} : V_\alpha \xrightarrow{\simeq} T_{i(\alpha)} \subset U_{i(\alpha)}$.
- Choose an imbedding $j : N \to \mathbb{R}^{2k}$.
- Replace M by $M \times \mathbb{R}^{2k}$, $\mathcal{G}$ by $\mathcal{G} \times \mathbb{R}^{2k}$, F by $\widehat{F}$ on $M \times \mathbb{R}^{2k}$. Leaves are $L \times \{a\}$, where L is a leaf of F and $a \in \mathbb{R}^{2k}$.
- Replace f by $\widehat{f} = \{(f_{\alpha\beta}, j)\}$, i.e. for $x \in V_\alpha \cap V_\beta \subset N$,

$$\widehat{f}_{\alpha\beta}(x) = (f_{\alpha\beta}(x), j(x)) \in \mathcal{G} \times \mathbb{R}^{2k}.$$

- $D_i \to T_i$ normal disc bundle in $M \times \mathbb{R}^{2k}$.
- Coord x on $V_\alpha \subset N$ gives coords (x, y_α) on $D_{i(\alpha)} \subset M \times \mathbb{R}^{2k}$.

$$D_{i(\alpha)} \simeq T_{i(\alpha)} \times \mathbb{D}^\ell \simeq V_\alpha \times \mathbb{D}^\ell.$$

- $U_{\alpha\beta}$ = classes of paths γ where
  - $s(\gamma) \in D_{i(\alpha)}$, $r(\gamma) \in D_{i(\beta)}$, $\gamma \,||\, \widehat{f}_{\alpha\beta}(x_o) \in \mathcal{G} \times \mathbb{R}^{2k}$, $x_o \in V_\alpha \cap V_\beta$.
  - $U_{\alpha\beta}$ charts on $\mathcal{G} \times \mathbb{R}^{2k}$, with coords (x, y_α, y_β).



$$(x, y_\alpha, y_\beta) \in U_{\alpha\beta} \subset \mathcal{G} \times \mathbb{R}^{2k}$$

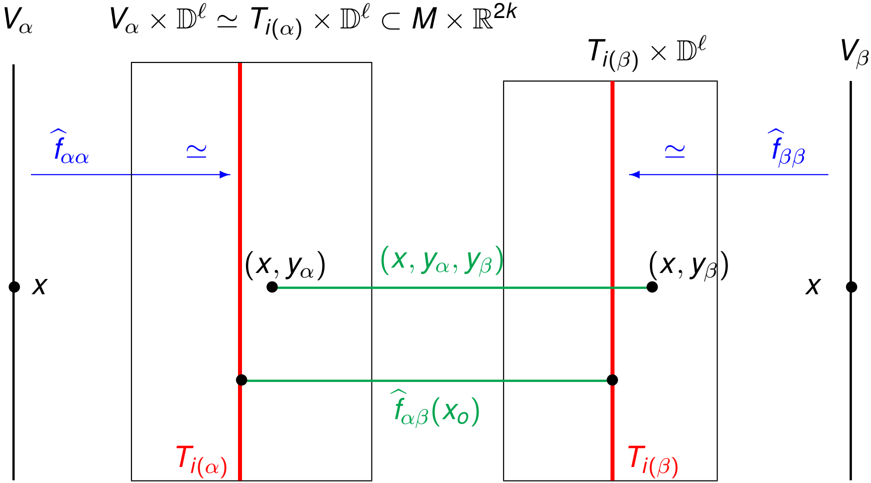
- Choose $\psi : \mathbb{D}^\ell \to \mathbb{R}$ with cpt sup, and $\displaystyle\int_{\mathbb{D}^\ell} \psi^2 = 1$.
- $\{\phi_\alpha\}$ a partition of unity on N subordinate to $\{V_\alpha\}$.
- $f_! : C_c^\infty(N) \to C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})$ given by:
  - on $U_{\alpha\beta}$, $\quad f_!(g)(x, y_\alpha, y_\beta) = g(x)\psi(y_\alpha)\psi(y_\beta)\sqrt{\phi_\alpha(x)\phi_\beta(x)}$
  - otherwise $\quad f_!(g) = 0$.

That is, $f_!(g)$ is g "smeared out to a nbhd of $f(N)$".

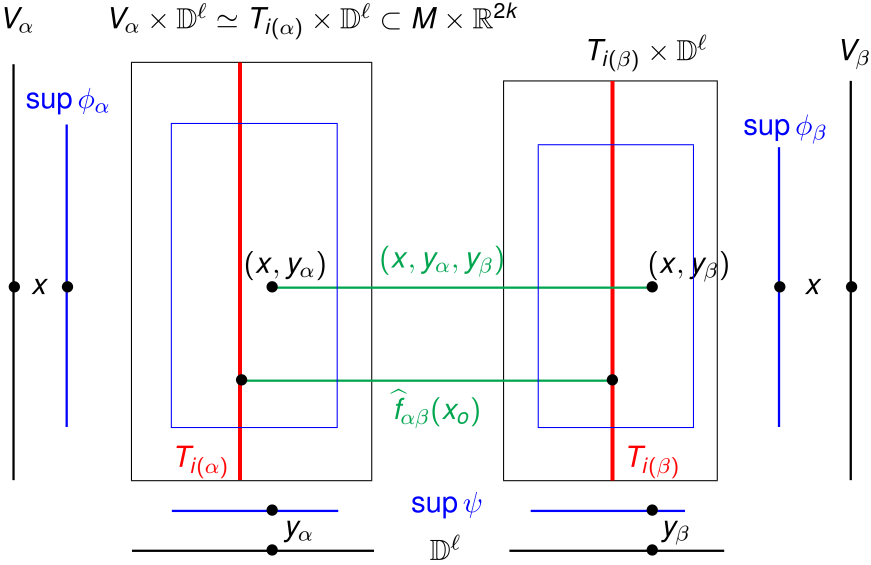
$f_!$ is an algebra map since the product on $C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})$ is the convolution product, not the pointwise product.
Get Connes-Skandalis map

$$f_! : K_c^0(N) \to K_0(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})).$$

Some diagrams to explain $f_!$



Want to define $f_!(g)((x, y_\alpha, y_\beta))$ for $(x, y_\alpha, y_\beta) \in U_{\alpha\beta}$.



On $U_{\alpha\beta}$, $\quad f_!(g)((x, y_\alpha, y_\beta)) = g(x)\psi(y_\alpha)\psi(y_\beta)\sqrt{\phi_\alpha(x)\phi_\beta(x)}$

The Chern Character ch_a

Some motivation.

- $E \to M$ a $\mathbb{C}$ bundle, then $E \oplus E^\perp = |^N \to M$, where $|^N$ is trivial.
- Implies E is image of fiberwise idempotent e.
- So $e = [e_{ij}]$ where $i, j = 1, ..., N$, and $e_{ij} \in C^\infty(M)$.
- $de = [de_{ij}]$ is a matrix of one forms.
- $de^2 = [de_{ij}] \wedge [de_{ij}]$ a matrix of two forms.
- $$\mathrm{ch}(E) := \mathrm{tr}\Big[e \exp\Big(\frac{-(de^2)}{2\pi i}\Big)\Big].$$

Extension to our case

- $E_1 \to M$ a $\mathbb{C}$ bundle induces $E = r^*(E_1) \to \mathcal{G}$.
- Any connection on E_1 induces a connection on E,

$$\nabla : C^\infty(E \otimes \Lambda T^*\mathcal{G}) \to C^\infty(E \otimes \Lambda T^*\mathcal{G}).$$

- $\nu_s^* \equiv s^*(T^*M) \subset T^*\mathcal{G}$ is dual normal bundle of F_s.
- $p_\nu : \Lambda T^*\mathcal{G} \to \Lambda \nu_s^*$, the projection.
- ∇^ν is the composite

$$C^\infty(E \otimes \Lambda \nu_s^*) \xrightarrow{i} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{\nabla} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{p_\nu} C^\infty(E \otimes \Lambda \nu_s^*).$$

- $C^\infty(E \otimes \Lambda\nu_s^*)$ is an $\Omega^*(M)-$module through the souce map s.
- $\mathcal{A}^k$ is the space of $\Omega^*(M)-$equivariant $\mathcal{G}-$invariant endomorphisms of $C^\infty(E \otimes \Lambda\nu_s^*)$ which have degree k.
  Set $\mathcal{A}^* = \bigoplus_k \mathcal{A}^k$.
- $T \in \mathcal{A}^*$ commutes with action of $C^\infty(M)$, so T acts **leafwise.**
- T is $\mathcal{G}-$invariant means T on $\widetilde{L}_x$ same as T on $\widetilde{L}_y$ if $L_x = L_y$.

Definition

$$\partial_\nu : \mathrm{End}(C^\infty(E \otimes \Lambda\nu_s^*)) \to \mathrm{End}(C^\infty(E \otimes \Lambda\nu_s^*))$$

is the linear operator defined by $\partial_\nu(T) = [\nabla^\nu, T]$.

Think of as a transverse deRham operator on Schwartz kernels.

Lemma

∂_ν preserves the subspace $\mathcal{A}^*$ and $(\partial_\nu)^2$ is given by the commutator with the curvature $\theta = (\nabla^\nu)^2 \in \mathcal{A}^2$.

Definition

For a leafwise smoothing operator $T \in \mathcal{A}^k$, with Schwartz kernel K, set

$$\mathrm{Tr}(T) = \int_F \mathrm{tr}(K(\overline{x}))dx \quad \in \quad \Omega_c^k(M/F).$$

$K \in C^\infty(\mathcal{G}, \mathrm{Hom}(E))$, so $K(\overline{x}))$ is a square matrix, well defined up to conjugation, and $\mathrm{tr}(K(\overline{x}))$ is well defined.

Proposition

Tr *is a graded trace and* $\mathrm{Tr} \circ \partial_\nu = d_H \circ \mathrm{Tr}$.

In general, $(\partial_\nu)^2 \neq 0$ so we need Connes' X-trick.

- Consider $M_2(\mathcal{A}^*)$ with the product

$$\widetilde{T} * \widetilde{T}' = \widetilde{T}\Theta\widetilde{T}' \text{ where } \Theta = \begin{pmatrix} 1 & 0 \\ 0 & \theta \end{pmatrix} \text{ and } \theta = (\nabla^\nu)^2.$$

- ∂_ν extends to a derivation δ on $M_2(\mathcal{A}^*)$ and $\delta^2 = 0$.
- For $T \in \mathcal{A}^k$, $\quad \delta\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \partial_\nu T & (-1)^k T \\ T & 0 \end{pmatrix}.$

Definition

For homogeneous $\widetilde{T} \in M_2(\mathcal{A}^*)$ of degree k define

$$\Phi(\widetilde{T}) \;=\; \mathrm{Tr}(\widetilde{T}_{11}) - (-1)^k\,\mathrm{Tr}(\widetilde{T}_{22}\theta).$$

Theorem

$\Phi : M_2(\mathcal{A}^*) \to \Omega_c^*(M/F)$ is a graded trace and $\Phi \circ \delta = d_H \circ \Phi$.

Definition of the Chern character ch_a

$C_c^\infty(\mathcal{G}; \mathrm{Hom}(E)) \subset \mathcal{A}^0$ as a set of regularizing operators.

Theorem (Benameur-H)

Let $B = [e_1] - [e_2]$ be an element of $K_0(C_c^\infty(\mathcal{G}; \mathrm{Hom}(E)))$. Then the Haefliger form

$$(\Phi \circ \mathrm{tr})\Big(e_1 \exp\Big[\frac{-(\delta e_1)^2}{2i\pi}\Big]\Big) - (\Phi \circ \mathrm{tr})\Big(e_2 \exp\Big[\frac{-(\delta e_2)^2}{2i\pi}\Big]\Big)$$

is closed and its Haefliger cohomology class depends only on B.

This Haefliger cohomology class is denoted $\mathrm{ch}_a(B)$ and is called the Chern character of B.

Proof is a fairly lengthy algebraic computation.

Theorem (Benameur-H)

M compact manifold, F oriented foliation.
$f : N \to M/F$ a K-oriented map.
For k large, the following diagram commutes.

$$
\begin{array}{ccc}
\mathrm{K}^0_c(N) & \xrightarrow{\ f_!\ } & \mathrm{K}_0(C^\infty_c(\mathcal{G} \times \mathbb{R}^{2k})) \\
\mathrm{ch}(\cdot) \wedge \mathrm{Td}(f) \ \downarrow & & \downarrow \ \mathrm{ch}_a \\
\mathrm{H}^*_c(N;\mathbb{R}) & \xrightarrow{\ f_{**}\ } & H^*_c(M \times \mathbb{R}^{2k}/\widehat{F}).
\end{array}
$$

Proof uses naturalness of *ch*, *Td*, and ch_a to reduce to a complicated direct computation.

Note that: $H^*_c(M \times \mathbb{R}^{2k}/\widehat{F}) \simeq H^*_c(M/F \times \mathbb{R}^{2k}) \simeq$

$$H^*_c(M/F) \otimes H^*_c(\mathbb{R}^{2k};\mathbb{R}) \simeq H^*_c(M/F),$$

so we get the original theorem.

Theorem (Benameur-H)

$Ind_t : K_c^0(TF) \longrightarrow K_0(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k}))$ Connes-Skandalis top index map. $\pi_F : H_c^*(TF) \to H^*(M)$ integration over fibers. Then in $H_c^*(M/F)$,

$$ch_a[Ind_t(u)] = (-1)^p \int_F \pi_F(ch(u))Td(TF \otimes \mathbb{C}).$$

Direct corollary of theorem above using classical results of Atiyah-Singer and Connes-Skandalis.

Implied by a result of Nistor if F is the natural foliation of a flat bundle.

- $Ind_a : K_c^0(TF) \longrightarrow K_0(C_c^\infty(\mathcal{G}))$, Connes-Skandalis analytic index map.
- $B : K_0(C_c^\infty(\mathcal{G})) \longrightarrow K_0(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k}))$, the Bott map.

  In general $B \circ Ind_a \neq Ind_t$.
- $\rho : C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k}) \hookrightarrow C^*(M \times \mathbb{R}^{2k}, \widehat{F})$, Connes C^* algebra of $\widehat{F}$.
- Get $\rho_* : K_0(C_c^\infty(\mathcal{G} \times \mathbb{R}^{2k})) \longrightarrow K_0(C^*(M \times \mathbb{R}^{2k}, \widehat{F})) \simeq$

$$K_0(C^*(M, F) \otimes C_0(\mathbb{R}^{2k})) \simeq K_0(C^*(M, F)).$$

Theorem (Connes-Skandalis)

$$\rho_* \circ B \circ Ind_a = \rho_* \circ Ind_t.$$

Theorem (Benameur-H)

$$ch_a \circ Ind_a = ch_a \circ Ind_t.$$

Proof depends on the deep extension theorem of Connes.
See also thesis of Carillo Rouse.

ch_a and the "index bundle"

- g leafwise metric on M and E Hermitian bundle over M.
- Assume that F is even dimensional, oriented and spin.
- D_E leafwise Dirac operator on F_s determined by $r^*(g)$ and $r^*(E)$.
- $P =$ proj onto ker D_E^2, $P_\epsilon =$ spectral proj of D_E^2 for $(0, \epsilon)$.
- D_E is regular near zero if both P and P_ϵ (for $\epsilon \sim 0$) have smooth Schwartz kernels, and their transverse derivatives define bounded smoothing operators along leaves of F_s.
- Novikov-Shubin invariants of D_E are $> \alpha$ if there is $\beta > \alpha$ so that $\mathrm{Tr}(P_\epsilon) = \mathcal{O}(\epsilon^\beta)$ as $\epsilon \to 0$.

Theorem (Benameur-H)

Assume $\mathcal{G}$ is Hausdorff, Novikov-Shubin invariants of $D_E >$ codim $F/2$, and D_E is regular near zero. Then

$$\mathrm{ch}_a(\mathrm{Ind}_a(D_E)) = \mathrm{ch}_a(P).$$

- Note $\mathrm{ch}_a(P)$ not yet defined! Will do soon.
- Proof uses the Wassermann idempotent and requires very careful analysis of $e^{-tD_E^2}$ as $t \to \infty$.
- H-Lazarov proved this for NS $> 3\,\mathrm{codim}\, F$.
- Actual Benameur-H result is stronger:

$$\mathrm{ch}_a^k(\mathrm{Ind}_a(D_E)) = \mathrm{ch}_a^k(P) \in H^{2k}(M/F),$$

  provided Novikov-Shubin invariants of $D_E > k$.
  So always true when $k = 0$.
- As

$$\mathrm{ch}_a(\mathrm{Ind}_a(D_E)) = \mathrm{ch}_a(\mathrm{Ind}_t(D_E)) = \int_F \widehat{A}(TF)\mathrm{ch}(E),$$

  and $\mathrm{ch}_a(P)$ carries geometric/topological information about F, this relates characteristic classes of F to its geometry and topology. More on this below.

Some conjectures

Conjecture

For any foliation of a compact manifold, the NS invariants are positive.

Conjecture

The theorem above is true provided the NS invariants are positive.

Conjecture

If $\mathcal{G}$ is Hausdorff, then D_E is regular near zero.

Chern Character of Transversely Smooth Idempotents

Want to extend Chern character to objects like P, so ...

- $E_1 \to M$ a $\mathbb{C}$ bundle; $E = r^*(E_1) \to \mathcal{G}$.
- Recall that ∇^ν is the composite

$$C^\infty(E \otimes \Lambda \nu_s^*) \xrightarrow{i} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{\nabla} C^\infty(E \otimes \Lambda T^*\mathcal{G}) \xrightarrow{p_\nu} C^\infty(E \otimes \Lambda \nu_s^*).$$

Definition

$$\partial_\nu(\cdot) = [\nabla^\nu, \cdot] : \mathrm{End}(C^\infty(E \otimes \Lambda \nu_s^*)) \to \mathrm{End}(C^\infty(E \otimes \Lambda \nu_s^*)).$$

- $s^*(TM) \simeq \nu_s \subset T\mathcal{G}$, the normal bundle of F_s.
- Any $Y \in C^\infty(TM)$ defines $s^*(Y) \in C^\infty(\nu_s)$.
- Set

$$\partial_\nu^Y = i_{s^*(Y)} \circ \partial_\nu.$$

Definition

A BLS (Bounded Leafwise Smoothing) operator H on E is transversely smooth if $\forall Y_1, ..., Y_\ell \in C^\infty(TM)$, the operator $\partial_\nu^{Y_1} ... \partial_\nu^{Y_\ell}(H)$ is BLS on E.

Theorem (Benameur-H)

Let e be a $\mathcal{G}$ invariant transversely smooth idempotent on E. Then $\Phi \circ \operatorname{tr}\left(e\left[\exp\left(\frac{-(\delta e)^2}{2i\pi}\right)\right]\right)$ is a closed Haefliger form whose cohomology class is denoted $\operatorname{ch}_a(e)$. Moreover, if e_t, $0 \leq t \leq 1$, is a smooth family of such idempotents, then $\operatorname{ch}_a(e_0) = \operatorname{ch}_a(e_1)$.

The higher harmonic signature for foliations

- M a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ.
- $\Omega^*_{(2)}(F_s) \to M$ is the bundle of L^2 differential forms on leaves of F_s.
- τ the usual leafwise involution gives $\Omega^*_{(2)}(F_s) = \Omega^*_+(F_s) \oplus \Omega^*_-(F_s)$.
- The leafwise Laplacian Δ preserves this splitting.
- The leafwise operator $D = d + d^*$ reverses splitting, $D^2 = \Delta$.
- D defines $D^+ : \Omega^*_+(F_s) \to \Omega^*_-(F_s)$, the leafwise signature operator.
- $\mathrm{Ind}_c^\infty(D^+) \in K_0(C_c^\infty(\mathcal{G}; \mathrm{Hom}(\Lambda^{2\ell} T^* F_s)))$.
- Assume projection $P : \Omega^*_{(2)}(F_s) \to \mathrm{Ker}(\Delta)$ transversely smooth.
- Implies $P_\pm : \Omega^*_{(2)}(F_s) \to \mathrm{Ker}(\Delta^\pm) = \mathrm{Ker}(\Delta) \cap \Omega^*_\pm(F_s)$, and $\pi_\pm : \Omega^{2\ell}_{(2)}(F_s) \to \mathrm{Ker}(\Delta^\pm_{2\ell})$ are transversely smooth.

Definition

The Higher Harmonic Signature $\sigma(F)$ of (M, F) is the Haefliger class

$$\sigma(F) = \mathrm{ch}_a(\pi_+) - \mathrm{ch}_a(\pi_-).$$

Proposition

The classes $\mathrm{ch}_a(\pi_\pm)$ *do not depend on the metric.*

Definition

The Chern character of the index bundle of D^+ is

$$\mathrm{ch}_a(P_+) - \mathrm{ch}_a(P_-).$$

Theorem

The Chern character of the index bundle of the leafwise signature operator coincides with the Higher Harmonic Signature of F*, that is*

$$\mathrm{ch}_a(P_+) - \mathrm{ch}_a(P_-) = \mathrm{ch}_a(\pi_+) - \mathrm{ch}_a(\pi_-).$$

P_ϵ the spectral projection of $D^2 = \Delta$ for the interval $(0, \epsilon)$.

Recall that the Novikov-Shubin invariants of D are larger than α provided that there is $\beta > \alpha$ so that

$$\mathrm{Tr}(P_\epsilon) \sim \mathrm{O}(\epsilon^\beta) \text{ as } \epsilon \to 0.$$

Theorem (Benameur-H)

Suppose that M is a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ. Assume P and P_ϵ are transversely smooth, and that the Novikov-Shubin invariants of D are larger than $\operatorname{codim} F/2$. Then

$$\mathrm{ch}_a(\mathrm{Ind}_c^\infty(D^+)) = \mathrm{ch}_a(P_+) - \mathrm{ch}_a(P_-) = \sigma(F).$$

Theorem

$$\mathrm{ch}_a(\mathrm{Ind}_c^\infty(D^+)) = \int_F \mathrm{L}(F).$$

The leafwise homotopy invariance theorem

Theorem (Benameur-H)

Suppose that M is a compact Riemannian manifold with oriented Riemannian foliation F of dimension 4ℓ. *Assume that the projection onto the leafwise harmonic forms in dimension* 2ℓ *of the associated foliation* F_s *of its holonomy graph is transversely smooth. Then the leafwise signature* $\sigma(F)$ *of* F_s *is a leafwise homotopy invariant.*

Corollary

Assume in addition that P_ϵ *is transversely smooth and that the Novikov-Shubin invariants of D are larger than* $\operatorname{codim}(F)/2$. *Then for any closed invariant current C for F,*

$$< \sigma(F), C > = < \int_F \mathrm{L}(F), C > .$$

As the number on the left is a leafwise homotopy invariant, so is the number on the right.

Outline of proof of the LHI theorem

- $f : M, F \to M', F'$ a LHE induces a leafwise map $\widetilde{f} : \mathcal{G}, F_s \to \mathcal{G}', F'_s$, and an isomorphism $f^* : H^*_c(M'/F') \to H^*_c(M/F)$.
- Use $\widetilde{f}^*$ to pull back the transversely smooth idempotents $\pi'_\pm$ to transversely smooth idempotents $\pi^f_\pm$.
- Define connections on idempotents (à la Bismut), and show they give the Chern characters $ch_a(\pi^f_\pm)$ and $ch_a(\pi'_\pm)$.
- Prove
$$f^* \operatorname{ch}_a(\pi'_\pm) = \operatorname{ch}_a(\pi^f_\pm).$$
- Prove
$$\operatorname{ch}_a(\pi^f_\pm) = \operatorname{ch}_a(P_{2\ell}\pi^f_\pm).$$
- Prove
$$\operatorname{ch}_a(P_{2\ell}\pi^f_\pm) = \operatorname{ch}_a(\pi_\pm).$$

Use $\widetilde{f}^*$ to pull back the idempotents $\pi'_\pm$ to idempotents $\pi^f_\pm$.

Problem. In general, $\widetilde{f}^*$ does NOT induce a map on L^2 leafwise forms.
Solution. Adapt Hilsum-Skandalis to define $\widetilde{f}^*$ on L^2 leafwise forms.

HS essentially says can assume that $\widetilde{f}$ is a leafwise submersion. We prove such $\widetilde{f}$ induce bounded maps on all leafwise Sobolev spaces.

Problem. Action of HS $\widetilde{f}^*$ on the algebra of forms not so obvious.

Solution. Use the results of H-Lazarov (à la Dodziuk) to construct another $\widetilde{f}^*$, show that on leafwise cohomology it is the same as the HS $\widetilde{f}^*$, and that the HL $\widetilde{f}^*$ has the needed algebraic properties.

Definition

Let $g : M', F' \to M, F$ be a homotopy inverse for f. Set

$$\pi^f_\pm = \widetilde{f}^* \pi'_\pm \widetilde{g}^* P_{2\ell}.$$

Proposition

The $\pi^f_\pm = \widetilde{f}^*\pi'_\pm\widetilde{g}^*P_{2\ell}$ *are transversely smooth idempotents.*

Proof.

$P_{2\ell}$ and $\pi'_\pm$ are TS, so take any Sobolev space to any Sobolev space.
The operator $\nabla^\nu = d_\nu + A$ where $d_\nu = p_\nu d_{\mathcal{G}}$, and A is a leafwise operator of order zero. Similarly $\nabla^{\nu'} = d_{\nu'} + A'$.
So A and A' take any leafwise Sobolev k space to itself.
$\widetilde{f}^*$ and $\widetilde{g}^*$ are bounded maps on all leafwise Sobolev k spaces.

Lemma

$$d_\nu\widetilde{f}^* = \widetilde{f}^*d'_\nu + \widetilde{f}^*d'_s, \quad \text{and} \quad d'_\nu\widetilde{g}^* = \widetilde{g}^*d_\nu + \widetilde{g}^*d_s.$$

d_s and d'_s are the leafwise de Rham operators, so take leafwise Sobolev k spaces to leafwise Sobolev $k-1$ spaces.
This allows us to relate $\partial^\nu = [\nabla^\nu, \cdot]$ and $\partial^{\nu'} = [\nabla^{\nu'}, \cdot]$.
A good deal of functional analysis finishes the proof. □

Define connections on transversely smooth idempotents, and show they give the Chern characters $ch_a(\pi^f_\pm)$ and $ch_a(\pi'_\pm)$.

Definition

Suppose $e : \Omega^*_{(2)}(F_s) \to \Omega^*_{(2)}(F_s)$ is a transversely smooth idempotent.

$$\xi \in C^\infty(e) \iff \xi \in \Omega^*_{(2)}(F_s) \cap \Omega^*(\mathcal{G}) \text{ and } e(\xi) = \xi.$$

Definition

A connection ∇ on e is a linear map
$\nabla : C^\infty(e) \otimes \Omega^*(M) \to C^\infty(e) \otimes \Omega^*(M)$ of order one, satisfying

1. $\nabla(\xi \otimes \omega) = \nabla\xi \wedge \omega + \xi \otimes d_M\omega$.
2. for local invariant $\xi \in C^\infty(e)$, and $X \in C^\infty(TF)$, $\nabla_X\xi = 0$.
3. ∇ is $\mathcal{G}$-invariant.
4. $A = e\Big(\nabla - \nabla^\nu\Big)e : \Omega^*_{(2)}(F_s) \otimes \Omega^*(M) \to C^\infty(e) \otimes \Omega^*(M)$ is transversely smooth.

Some Comments

- ξ local invariant if $\xi([\gamma\gamma_1]) = \xi([\gamma])$, i.e. $\xi([\gamma])$ depends only on $r[\gamma]$, so ∇ is flat along F.
- ∇ is $\mathcal{G}$-invariant says $\nabla \,|\, \widetilde{L}_x = \nabla \,|\, \widetilde{L}_y$, where $x, y \in L$.
- ∇ flat along F and $\mathcal{G}$-invariant says ∇ looks like the pull back under r of an operator on leaves of F.
- $e\nabla^\nu e$ is a connection on e.
- Condition 4 says ∇ is $e\nabla^\nu e$ plus a transversely smooth operator.

Proposition

- *∇ is determined by $\nabla \,|\, T$. That is, ∇ really is a "connection" on the "bundle" e defined over the "space of leaves of F."*
- *If ∇ is a connection on e with curvature θ then* $\mathrm{Tr}(e[\exp(-\theta/2i\pi)])$ *is a closed Haefliger form whose class is independent of ∇.*
- $[\mathrm{Tr}(e[\exp(-\theta/2i\pi)])] = \mathrm{ch}_a(e)$.

Proposition

$$f^* \, \mathrm{ch}_a(\pi'_\pm) = \mathrm{ch}_a(\pi^f_\pm).$$

Proof.

If ∇' is a connection on π'_+, it defines the pull-back connection $\nabla = \widetilde{f}^*(\nabla')$ on π^f_+. Fundamental idea is that we only need to know what ∇' does on $s^{-1}(T')$, combined with fact that $f \mid T \to T'$ is locally a diffeo, so can push transverse vectors forward.

Also use fact that $\widetilde{f}^*$ and $\widetilde{g}^*$ are inverses of each other on leafwise cohomology to push forward sections of $\pi^f_\pm$ to sections of $\pi'_\pm$, and vice-versa.

Then $\theta = \widetilde{f}^*(\theta')$ and $\mathrm{Tr}(\theta^k) = f^* \, \mathrm{Tr}({\theta'}^k)$ for all k, which gives the result. □

Proposition

$$\mathrm{ch}_a(\pi^f_\pm) = \mathrm{ch}_a(P_{2\ell}\pi^f_\pm).$$

Proof.

$(1-t)P_{2\ell}\pi^f_\pm + t\pi^f_\pm$ is a smooth family of TS idempotents. □

Finally,

Proposition

$$\mathrm{ch}_a(P_{2\ell}\pi^f_\pm) = \mathrm{ch}_a(\pi_\pm).$$

Proof.

Restriction of $\pi_\pm$ to $\mathrm{Im}(P_{2\ell}\pi_\pm^f)$ is an isomorphism onto $\mathrm{Im}(\pi_\pm)$ with uniformly bounded inverse.

Main Step: $\varrho_\pm = \pi_\pm^{-1} \circ \pi_\pm : \Omega^{2\ell}_{(2)}(F_s) \to \mathrm{Im}(P_{2\ell}\pi_\pm^f)$ is a TS idempotent.

Proof involves a good deal of heavy functional analysis.

To finish we need two easy results.

1. The TS idempotents $\varrho_\pm$ and $P_{2\ell}\pi_\pm^f$ have the same image, so $t\varrho_\pm + (1-t)P_{2\ell}\pi_\pm^f$ is a smooth family of TS idempotents, and

$$\mathrm{ch}_a(P_{2\ell}\pi_\pm^f) = \mathrm{ch}_a(\varrho_\pm).$$

2. Since $\varrho_\pm$ is projection onto $\mathrm{Im}(P_{2\ell}\pi_\pm^f)$ along $\mathrm{Ker}(\pi_\pm)$, we have $\varrho_\pm\pi_\pm = \varrho_\pm$ and $\pi_\pm\varrho_\pm = \pi_\pm$. Thus, $t\varrho_\pm + (1-t)\pi_\pm$ is a smooth family of TS idempotents, and

$$\mathrm{ch}_a(\varrho_\pm) = \mathrm{ch}_a(\pi_\pm).$$

Same techniques prove:

Theorem (Benameur-H)

M a cpt Riemannian mfd; F an oriented Riemannian dim 2ℓ foliation. $E \to M$ a leafwise flat $\mathbb{C}$ bundle with an (indefinite) non-degenerate Hermitian metric, preserved by the leafwise (for F) flat structure. Assume that the proj. to $\mathrm{Ker}(\Delta_\ell^E)$ (for F_s) is transversely smooth. Then $\sigma(F, E)$, the leafwise (for F_s) signature with coefficients in $r^*(E)$, is a leafwise homotopy invariant.

Corollary

Assume that P_ϵ^E is transversely smooth ($\epsilon \sim 0$), and the N-S invariants of D_E are $> \mathrm{codim}(F)/2$. Then for any closed invariant current C for F,

$$< \sigma(F, E), C > = < \int_F \mathrm{L}(F)\,\mathrm{ch}(E), C > .$$

As the number on the left is a leafwise homotopy invariant, so is the number on the right.