Applications of Lie Manifolds to PDEs, to Analysis on Polyhedral Domains, and to Numerical Methods

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(left from last time)

• **The integration problem:**

Given a Lie algebroid $A \to \overline{M}$, to find a Lie groupoid $\mathcal{G}$ with $A(\mathcal{G}) \simeq A$. Not always possible. It is possible for Lie algebroids coming from Lie manifolds: Pradines, N., Debord, Crainic-Fernandez.

• Idea: if we can integrate $A \to Y$ and $A \to Y^c$, $Y^c := \overline{M} \smallsetminus Y$, and if we choose *connected, simply connected* integrating groupoids, then we can glue them and get a Lie groupoid (N, '89).

• It is important to be able to integrate "simple groupoids."

• Let $A \to \overline{M}$ be an *arbitrary* Lie algebroid. Denote for each $p \in \overline{M}$ by $k_p := \ker(A_p \to T_p\overline{M})$. Then k_p is naturally a Lie algebra (finite dimensional).

• Assume $\Gamma(A)$ consists of vector fields tangent to all faces of $\overline{M}$, then $\exp(X)$ is defined for all $X \in \mathcal{V}$ and gives rise to a commutative diagram

$$\begin{array}{ccc} \exp(X) : A & \xrightarrow{\simeq} & A \\ \uparrow & & \downarrow \\ \exp(X) : T\overline{M} & \xrightarrow{\simeq} & T\overline{M} \end{array}$$

• Let S be an orbit of a point $p \in \overline{M}$, then it is an immersed manifold and the Lie algebras k_p, $p \in S$ are non-canonically isomorphic. (Obstruction to integration ... ?)

• Assume $A|_S = TS \oplus k$ as Lie algebroids, for some Lie algebra k. Then we can integrate to obtain $\mathcal{G} = S \times S \times K$, where K is *any* Lie group with Lie algebra k. Not simply connected in general, choose $\mathcal{P}S \times K$ with K simply connected.

Index Theory

Let us fix $(\overline{M}, \mathcal{V})$ Lie manifolds, M the interior of $\overline{M}$.

Problem: Prove an index theorem for Fredholm operators on Lie M.

What is the structure and the meaning of the non-local invariants coming from the structure at infinity? (Piazza program,...)

A possible approach: let $\mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$, and consider the quotient

$$\mathfrak{B} := \mathfrak{A}/\mathcal{K}.$$

This quotient is $\mathcal{C}^\infty(S^*M)$ in the compact case, but it is non-commutative in the non-compact case.

An elliptic, Fredholm operator $P \in \mathrm{Diff}(\mathcal{V})$ on M gives rise in the usual way to a Fredholm operator $Q = P(1 + P^*P)^{-1/2} \in \mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$, as in the previous lecture, which will be invertible in $\mathfrak{B}$.

As we have seen, both $\mathfrak{A}$ and $\mathfrak{B}$ can be described explicitly using groupoids.

The map index map

$$ind = \partial : K_1(\mathfrak{B}) \to \mathbb{Z} = K_0(\mathcal{K})$$

coming from $0 \to \mathcal{K} \to \mathfrak{A} \to \mathfrak{B} \to 0$ can be described *in principle* using cyclic homology (using “smooth algebras.”)

This is due to many people, on different levels of generality and applicability: Bunke, Connes, Karoubi, Quillen, Tsygan, N., in general, for cylindical ends Lauter-Moroianu, Leichtnam-Piazza, Melrose-N., Moroianu-N. ... , foliations Benameur-Heitch, Connes, Gorokhovsky-Lott, N., Piazza, ...

The *explicit form* of the index cocycle is however not understood well enough.

Applications to PDEs

Reduction to the boundary

Regular elliptic boundary value problems of the form

$$Du = f \text{ in } \Omega, \quad Bu = f \text{ on } \partial\Omega$$

on a smooth, bounded domain can be reduce to an elliptic pseudodifferential equation $Pv = h$ on $\partial\Omega$. (Layer potentials, Cauchy Data spaces: Seeley, Ballmann-Brunning-Carron, ...)

The first obstruction to solving this equation is the index of P.

The index of P is determined by the Atiyah-Singer Index Theorem.

Natural question: what about polyhedral domains?

The method of Layer Potentials

Consider the Poisson problem

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^3$.

Equivalent to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Let

$$E(x) = c_n |x|^{2-n}$$

$=$ the usual fundamental solution of Δ on $\mathbb{R}^n$.

We try our solution in the form of a single layer potential

$$u = \mathcal{S}(h) := \int_{\partial\Omega} E(x-y)h(y)d\sigma(y)$$

with σ the surface measure.

Then u is harmonic: $\Delta u = 0$ and its boundary values are $u|_{\partial\Omega} = Sh$, with S an elliptic pseudodifferential operator of order -1. It is also self-adjoint.

Our equation reduces to $Sh = g$. Since S is Fredholm of index zero, it is enough to prove that S is injective. PDE tricks.

For polyhedral domains the single layer operator S is in a groupoid pseudodifferential operators algebra.

Question: extend the usual results on layer potentials to the polyhedral case. (Fredholm conditions.)

Numerical methods: the "Boundary Element Method"

Applications to Schrödinger operators

(to be included)

Optimal Rates of Convergence for FEM

Numerical method: the "Finite Element Method" (FEM)

Consider the Poisson problem

$$\begin{cases} \Delta u := \partial_x^2 u + \partial_y^2 u + \partial_z^2 u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

on a bounded, polyhedral domain $\Omega \subset \mathbb{R}^3$.

We construct a sequence of tetrahedralizations (*i.e.* meshes) $\mathcal{T}_k'$ of Ω with the property

$$\|u - u_k\|_{H^1(\Omega)} \leq C \dim(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)},$$

with C independent of k and f. Here:

- S_k is the associated *finite element space* of continuous, piecewise polynomials of degree $m \geq 2$.

- $u_k \in S_k$ is the *finite element approximation* of the solution of our Poisson problem above.

u_k has quasi-optimal approximation properties with respect to the dimension of S_k ($m \geq 2$).

Our method relies on the a priori estimate

$$\|u\|_{\mathcal{D}^{m+1}_{a+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)}$$

in anisotropic weighted Sobolev spaces $\mathcal{D}^{m+1}_{a+1}(\Omega)$, with $a > 0$ small and determined by Ω. The weight is the distance to the set of singular boundary points (*i.e.* edges).

This estimate is proved using Lie manifolds (the regularity theorem with Ammann & Ionescu).

For smooth domains this estimate becomes simply

$$\|u\|_{H^{m+1}(\Omega)} \leq C\|f\|_{H^{m-1}(\Omega)},$$

which, we have discussed many times, does *not* extend to *non-smooth* domains.

Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be an open subset, $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$ be a basic differential monomial, $|\alpha| = \alpha_1 + \dots + \alpha_d$.

$$L^2(\Omega) = \{u : \Omega \to \mathbb{C},\ \int_\Omega |u(x)|^2 dx < \infty\}.$$

mth Sobolev spaces $H^m(\Omega)$:

$$H^m(\Omega) := \{u,\ \partial^\alpha u \in L^2(\Omega),\ \forall |\alpha| \le m\}.$$

An important variant (for Ω nice)

$$H_0^1(\Omega) := H^1(\Omega) \cap \{u = 0,\ \text{on } \partial\Omega\}.$$

Let

$$\nabla u = (\partial_1 u, \partial_2 u, \dots, \partial_d u).$$

Basic bilinear form:

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$$
$$= \int_\Omega \Big(\partial_1 u \partial_1 v + \partial_2 u \partial_2 v + \dots + \partial_d u \partial_d v\Big) dx.$$

Weak formulation

We shall take $u \in H_0^1(\Omega)$, which includes the (Dirichlet) boundary conditions.

$$
\begin{aligned}
&-\Delta u = f,\ u \in H_0^1(\Omega) \\
&\quad \Leftrightarrow -\int_\Omega (\Delta u) v dx = \int_\Omega f v dx,\ \forall v \in H_0^1(\Omega), \\
&\qquad \Leftrightarrow \int_\Omega \nabla u \cdot \nabla v dx =: B(u,v) = \int_\Omega f v dx.
\end{aligned}
$$

The weak formulation of our Poisson problem:

Find $u \in H_0^1(\Omega)$ *such that*

$$
B(u,v) = \int_\Omega f v dx,
$$

for all $v \in H_0^1(\Omega)$.

(Poincaré inequality and Lax-Milgram Lemma imply that u exists, is unique, and depends continuously on $f \in H^{-1}(\Omega) := H_0^1(\Omega)^*$, well posedness for the Poisson problem with Dirichlet boundary conditions.)

Discretization

Assume we are given a finite dimensional subspace $S \subset H_0^1(\Omega)$. Then we define the discrete solution of our Poisson problem (1) as the unique $u_S \in S$ satisfying

$$B(u_S, v_S) = \int_\Omega f v_S dx, \ \forall v_S \in S.$$

Consider a basis ϕ_j of S, so that $u_S = \sum x_j \phi_j$. Take then $v_S = \phi_k$ for each k to obtain a system

$$K_S u_S = f_S$$

of size $\dim(S) \times \dim(S)$.

Basic question: What is the relation between the discrete solution $u_S \in S$ and the actual solution $u \in H_0^1(\Omega)$ of our Poisson equation:

u_S is the projection of the solution u onto S (in the inner product defined by the bilinear form B).

Consider a sequence of subspaces $S_k \in H_0^1(\Omega)$ and denote $u_k = u_{S_k}$.

If $S_k \subset S_{k+1} \subset \ldots$ and $\cup S_k$ is dense in $H_0^1(\Omega)$, then

$$\|u - u_k\|_{H^1(\Omega)} \to 0, \text{ as } k \to \infty.$$

Solving the system $K_S u_S = f_S$ is expensive and the amount of work required growth with $\dim(S)$.

The amount of work needed to solve the system is $\sim \dim(S)^{7/3}$ for direct elimination, $d = 3$ (L. Greengard) and $\sim \dim(S)$ for Multi-grid (optimal).

We want $\dim(S)$ as small as possible.

More on solving this system below, when we will compare several methods.

Basic FEM Estimate

We partition Ω in triangles or tetrahedra (mesh) $\mathcal{T}$.

Take $S =$the space of continuous, piecewise polynomials of degree m on some mesh $\mathcal{T}$ on Ω.

We assume:

• the angles appearing in mesh are bounded from below by $\theta > 0$ and

• the sizes of the triangles or tetrahedra T appearing in the mesh are comparable:

$$\text{diam}(T)/\,\text{diam}(T') \leq R.$$

A quasi-uniform sequence of meshes $\mathcal{T}_k$ if we can chose θ and R *independent* of n.

We let S_k be the associated finite element spaces of continuous piecewise polynomials of degree m and denote by $u_k = u_{S_k}$ the associated discrete solution.

Assume that $u \in H^{m+1}(\Omega)$ (i.e. $(m+1)$-square integrable derivatives) and let h_k be the maximum diameter of $T \in \mathcal{T}_k$. Then we have the following quasi-optimal rate of convergence

$$\|u - u_k\|_{H^1} \leq C h_k^m \|u\|_{H^{m+1}},$$

with C independent of k and f ($\Omega \subset \mathbb{R}^d$, $d = 2, 3$ in our case).

Fortunately, a basic result in Partial Differential Equations provides us with conditions on f that will insure $u \in H^{m+1}(\Omega)$. Let us recall this basic result next.

Theorem. If $f \in H^{m-1}(\Omega)$ and Ω has smooth boundary, then the solution u of the Poisson equation (1) satisfies $u \in H^{m+1}(\Omega)$ and, moreover

$$\|u\|_{H^{m+1}} \leq C\|f\|_{H^{m-1}},$$

This result is the well posedness of the Poisson problem with Dirichlet boundary conditions in the H^{m+1} spaces. The inequality of the above theorem is called an *a priori estimate*.

Combining the basic Finite Element Estimate with the apriory estimate of the above theorem yields that for quasi-uniform meshes we have

$$\|u - u_k\|_{H^1} \leq Ch_k^m \|u\|_{H^{m+1}} \leq Ch_k^m \|f\|_{H^{m-1}}.$$

As we will see, the framework of quasi-uniform meshes is not sufficient, so we want to replace the above estimate with something that is independent of the mesh size. Indeed, taking into account that, for a quasi-uniform family of meshes, the number of vertices, edges, triangles (and tetrahedra in 3D, i.e. $d = 3$) has the order of h_k^{-d}, we obtain that $\dim(S_k) \sim h_k^{-d}$, or $h_k \sim \dim(S_k)^{-1/d}$. Since the relevant quantity is $\dim(S_k)$, whether the sequence of meshes is quasi-uniform or not, we see that the above equation can be replaced with

$$\|u - u_k\|_{H^1} \leq C\dim(S_k)^{-m/d}\|f\|_{H^{m-1}}, \quad (2)$$

which is the equation that gives quasi-optimal rates of convergence for an arbitrary sequence of meshes, whether quasi-uniform or not.

We have seen that this happens for $u \in H^{m+1}(\Omega)$, but this is where our problem begins.

The problem is that, in general, $u \notin H^{m+1}(\Omega)$; we are guaranteed only $u \in H^{s+1-\delta}(\Omega)$, where $s = \pi/\alpha_{MAX}$ for polygonal domains. (For more general on Lipschitz domains see: **Costabel, Dauge, Grisvard, Jerison, Kenig, Mitrea, Verchota, Vogel, Taylor.**)

The loss of regularity can very easily be seen by looking again at the Poisson problem $\Delta u = f$, $u \in H^1(\Omega)$. The above well-posedness Theorem for the Poisson equation gives, in particular, that if f, g, and $\partial\Omega$ are smooth, then u is also smooth (including the boundary).

This is not true if $\partial\Omega$ is not smooth, as seen from the following simple example. Let $\Omega = (0,1)^2$ and assume that u is smooth. Then

$$\partial_x^2 u(0,0) = 0 = \partial_y^2 u(0,0)$$

and hence $f(0,0) = \Delta u(0,0) = 0$ is a necessary condition for $u \in \mathcal{C}^\infty(\overline{\Omega})$, which is however not always satisfied.

In fact, not only regularity is lost, we also obtain much lower convergence rate

$$\mathrm{dim}(S_k)^{-s/d}$$

when quasi-uniform (QU) meshes are used.

Comparison of methods

Assume our problem is to approximate u within ϵ.

First Main Result: a method to replace the quasi-uniform meshes with adaptive (AD) type of meshes, so that the quasi-optimal rate of convergence are restored.

Take $\alpha_{MAX} = 3\pi/2$, so $s = 2/3$ in $\dim(S_k)^{-s/d}$.

n	mesh	m	DE work	MG work
2	QU	–	$(1/\epsilon)^6$	$(1/\epsilon)^3$
2	AD	1	$(1/\epsilon)^4$	$(1/\epsilon)^2$
2	AD	3	$(1/\epsilon)^{1.3}$	$(1/\epsilon)^{0.6}$
3	QU	–	$(1/\epsilon)^{10.5}$	$(1/\epsilon)^{4.5}$
3	AD	1	$(1/\epsilon)^7$	$(1/\epsilon)^3$
3	AD	3	$(1/\epsilon)^{2.3}$	$(1/\epsilon)$

QU="Quasi-uniform meshes", AD="Adaptive meshes", DE="Direct elimination solver", MG="Multigrid solver".

most work $\sim$ (least work)^{10}.

If $(1/\epsilon) = 1000$, $m = 3$, then AD may work 1 billion times faster than QU.

In practice, $(1/\epsilon)$ can be $>> 1000$.

This completes our introduction to the Finite Element Method and motivates the problem of restoring the quasi-optimal rate of convergence:

$$\|u - u_k\|_{H^1(\Omega)} \leq C \, \mathrm{dim}(S_k)^{-m/3} \|f\|_{H^{m-1}(\Omega)},$$

- Apriori estimates for boundary value problems on polyhedral domains

- Mesh refinement

A priori estimates

Weighted Sobolev spaces (anisotropical and isotropical).

$\vartheta(x) =$ the distance from $x \in \Omega$ to the edges of Ω.

The isotropically weighted Sobolev spaces $\mathcal{K}^m_a(\Omega)$ are defined by

$$\mathcal{K}^m_a(\Omega) := \{u,\ \vartheta^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ \ |\alpha| \leq m\},$$

$m \in \mathbb{Z}_+$, $a \in \mathbb{R}$ (Kondratiev, Babuška, ...). For $a = n/2$ these are the Sobolev spaces associated to a Lie manifold with metric $\varrho^{-1} g_{euclidean}$.

For example, $H^m(\Omega) \subset \mathcal{K}^m_0(\Omega) \subset \mathcal{K}^m_b(\Omega)$, if $b \leq 0$.

The index m measures regularity, as usual, and the index a measures the decay towards the singular points (the edges).

The behavior of the spaces $\mathcal{K}_a^m(\Omega)$ with respect to the indices a and m is very similar. Thus, if P is a differential operator of order k with smooth coefficients, we have that

$$P : \mathcal{K}_a^m(\Omega) \to \mathcal{K}_{a-k}^{m-k}(\Omega).$$

Also,

$$\mathcal{K}_a^m(\Omega) \subset \mathcal{K}_{a'}^{m'}(\Omega)$$

if $m \geq m'$ and $a \geq a'$.

Theorem. Second Main Result. (Bacuta-Mazzucat-N.-Zikatanov)
Let $m \in \mathbb{Z}_+$ and $\Omega \subset \mathbb{R}^n$ be a bounded, polyhedral domain (curved faces). Then there exists $\eta > 0$ such that the our Poisson problem ($\Delta u = f$, $u|_\Omega = 0$) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega)$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$ which depends continuously on f:

$$\|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)} \leq C_{\Omega,a} \|f\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)}$$

for all $|a| < \eta$.

It extends to elasticity in three dimensions.

Equivalently, the map

$$\Delta : \mathcal{K}^{m+1}_{a+1}(\Omega) \cap \{u|_{\partial\Omega} = 0\} \to \mathcal{K}^{m-1}_{a-1}(\Omega)$$

is a continuous isomorphism. (**Well-posedness**, true also for combinations of Neumann and Dirichlet bdry. cond.)

This is where the geometry of **Lie manifolds** is used repeatedly.

In general, $u \notin H^{m+1}(\Omega)$, because Ω is not a smooth domain.

We need more regularity along the edge. (Skipped in the lecture).

Assume $\Omega = D_\alpha = \{0 < \theta < \alpha\}$, a dihedral angle with edge along the Oz–axis.

Assume $f \in H^{m-1}(D_\alpha)$. Then or previous theorem states that $u \in \mathcal{K}^{m+1}_{a+1}(D_\alpha)$, $0 < a$ small. Therefore

$$\partial_z u \in \mathcal{K}^m_a(D_\alpha).$$

However, we also have

$$\Delta \partial_z u = \partial_z \Delta u = \partial_z f \in H^{m-2}(\Omega).$$

Then the same theorem as before implies then

$$\partial_z u \in \mathcal{K}^m_{a+1}(\Omega),$$

which is better than above.

This suggests to introduce the following spaces

$$\mathcal{D}_a^1(D_\alpha) := \mathcal{K}_1^1(D_\alpha)$$
$$\mathcal{D}_a^m(D_\alpha) := \{u \in \mathcal{K}_a^m(D_\alpha),\ \partial_z u \in \mathcal{D}_a^{m-1}(D_\alpha)\}.$$

Thus our spaces $\mathcal{D}_a^1$ are, in fact, independent of a. We endow the space $\mathcal{D}_a^m(D_\alpha)$ with the norm

$$\|u\|^2_{\mathcal{D}_a^m(D_\alpha)} := \|u\|^2_{\mathcal{K}_a^m(D_\alpha)} + \|\partial_z u\|^2_{\mathcal{D}_a^{m-1}(D_\alpha)}.$$

If $\Omega = \mathcal{C}$, a cone centered at the origin. $\rho(x) = |x|$ is the distance from x to the origin. Then we let.

$$\mathcal{D}_a^1(\mathcal{C}) := \rho^{a-1}\mathcal{K}_1^1(\mathcal{C}) = \{\rho^{a-1}v,\ v \in \mathcal{K}_1^1(\mathcal{C})\},$$

with norm $\|u\|_{\mathcal{D}_a^1(\mathcal{C})} := \|u/\rho^{a-1}\|_{\mathcal{K}_a^1(\mathcal{C})}$.

For $m \geq 2$, let $\rho\partial_\rho = x\partial_x + y\partial_y + z\partial_z$ be the infinitesimal generator of dilations. Then, for $m \geq 2$, we define by induction

$$\mathcal{D}_a^m(\mathcal{C}) := \{u \in \mathcal{K}_a^m(\mathcal{C}),\ \rho\partial_\rho(u) \in \mathcal{D}_a^{m-1}(\mathcal{C})\}.$$

with a similarly defined norm.

For a general bounded polyhedral domain Ω, we define the *anisotropic weighted Sobolev spaces* $\mathcal{D}_a^m(\Omega)$ by localization around vertices, edges, such that in away from the edges we have the usual Sobolev spaces H^m.

Theorem. (Bacuta-N.-Zikatanov)
Let $m \geq 1$ and $f \in H^{m-1}(\Omega)$. Then there exists $\eta \in (0, 1]$ such that our Poisson problem has a unique solution $u \in \mathcal{D}_{a+1}^{m+1}(\Omega)$. This solution depends continuously on f, for any $0 \leq a < \eta$ and any m:

$$\|u\|_{\mathcal{D}_{a+1}^{m+1}(\Omega)} \leq C_{\Omega,a}\|f\|_{H^{m-1}(\Omega)}.$$

(**Arnold-Falk, Apel99, ApelNicaise, Babuška-Guo, Bacuta-Bramble-Xu, Buffo-Costabel-Dauge03, Kellogg-Osborn, ...**)

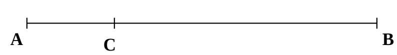
Refinement

• We shall construct a sequence of divisions (partitions) of Ω into polyhedral domains (mostly tetrahedra and prisms).

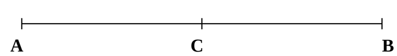
• Points are divided into type **V, E, S**. Type **V** is more singular than **E**, which in turn is more singular than **S**.

The edges will be of one of the types **VE, VS, ES, SS** and the triangles will be of one of the types **VES, VSS, ESS**.

• We divide the edges as explained



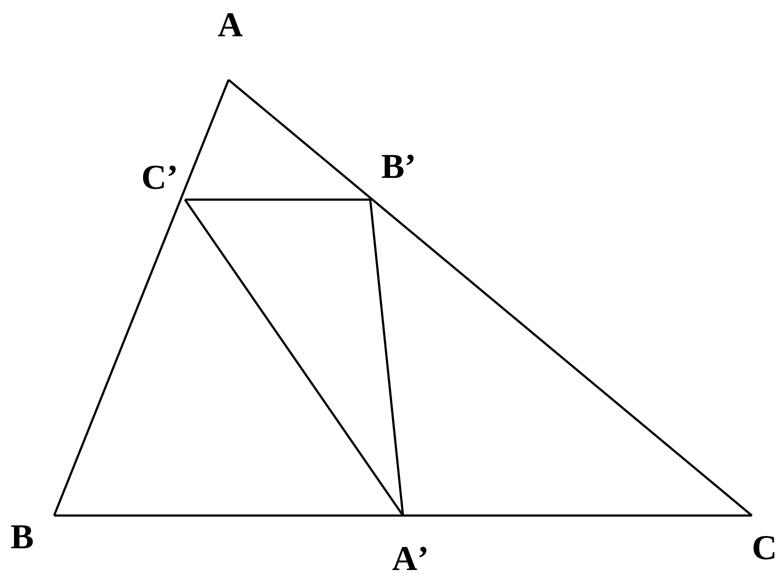
A more singular than B



A and B equally singular

• This leads to the following decompositions of triangles.

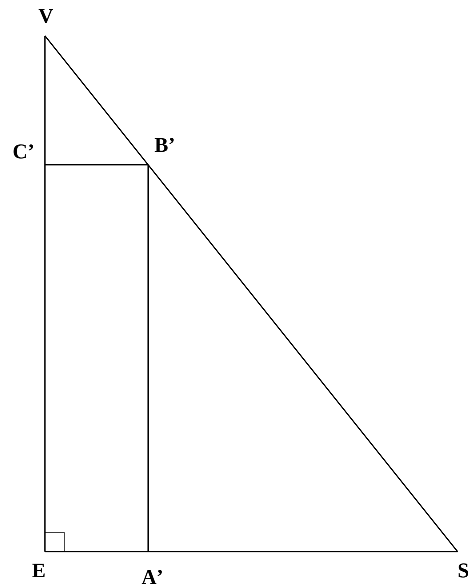
Type **VSS**:



Face decomposition:
A of type **V** or **E**,
B and C of type **S**,
$|AC'| = \kappa|AB|$, $|AB'| = \kappa|AC|$,
$|A'B| = |A'C|$, $\kappa = 1/4$

The decomposition of a triangle of type $\mathbf{S}^3$ is obtained by setting $\kappa = 1/2$ in the above picture. This gives four congruent triangles.

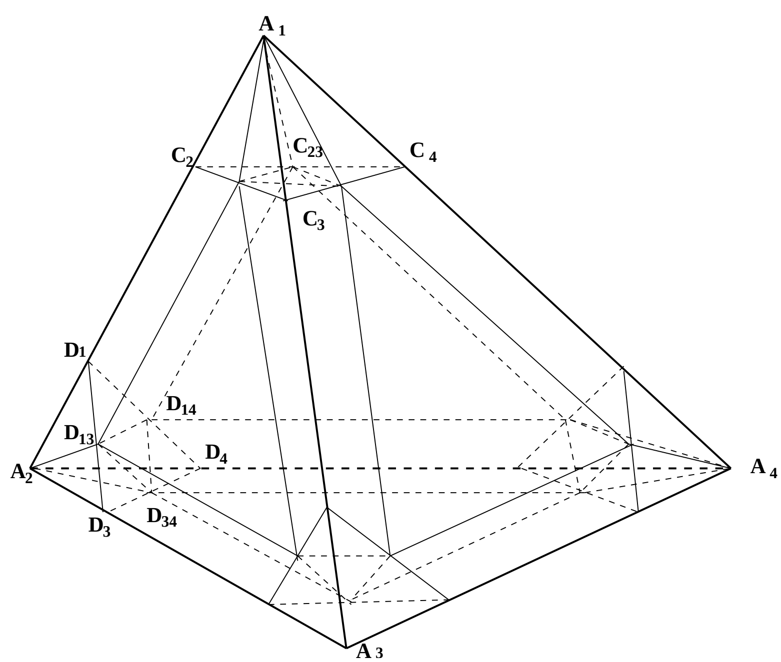
Finally, a triangle of type **VES** is decomposed as



$$|VC'| = \kappa|VE|,\ |VB'| = \kappa|VS|,\ |EA'| = \kappa|ES|,\ A'C'\ was\ removed,\ \angle E = 90^o$$

Division of Ω: a sequence of divisions $\mathcal{T}_n$ into tetrahedra and prisms for $n \geq 1$. The mesh $\mathcal{T}_n'$ is obtained from $\mathcal{T}_n$ by a canonical procedure, each prism $\longrightarrow$ 3 tetrahdedra, using some additional initial choices (marks on the prisms).

• $\mathcal{T}_0$ initial division of our polyhedral domains in straight triangular prisms, tetrahedra of types **VESS** and **VS**^{3} (thus having a vertex in common with Ω), and an interior region Λ_0.



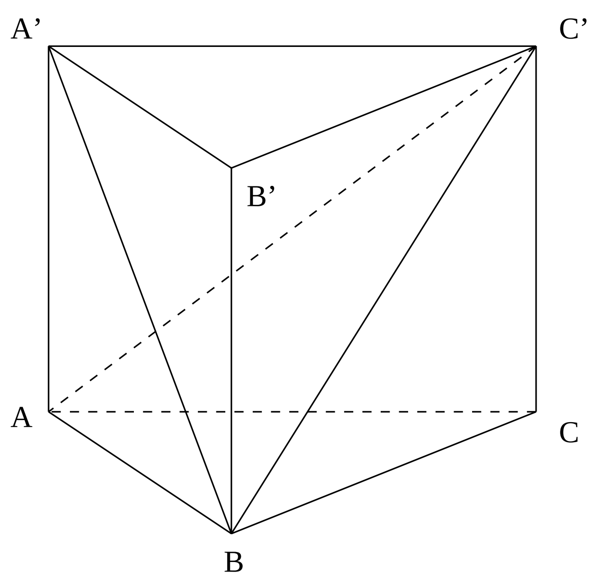
Initial decomposition.

We shall deform our edge points so that the prisms become straight triangular prisms (*i.e.*the edges are perpendicular to the bases).

• We tetrahedralize Λ_0 without introducing additional edges on the boundary of Λ (but allowing additional internal edges and vertices)

• We then apply uniform, semi-uniform, and non-uniform refinements to obtain the divisions $\mathcal{T}_n$ of Ω into marked prisms and tetrahedra.

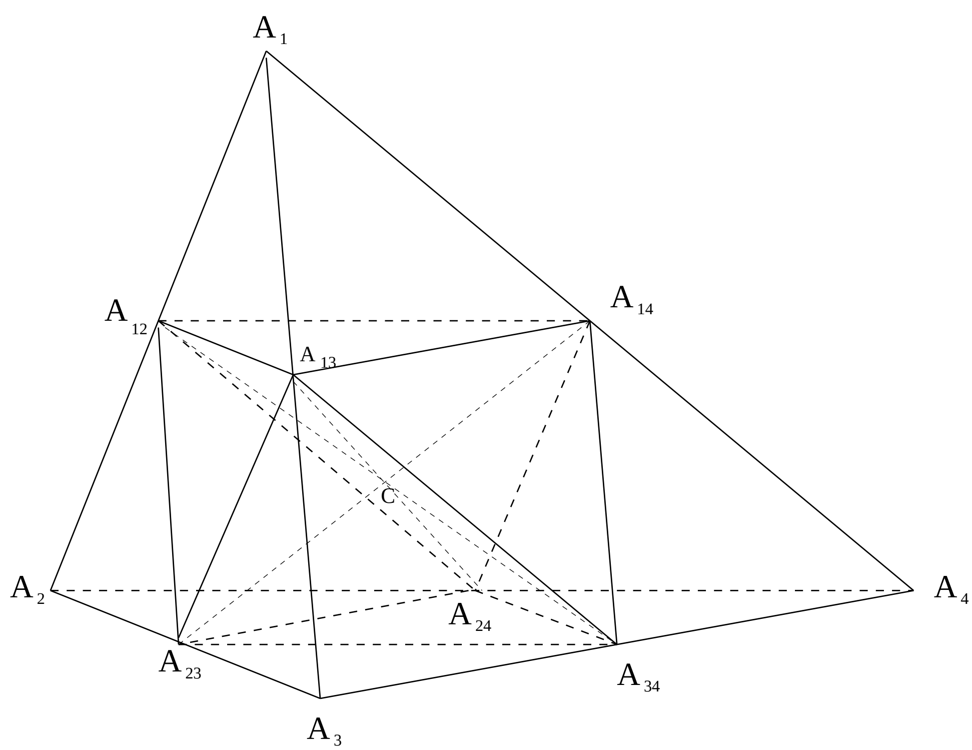
The meshes $\mathcal{T}_n'$ are obtained by dividing each prism into three tetrahedra using the mark.



$$BC' = mark,\ AA' \,||\, BB' \,||\, CC' \perp ABC || A'B'C'$$

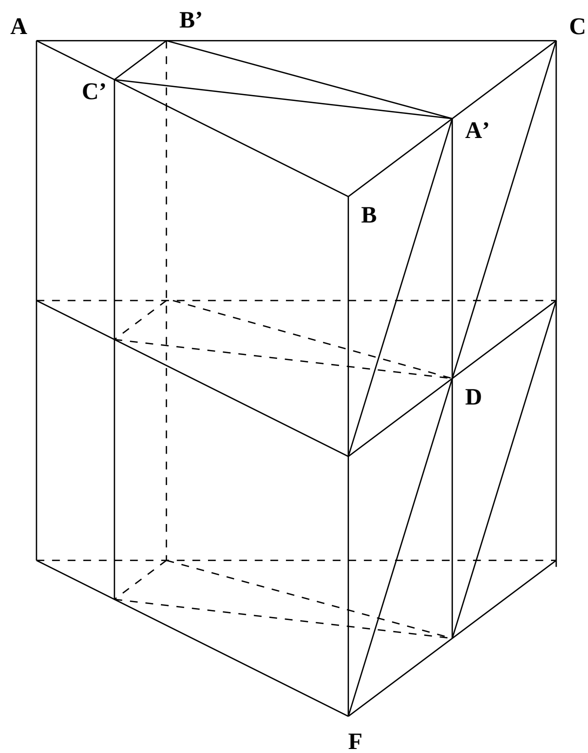
- Uniform refinement

For tetrahedra of type **S**^{4}. Use planes of the form $x_i + x_j = k/2^n$, $1 \le k \le 2^n$, where x_j are affine barycentric coordinates. Compatible with faces.



First level uniform barycentric refinement

- Semi-uniform refinement

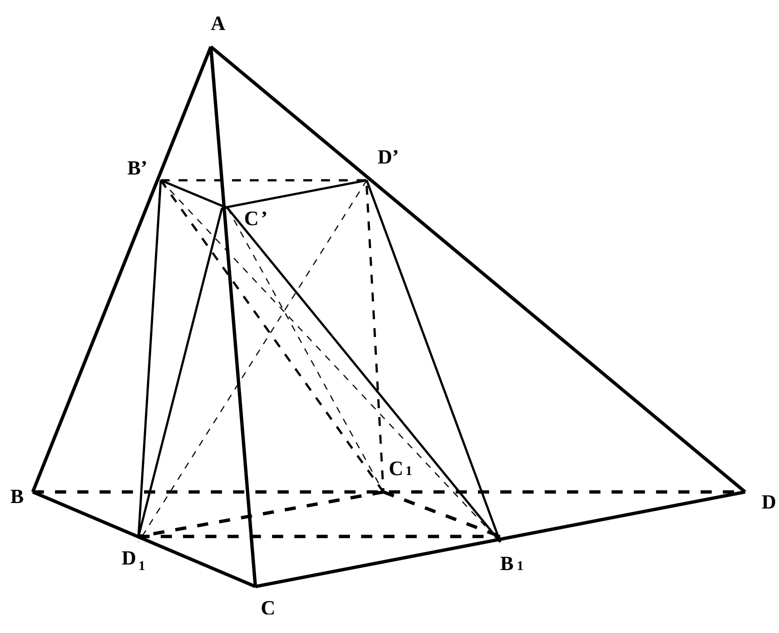


First level of semi-uniform refinement of a prism, $CD, DF =$ marks.

We divide the base in a non-uniform way (as a triangle of type **VSS**) n times, and we divide the edges in 2^n equal segments. $n = 1$ in the picture (level one).

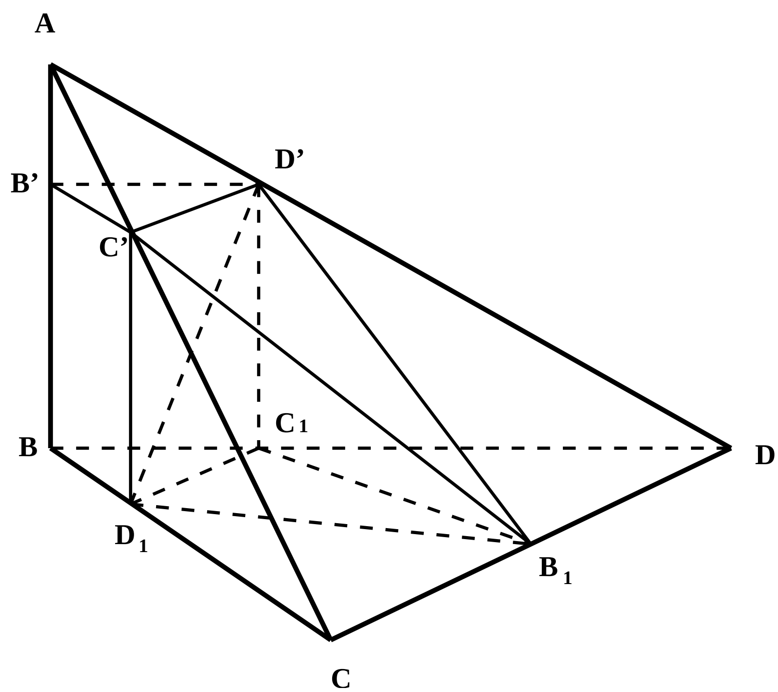
- Non-uniform refinement

We divide a tetrahedron T of type **VS**^{3} in 12 tetrahedra like in the uniform strategy, with the edges through the vertex of type **V** divided in the ratio κ. These tetrahedra will belong to $\mathcal{T}_{n+1}$. There will be a tetrahedron of type **VS**^{3} and 11 tetrahedra of type **S**^{4}. Then we iterate this construction for the small tetrahedron of type **VS**^{3}, whereas the tetrahedra of type **S**^{4} are divided according to the uniform strategy.



A of type **V**, B, C, D of type **S**

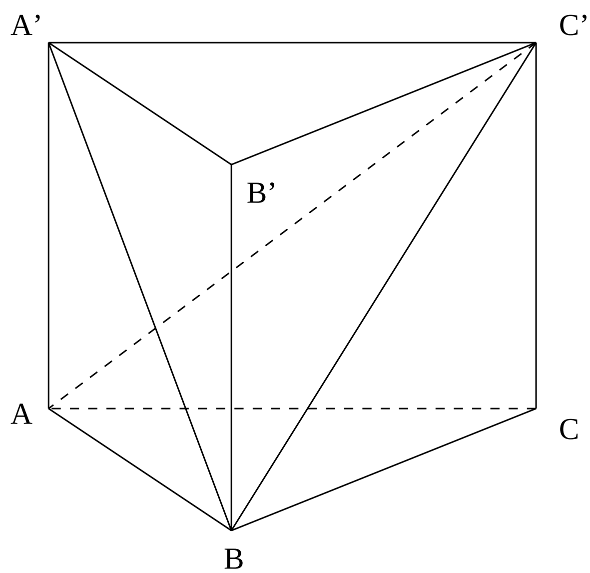
If T is a tetrahedron of type **VESS**, we divide it into 6 tetrahedra of type **S**^{4}, one tetrahedron of type **VS**^{3}, and a prism. The vertex of type **E** of T will belong only to the prism. This division is obtained by first dividing it into 12 pieces like in the uniform strategy. The union of the tetrahedron containing the vertex of type **E** and of the tetrahedra adjacent to it will form the prism.



A of type **V**, B of type **E**, C, D of type **S** and $D_1D' =$ mark for the prism $BD_1C_1D'C_1B'$

Interpolation on thin tetrahedra

Let $ABCA'B'C'$ be a straight prism divided into three tetrahedra.



$BC' = mark$, $AA' \,||\, BB' \,||\, CC' \perp ABC$ and $A'B'C'$

Let $\hat{\sigma}$ be any of these tetrahedra and $m \geq 2$. Let $u \in \mathcal{C}^1(\hat{\sigma})$ and $I(u) = u_I$ be interpolant associated to the linear m-simplex.

Theorem. We have

$$\|\partial_z(u-u_I)\|_{L^2(\hat\sigma)} \leq \widehat{C}|\partial_z u|_{H^m(\hat\sigma)}$$

and

$$\|\partial_x(u-u_I)\|_{L^2(\hat\sigma)} + \|\partial_y(u-u_I)\|_{L^2(\hat\sigma)} \leq \widehat{C}\Big(|\partial_x u|_{H^m(\hat\sigma)} + |\partial_y u|_{H^m(\hat\sigma)}\Big),$$

for a constant $\widehat{C}$ that depends only on ϵ and δ.

This theorem, valid only for $m \geq 2$, yields, through affine transformations (dilations), the needed interpolation estimates on the resulting thin tetrahedra.

The proof is based on:

$\partial_z u = 0$ implies $\partial_z I(u)$,

$\partial_x u = 0$ and $\partial_y u = 0$ imply $\partial_x I(u) = 0$ and $\partial_y I(u) = 0$,

and the Bramble-Hilbert Lemma.