Tying loose ends and applications

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Fredholm conditions

Assume $(\overline{M}, A)$ a "nice" Lie manifold, defined in terms of groupoids

Recall that we can associate to $(\overline{M}, A)$ a family M_α of manifolds such that, if $P \in \mathrm{Diff}(\mathcal{V})$, then there exist operators P_α on M_α satisfying:

$$P \text{ is Fredholm } \Leftrightarrow P \text{ is elliptic and all } P_\alpha \text{ are invertible.}$$

Each manifold $M_\alpha = Z_\alpha \times G_\alpha$, where Z_α is a *lower dimensional manifold* and G_α is a Lie group.

Each operator P_α is G_α–invariant and "of the same kind" as the operator P (Laplace, Dirac, ...).

Question: How to identify directly Z_α and G_α directly in terms of $(\overline{M}, A)$.

Groupoids and Fredholm conditions

The idea to prove the Fredholm property:

$P : H^{m+s} \to H^s$ is Fredholm for all s if, and only if, it is so for $s = 0$.

$P : H^m \to L^2 = H^0$ is Fredholm if, and only if,
$Q := P(1 + P^*P)^{-1/2} : L^2 \to L^2$ is Fredholm.

$Q := P(1 + P^*P)^{-1/2} \in \mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$ (norm closure), so to prove that Q is Fredholm is the same thing as proving that it is invertible in $\mathfrak{A}/\mathcal{K}$.

To prove that the image Q in $\mathfrak{A}/\mathcal{K}$ is invertible is equivalent to proving that $\chi(Q)$ is invertible for χ in a faithful family of representations of the C^*-algebra $\mathfrak{A}/\mathcal{K}$.

In the “nice situation,” a faithful family of representations $\{\chi\}$ is obtained by considering the principal symbol map and the regular representations associated to units in the boundary of $\overline{M}$, which then identify with representations on $L^2(M_\alpha) = L^2(Z_\alpha \times G_\alpha)$.

In the “nice situation,” the spaces Z_α and the groups G_α (as well as the operators P_α can be easily obtain from the Lie manifold $(\overline{M}, A)$ and the operator P.

This is easiest achieved (at least for now) using groupoids and the structure of the integrating groupoid.

Lie groupoids

$\mathcal{G}$, the domain map $d : \mathcal{G} \to \overline{M}$, the space of units, "right multiplication" diffeomorphisms $\mathcal{G}_x \ni y \to yg \in \mathcal{G}_y$, where $d(g) = y$, $r(g) = x$.

• The Lie algebroid associated to a Lie groupoid:

$T_d\mathcal{G} = \cup T\mathcal{G}_x = \ker(d_* : T\mathcal{G} \to T\overline{M})$ is the *vertical* tangent bundle, $A(\mathcal{G}) := T_d\mathcal{G}|_{\overline{M}}$. The sections of $A(\mathcal{G})$ identify with the *right invariant, vertical* vector fields on $\mathcal{G}$, so $\Gamma(A(\mathcal{G}))$ has a natural Lie algebra structure: $A(\mathcal{G})$ is a *Lie algebroid*.

• If all $\mathcal{G}_x$ are (connected and) simply connected, then we say that $\mathcal{G}$ is simply connected.

Two simply connected Lie groupoids with the same Lie algebroid are isomorphic (with a differentiable isomorphism).

We can say what it means for $(M, \overline{M}, A)$ to be "nice:"

1. We can find a groupoid $\mathcal{G}$ integrating A for which π is an isometric bijection.

2. $\mathcal{G}|_{\partial \overline{M}}$ is amenable.

Then $Q \in \mathfrak{A}$ is invertible, if and only if, its natural action Q_x on $L^2(\mathcal{G}_x)$ is invertible for all $x \in \partial \overline{M}$. But $Q_x = P_x(1 + P_x^* P_x)^{-1/2}$!

This gives $P_\alpha = P_x$ and $M_\alpha = \mathcal{G}_x$ for x in an equivalence class of units.

3. $\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x)$ acts on $\mathcal{G}_x$ and this action splits to a product (automatic if G_x^x is solvable, simply connected).

• **The integration problem:**

Given a Lie algebroid $A \to \overline{M}$, to find a Lie groupoid $\mathcal{G}$ with $A(\mathcal{G}) \simeq A$. Not always possible. It is possible for Lie algebroids coming from Lie manifolds: Pradines, N., Debord, Crainic-Fernande

• Idea: if we can integrate $A \to Y$ and $A \to Y^c$, $Y^c := \overline{M} \setminus Y$, and if we choose *connected, simply connected* integrating groupoids, then we can glue them and get a Lie groupoid (N, '89).

• It is important to be able to integrate "simple groupoids."

• Let $A \to \overline{M}$ be an *arbitrary* Lie algebroid. Denote for each $p \in \overline{M}$ by $k_p := \ker(A_p \to T_p\overline{M})$. Then k_p is naturally a Lie algebra (finite dimensional).

• Assume $\Gamma(A)$ consists of vector fields tangent to all faces of $\overline{M}$, then $\exp(X)$ is defined for all $X \in \mathcal{V}$ and gives rise to a commutative diagram

$$\begin{array}{ccc} \exp(X):\ A & \xrightarrow{\simeq} & A \\ \downarrow & & \downarrow \\ \exp(X):T\overline{M} & \xrightarrow{\simeq} & T\overline{M} \end{array}$$

• Let S be an orbit of a point $p \in \overline{M}$, then it is an immersed manifold and the Lie algebras k_p, $p \in S$ are non-canonically isomorphic. (Obstruction to integration ... ?)

• Assume $A|_S = TS \oplus k$ as Lie algebroids, for some Lie algebra k. Then we can integrate to obtain $\mathcal{G} = S \times S \times K$, where K is *any* Lie group with Lie algebra k. Not simply connected in general, choose $\mathcal{P}S \times K$ with K simply connected.

Examples. b- and c-calculi, boundary fibration.

Index Theory

Let us fix $(\overline{M}, \mathcal{V})$ Lie manifolds, M the interior of $\overline{M}$.

Problem: Prove an index theorem for Fredholm operators on Lie M.

What is the structure and the meaning of the non-local invariants coming from the structure at infinity? (Piazza program,...)

A possible approach: let $\mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$, and consider the quotient

$$\mathfrak{B} := \mathfrak{A}/\mathcal{K}.$$

This quotient is $\mathcal{C}^\infty(S^*M)$ in the compact case, but it is non-commutative in the non-compact case.

An elliptic, Fredholm operator $P \in \mathrm{Diff}(\mathcal{V})$ on M gives rise in the usual way to a Fredholm operator $Q = P(1+P^*P)^{-1/2} \in \mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$, as in the previous lecture, which will be invertible in $\mathfrak{B}$.

As we have seen, both $\mathfrak{A}$ and $\mathfrak{B}$ can be described explicitly using groupoids.

The map index map

$$ind = \partial : K_1(\mathfrak{B}) \to \mathbb{Z} = K_0(\mathcal{K})$$

coming from $0 \to \mathcal{K} \to \mathfrak{A} \to \mathfrak{B} \to 0$ can be described *in principle* using cyclic homology (using "smooth algebras.")

This is due to many people, on different levels of generality and applicability: Bunke, Connes, Karoubi, Quillen, Tsygan, N., in general, for cylindical ends Lauter-Moroianu, Leichtnam-Piazza, Melrose-N., Moroianu-N. ... , foliations Benameur-Heitch, Connes, Gorokhovsky-Lott, N., Piazza, ...

The *explicit form* of the index cocycle is however not understood well enough.

Applications to PDEs

Reduction to the boundary

Regular elliptic boundary value problems of the form

$$Du = f \text{ in } \Omega, \quad Bu = f \text{ on } \partial\Omega$$

on a smooth, bounded domain can be reduce to an elliptic pseudodifferential equation $Pv = h$ on $\partial\Omega$. (Layer potentials, Cauchy Data spaces: Seeley, Ballmann-Brunning-Carron, ...)

The first obstruction to solving this equation is the index of P.

The index of P is determined by the Atiyah-Singer Index Theorem.

Natural question: what about polyhedral domains?

The method of Layer Potentials

Consider the Poisson problem

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^3$.

Equivalent to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Let

$$E(x) = c_n |x|^{2-n}$$

$=$ the usual fundamental solution of Δ on $\mathbb{R}^n$.

We try our solution in the form of a single layer potential

$$u = \mathcal{S}(h) := \int_{\partial\Omega} E(x-y)h(y)d\sigma(y)$$

with σ the surface measure.

Then u is harmonic: $\Delta u = 0$ and its boundary values are $u|_{\partial\Omega} = Sh$, with S an elliptic pseudodifferential operator of order -1. It is also self-adjoint.

Our equation reduces to $Sh = g$. Since S is Fredholm of index zero, it is enough to prove that S is injective. PDE tricks.

For polyhedral domains the single layer operator S is in a groupoid pseudodifferential operators algebra.

Question: extend the usual results on layer potentials to the polyhedral case. (Fredholm conditions.)

Numerical methods: the "Boundary Element Method"

Applications to Schrödinger operators

Assume $H = -\Delta + V$, where V has only singularities of the form $1/r$ on $\mathbb{R}^3$, such that $|x|V(x)$ has radial limits at ∞.

We want to study

$$Hu = \lambda u, \quad \lambda < 0.$$

Write

$$r^2(Hu - \lambda u) = [(r\partial_r)^2 + r\partial_r + \Delta_{S^2} + r^2 V - \lambda r^2]u = 0$$

is in the b-calculus at r=0. Similar at any other point.

This gives $u \in \mathcal{K}^m_{5/2-\epsilon}(\mathbb{R}^3)$, so we obtain again optimal approximation rates. (Hunsicker-N.-Sofo, for periodic potentials.)