Lie Manifolds

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Lecture 2 is continued with Lecture 3 in the same file

The framework:

• $\overline{M}$ is a compact manifold *with corners*.

An example is $[0,1]^n$.

We assume that each proper face $H \subset \overline{M}$ of maximal dimension (hyperface) is given by $H = \{x_H = 0\}$, $x_H \geq 0$ on $\overline{M}$, and $dx_H \neq 0$ on H.

(Each hyperface has a *defining function*.)

• M is the *interior* of $\overline{M}$:

$$M = \overline{M} \smallsetminus \cup H.$$

We denote by $\Gamma(E)$ the space of smooth sections of E, so $\Gamma(T\overline{M})$ is the space of smooth vector fields on $\overline{M}$.

• $\mathcal{V} \subset \Gamma(T\overline{M})$ and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$.

Definition of Lie manifolds

A Lie manifold is a manifold $M =$ the interior of a compact manifold with corners $\overline{M}$, together with a subspace $\mathcal{V} \subset \Gamma(T\overline{M})$, consisting of vector fields tangent to all faces of $\overline{M}$ and satisfying:

1. $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$;

2. $\mathcal{V}$ is a $\mathcal{C}^\infty(\overline{M})$ module that is generated locally in the neighborhood of each point $p \in M$ by n linearly independent vector fields $X_1, \dots, X_n$.

3. If above $p \in M$ (= interior of $\overline{M}$), then the vector fields $X_1, \dots, X_n$, locally generating $\mathcal{V}$ around p, also give a local basis of T_pM.

Many interesting particular cases in **Melrose**'s, *Geometric scattering theory*, 1995. Also, Cordes and Schulze.

Formalized by Ammann-Lauter-N. and A.-L.-N.-Vasy in order to study axiomatically the geometry of these spaces.

The Lie algebroid $A_{\mathcal{V}}$ associated to a Lie manifold

Recall that a *Lie algebroid* $A \to \overline{M}$ is a vector bundle over $\overline{M}$ together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A and a bundle map $\varrho : A \to T\overline{M}$ such that

$\varrho([X,Y]) = [\varrho(X), \varrho(Y)]$ and

$[X, fY] = f[X,Y] + (\varrho(X)f)Y$, for any smooth sections X and Y of A and any smooth function f on $\overline{M}$.

The map ϱ is called the *anchor of map* of A. In the definition, we extended it to a map $\varrho : \Gamma(A) \to \Gamma(T\overline{M})$ between sections of these bundles by a pointwise action.

The Lie algebroid $A_{\mathcal{V}}$ associated to a Lie manifold

• $\mathcal{V}$ is a finitely generated, projective $\mathcal{C}^\infty(\overline{M})$–module.

The Serre–Swan Theorem implies that there exists a finite dimension bundle $A_{\mathcal{V}} \to \overline{M}$, uniquely defined up to isomorphism, such that $\mathcal{V} \simeq \Gamma(A_{\mathcal{V}})$.

We obtain that if $(M, \overline{M}, \mathcal{V})$ is a Lie manifold, there is a Lie algebroid $A_{\mathcal{V}} \to T\overline{M}$ such that:

• the anchor map $\varrho : A_{\mathcal{V}} \to T\overline{M}$ is an isomorphism over $M := \overline{M} \smallsetminus \partial M$ and

• the Lie algebra of vector fields

$$\mathcal{V} := \Gamma(A) = \varrho(\Gamma(A))$$

consists of vector fields *tangent* to all faces of the manifold with corners $\overline{M}$.

Examples of $\mathcal{V}$

Example One. **(a)** $\overline{M}$ = a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \smallsetminus \partial\overline{M}$.

(b) $\mathcal{V} = \mathcal{V}_v :=$ the space of vector fields on $\overline{M}$ that are *tangent* to $\partial\overline{M}$.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x\partial_x$, ∂_{y_2}, ... , ∂_{y_n}.

($y_2, \ldots, y_n$ are local coordinates on $\partial\overline{M}$.)

The fibers $\mathcal{G}_x := d^{-1}(x)$ are either M, the interior of $\overline{M}$ or $\partial\overline{M} \times \mathbb{R}$.

(Anticipating) The geometry is that of a manifold with cylindrical ends. The groupoid $\mathcal{G}$ integrating the resulting Lie algebroid is the disjoint union of $M \times M$ and $\partial\overline{M} \times \partial\overline{M} \times \mathbb{R}$.

Metric and geometry

Fix a Lie manifold $(\overline{M}, \mathcal{V})$ and a vector bundle $A \to \overline{M}$ such that

$$\mathcal{V} \simeq \Gamma(A),$$

where A extends TM to $\overline{M}$, namely

$$A|_M \simeq TM.$$

($A \to \overline{M}$ is the Lie algebroid associated to $(\overline{M}, \mathcal{V})$).

In particular, a metric on A will induce a Riemannian metric on TM (i.e. a metric on M). $\Rightarrow$ cylindrical ends for our first example.

Connections

The Levi-Civita connection

$$\nabla : \Gamma(TM) \to \Gamma(TM \otimes T^*M),$$

extends to an A^*-valued connection

$$\nabla : \Gamma(A) \to \Gamma(A \otimes A^*),$$

satisfying the usual Leibnitz rule:

$$\nabla_X(fY) = X(f)Y + f\nabla_X(Y) \text{ and}$$
$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

for all $X, Y, Z \in \mathcal{V} = \Gamma(A)$.

In particular, the covariant derivative $\nabla^k R$ of the curvature R extends to a section of $A^{*\otimes k} \otimes \Lambda^2 A^* \otimes \mathrm{End}(A)$ over $\overline{M}$ and hence they are all bounded on M.

In fact, Lie manifolds have bounded geometry ($r > 0$).

The A^*-valued connection is obtained as follows.

Recall that the formula for the Levi-Civita connection, $\nabla_X Y$, is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X,Y],Z\rangle - \langle [Y,Z],X\rangle + \langle [Z,X],Y\rangle \\ + X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle.$$

Suppose $X, Y, Z \in \Gamma(A)$ in the above formula.

The function $2\langle \nabla_X Y, Z\rangle$, defined initially only on M, extends to a smooth function on $\overline{M}$. Since the inner product $\langle\ ,\ \rangle$ is the same on A and on TM, the above equation determines $\nabla_X Y$ as a smooth section of A.

Differential operators

Recall that $\mathsf{Diff}(\mathcal{V}) =$ the algebra of differential operators on M generated by $\mathcal{C}^\infty(\overline{M})$ and vector fields $X \in \mathcal{V}$.

We can extend the definition of $\mathsf{Diff}(\mathcal{V})$ to include operators $\mathsf{Diff}(\mathcal{V}; E, F)$ acting between vector bundles $E, F \to \overline{M}$.

$$d \in \mathsf{Diff}(\mathcal{V}; \Lambda^q A^*, \Lambda^{q+1} A^*)$$

and

$$\nabla \in \mathsf{Diff}(\mathcal{V}; A, A \otimes A^*).$$

Then (Ammann–Lauter–N.)

$$\Delta \in \mathsf{Diff}(\mathcal{V}).$$

Similarly, all geometric differential operators on $\overline{M}$ are generated by $\mathcal{V}$.

(Done "by hand" for earlier particular examples of manifolds with a Lie structure at infinity.)

Fix a local orthonormal basis $X_1, \ldots, X_n$ of A (on some open subset of M). Then $\nabla_{X_i} X = \sum c_{ij}(X) X_j$, for some smooth functions $c_{ij}(X)$. Consequently, $\mathrm{div}(X) := -\sum c_{jj}(X)$ is well defined and gives rise to a smooth function on M.

Lemma. Let $X \in \Gamma(A)$ and $f \in \mathcal{C}_c^\infty(M_0)$. Then

$$\int_{M_0} X(f)\mu = \int_{M_0} f \, \mathrm{div}(X)\mu.$$

In particular, the formal adjoint of X is $X^* = -X + \mathrm{div}(X) \in \mathrm{Diff}\, \mathcal{V}$.

Applications

Assume $(\overline{M}, A)$ "nice".

Theorem. We can associate to $(\overline{M}, A)$ a family M_α of manifolds such that, if $P \in \mathrm{Diff}(\mathcal{V})$, then there exist operators P_α on M_α satisfying:

P is Fredholm $\Leftrightarrow$ P is elliptic and all P_α are invertible.

(Lauter-Monthubert-N., earlier: **Kondratiev, Mazya, Plamenevski, Mazzeo, Melrose, Mendoza, Piazza, Schrohe, Schulze ...**)

Each manifold $M_\alpha = Z_\alpha \times G_\alpha$, where Z_α is a *lower dimensional manifold* and G_α is a Lie group.

Each operator P_α is G_α–invariant and "of the same kind" as the operator P (Laplace, Dirac, ...).

Questions about M are reduced to questions about its P_α and M_α $\Rightarrow$ Harmonic analysis on various Lie groups.

$\Rightarrow$ an inductive procedure to study geometric operators on M.

In this and the following example we anticipate the construction of pseudodifferential operators on Lie manifolds, to be introduced later.

Example One Revisited. (a) $\overline{M}$ = a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \setminus \partial\overline{M}$.

(b) $\mathcal{V} = \mathcal{V}_v :=$ the space of vector fields on $\overline{M}$ that are *tangent* to $\partial\overline{M}$.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$, a local basis is given by $x\partial_x, \partial_{y_2}, \ldots, \partial_{y_n}$.

The transformation $x = e^t$ maps $x\partial_x$ to ∂_t. If we set the length of ∂_t to be one, this shows that the resulting geometry is that of a manifold with cylindrical ends.

The resulting differential operators are almost translation invariant. We have that $M_\alpha = C \times \mathbb{R}$, where C are the connected components of the boundary, with $G_\alpha = \mathbb{R}$.

Example Two. (a) $\overline{M}$ = a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \smallsetminus \partial\overline{M}$.

(b) $\mathcal{V}$ = the space of vector fields on $\overline{M}$ that vanish on $\partial\overline{M}$.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$ a local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_n}$.

The resulting geometry is that of an asymptotically hyperbolic manifold.

$\{\alpha\} = \partial\overline{M}$, each $M_\alpha = G_\alpha = T_\alpha(\partial\overline{M}) \rtimes \mathbb{R}$ is a solvable Lie group, and each P_α is G_α invariant.

Recently these manifolds have been used in Math. physics in connection to the AdS–CFT correspondence (Witten, Anderson, Lee, Mazzeo, ...).

Earlier pseudodifferential calculus: Mazzeo's, Schulze's edge-calculus.

Example Three. (a) $\overline{M}$ $=$ a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \setminus \partial\overline{M}$.

(b) $\mathcal{V}$ $=$ the space of vector fields on $\overline{M}$ that vanish on $\partial\overline{M}$ and the normal covariant to the boundary also vanishes at the boundary.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$ a local basis is given by $x^2\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_n}$.

The resulting geometry is that of an asymptotically Euclidean manifold.

$\{\alpha\} = \partial\overline{M}$, each $M_\alpha = G_\alpha = T_\alpha(\partial\overline{M}) \times \mathbb{R}$ is an abelian Lie group, and each P_α is G_α invariant.

Earlier pseudodifferential calculus: Parenti's SG-calculus= Melrose's scattering-calculus.

Example Four. **(a)** $\overline{M}$ = a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \setminus \partial\overline{M}$, together with a fibration $\pi : \partial M \to B$.

(b) $\mathcal{V} =$ the space of vector fields on $\overline{M}$ that are tangent to the fibers of $\partial\overline{M} \to B$.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$ a local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_k}$, $\partial_{y_{k+1}}, \ldots \partial_{y_n}$.

The resulting geometry is related to that of an locally symmetric spaces. It is also relevant for boundary value problems.

$\{\alpha\} = B$, each $Z_\alpha = \pi^{-1}(\alpha), \alpha \in B$, $M_\alpha = Z_\alpha \times G_\alpha$, $G_\alpha = T_\alpha B \rtimes \mathbb{R}$ is a solvable Lie group, and each P_α is G_α invariant.

Earlier pseudodifferential calculus: Mazzeo's, Schulze's edge-calculus.

It is important to generalize this example to higher rank spaces (=corners of higher codimension).

Example 5. (a) $\overline{M}$ = a manifold with smooth boundary $\partial\overline{M}$, $M = \overline{M} \smallsetminus \partial\overline{M}$, together with a foliation $F \subset T\partial\overline{M}$.

(b) $\mathcal{V} =$ the space of vector fields on $\overline{M}$ that are tangent to the leaves of the foliation F.

(c) There is no condition on these vector fields in the interior.

(d) At the boundary $\partial\overline{M} = \{x = 0\}$ a local basis is given by $x\partial_x$, $x\partial_{y_2}$, ... , $x\partial_{y_k}$, $\partial_{y_{k+1}}, \ldots \partial_{y_n}$.

This manifold with a Lie structure at infinity is, however, not "nice." There is no description of Fredholm operators in terms of lower dimensional spaces and operators invariant with respect to groups. (There exists however a different characterization of Fredholm operators.)

Earlier pseudodifferential calculus?

Questions on the analysis on foliations spaces arise also from Riemannian geometry.

Lecture Two also included some of the things reviewed in Lecture Three

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Singular Integral Kernels

(partial review)

Fix $(\overline{M}, \mathcal{V})$ = a manifold with a Lie structure at infinity, with Lie algebroid $A \to \overline{M}$.

Let P be a properly supported, classical pseudodifferential operator on M. Then its distribution kernel is in $I^m(M^2, M)$.

Fix $r > 0$ small so that the exponential map defines a bijection

$$(TM)_r := \{v \in TM, \|v\| < r\} \to M_r^2 \\ := \{(x,y) \in M^2, d(x,y) < r\}.$$

We have that

$$(TM)_r \subset (A)_r := \{w \in A, \|w\| < r\}.$$

So $(A)_r$ extends TM and $\overline{M}$ extends M.

Definition. We say that P extends to A if its distribution kernel extends to a compactly supported distribution in $I^m((A)_r, \overline{M})$.

Pseudodifferential operators

Denote by $\Psi^m(\overline{M})$ the pseudodifferential operators on M whose kernel extends to $I^m((A)_r, \overline{M})$.

Define:

$$\Psi_{\mathcal{V}}^{-\infty}(\overline{M}) = \{\sum P \exp(X_1) \ldots \exp(X_k), \\ P \in \Psi^{-\infty}(\overline{M}) \text{ and } X_j \in \mathcal{V}\}.$$

and

$$\Psi_{\mathcal{V}}^{m}(\overline{M}) = \Psi^m(\overline{M}) + \Psi_{\mathcal{V}}^{-\infty}(\overline{M}).$$

Theorem. $\Psi_{\mathcal{V}}^{\infty}(\overline{M}) := \cup_{m\in\mathbb{Z}} \Psi_{\mathcal{V}}^{m}(\overline{M})$ is an algebra closed under adjoints that "quantizes" the Lie algebra $\mathcal{V}$.

(Ammann-Lauter-N, answers a question of Melrose; based on important earlier work of **Melrose**, **Mazzeo**, **Weinstein**, and **Ping Xu**. Also: **Schulze, Schrohe, Grisvard, Mazya, Plamenevsk ...**)

A major ingredient in the proof is to establish first Lie's third theorem for $\mathcal{V}$ (or A), N. in the cases of interest for Ψdo's, and then completely by Crainic-Fernandez (also Pradines, Debord).

Comments on the last theorem (with Ammann and Lauter).
"Quantizes" means:

1. The differential operators in $\Psi_{\mathcal{V}}^{\infty}(\overline{M})$ are exactly the ones "generated" by $\mathcal{V}$:

$$\Psi_{\mathcal{V}}^{\infty}(\overline{M}) \cap \mathrm{Diff}(M) = \mathrm{Diff}(\overline{M}, \mathcal{V}).$$

2. It has the usual symbolic and analytic properties of Ψdo's.

The principal symbol defines isomorphisms

$$\sigma_m : \Psi_{\mathcal{V}}^{m}(\overline{M})/\Psi_{\mathcal{V}}^{m-1}(\overline{M}) \to S^m(A^*)/S^{m-1}(A^*)$$

(choose your class of symbols: classical, "(1,0)," ...)

We also have L^2-mapping properties (next).

Fix an arbitrary metric on TM coming from a metric on $A \to \overline{M}$.

Then $L^2(M)$ is (as a vector space) independent of the metric on A and $\Psi^0_{\mathcal{V}}(\overline{M})$ acts by bounded operators on $L^2(M)$.

The norm closure of $\Psi^0_{\mathcal{V}}(\overline{M})$ generalizes the comparison algebras considered by **Cordes**. Relevant in proving the Fredholm property.

Similarly, let

$$H^s(M) := \{u \in L^2(M),\ (1+\Delta)^{s/2}u \in L^2(M)\},\ s \geq 0.$$

Then these spaces are *independent of the metric* on A and $P : H^s(M) \to H^{s-m}(M)$ if $P \in \Psi^m_{\mathcal{V}}(\overline{M})$.

We shall say (as usual) that $P \in \Psi^m_{\mathcal{V}}(\overline{M})$ is *elliptic* if $\sigma_m(P)(\xi)$ is invertible for $\xi \in A^*$ large.

If $P \in \Psi^m_{\mathcal{V}}(\overline{M})$ is elliptic and symmetric, then it is essentially self-adjoint (Ammann–Ionescu–N.)

Regularity

Theorem. (Ammann-Ionescu-N.) Let $P \in \Psi_{\mathcal{V}}^m(\overline{M})$ and $u \in H^r(M)$ be such that $Pu \in H^s(M)$, then $u \in H^{s+m}(M)$.

Proof. Let $Q \in \Psi_{\mathcal{V}}^m(\overline{M})$ be a parametrix of P, that is $QP - 1$ and $PQ - 1$ are in $\Psi_{\mathcal{V}}^{-\infty}(\overline{M})$ ...

Alternatively, use the bounded geometry to obtain a nice partition of unity (Aubin, Gromov, Shubin, ...).

We can extend this to the case of boundary value problems. Let $\Omega \subset \overline{M}$. We require the boundary $\partial\Omega$ to have a tubular neighborhood (Shick). Sobolev spaces are defined by restriction.

Theorem. (Ammann-Ionescu-N.) Assume $P \in \Psi_{\mathcal{V}}^m(\overline{M})$ is a second order differential operator with a strictly positive principal symbol (i.e. it is strongly elliptic). Let $u \in H^r(\Omega)$, $r \geq 1$, be such that $Pu \in H^s(\Omega)$ and $u = 0$ on $\partial\Omega$, then $u \in H^{s+2}(\Omega)$.

The same holds for Neumann boundary value problems.

Spectra

Let $P = \Delta - \lambda = \Delta_M - \lambda$. Then

$$\widehat{P}(\tau) = \Delta_{\partial M} + \tau^2 - \lambda.$$

Since the spectrum of $\Delta_{\partial M}$ is

$$\Delta_{\partial M} = \{0, \lambda_1, \lambda_2, \ldots\} \subset [0, \infty),$$

we obtain that $\widehat{P}(\tau) = \Delta_{\partial M} + \tau^2 - \lambda$ is invertible for any $\tau \in \mathbb{R}$ if, and only if, $\lambda < 0$. Hence $\Delta_M - \lambda$ is Fredholm, if, and only if, $\lambda < 0$.

This shows that $\sigma_e(\Delta_M) = [0, \infty)$. But then

$$[0, \infty) \subset \sigma_e(\Delta_M) \subset \sigma(\Delta_M) \subset [0, \infty)$$

and hence

$$\sigma(\Delta_M) = [0, \infty).$$

This argument generalizes to higher “rank spaces.”

Example 6. (a) $\overline{M}$ = a compact manifold with corners, M = the interior of $\overline{M}$.

(b) $\mathcal{V}$ = the space of vector fields on $\overline{M}$ that are tangent to all hyperfaces of $\overline{M}$.
(generalizes our first example)

(c) There is no condition on these vector fields in the interior.

(d) At the codimension k face

$$\{x_1 = \ldots = x_k = 0\},$$

a local basis is

$$x_1 \partial_{x_1}, \ldots, x_k \partial_{x_k}, \partial_{y_{k+1}}, \ldots, \partial_{y_n}.$$

Earlier pseudodifferential calculus: Melrose and Piazza.

The manifolds Z_α are the open faces of $\overline{M}$. The group G_α is $\mathbb{R}^k$, with k the *codimension* of Z_α.

Theorem.[Lauter-N]

$$\sigma(\Delta_M) = [0, \infty)$$

The complete characterization of the spectrum (multiplicity of the spectral measure, discreteness of the point spectrum, absence of continuous singular spectrum) is still open, although it is of great importance in Number Theory and Representation Theory (automorphic forms, Langlands program).

Similarly, let D be the Dirac operator associated to a Clifford(A)-bundle over $\overline{M}$. Then

Theorem.[N] The Dirac operator D is invertible if, and only if, the Dirac operator D_F associated to any open face F of $\overline{M}$ (including M), has no harmonic spinors (=zero kernel).

Groupoids and Fredholm conditions

The idea to prove the Fredholm property:

$P : H^{m+s} \to H^s$ is Fredholm for all s if, and only if, it is so for $s = 0$.

$P : H^m \to L^2 = H^0$ is Fredholm if, and only if,
$Q := P(1 + P^*P)^{-1/2} : L^2 \to L^2$ is Fredholm.

$Q := P(1 + P^*P)^{-1/2} \in \mathfrak{A} := \overline{\Psi^0_{\mathcal{V}}(\overline{M})}$ (norm closure), so to prove that Q is Fredholm is the same thing as proving that it is invertible in $\mathfrak{A}/\mathcal{K}$.

To prove that the image Q in $\mathfrak{A}/\mathcal{K}$ is invertible is equivalent to proving that $\chi(Q)$ is invertible for χ in a faithful family of representations of the C^*-algebra $\mathfrak{A}/\mathcal{K}$.

In the “nice situation,” a faithful family of representations $\{\chi\}$ is obtained by considering the principal symbol map and the regular representations associated to units in the boundary of $\overline{M}$, which then identify with representations on $L^2(M_\alpha) = L^2(Z_\alpha \times G_\alpha)$.

In the “nice situation,” the spaces Z_α and the groups G_α (as well as the operators P_α can be easily obtain from the Lie manifold $(\overline{M}, A)$ and the operator P.

This is easiest achieved (at least for now) using groupoids and the structure of the integrating groupoid.

Plan (detailed):

• Lie groupoids:

$\mathcal{G}$, the domain map $d : \mathcal{G} \to \overline{M}$, the space of units, "right multiplication" diffeomorphisms $\mathcal{G}_x \ni y \to yg \in \mathcal{G}_y$, where $d(g) = y$, $r(g) = x$.

• The Lie algebroid associated to a Lie groupoid:

$T_d\mathcal{G} = \cup T\mathcal{G}_x = \ker(d_* : T\mathcal{G} \to T\overline{M})$ is the *vertical* tangent bundle, $A(\mathcal{G}) := T_d\mathcal{G}|_{\overline{M}}$. The sections of $A(\mathcal{G})$ identify with the *right invariant, vertical* vector fields on $\mathcal{G}$, so $\Gamma(A(\mathcal{G}))$ has a natural Lie algebra structure: $A(\mathcal{G})$ is a *Lie algebroid*.

• If all $\mathcal{G}_x$ are (connected and) simply connected, then we say that $\mathcal{G}$ is simply connected.

Two simply connected Lie groupoids with the same Lie algebroid are isomorphic (with a differentiable isomorphism).

- **Pseudodifferential operators on $\mathcal{G}$:**

$\Psi^m(\mathcal{G})$ consists of families $(P_x), P_x \in \Psi^m(\mathcal{G}_x),$ satisfying

- right invariant,
- smooth,
- with support in a compact neighborhood of the units of $\mathcal{G}$.

For instance, $\Psi^{-\infty}(\mathcal{G}) = \mathcal{C}_c^\infty(\mathcal{G})$ with the convolution product.

$\Psi^m(\mathcal{G})$ acts on $\mathcal{C}_c^\infty(M)$ by the following. Let $P = (P_x) \in \Psi^m(\mathcal{G})$. Any right invariant function on $\mathcal{G}$ is of the form $f \circ r$. This gives $\pi(P) : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)$ by the formula

$$(\pi(P)f) \circ r|_{\mathcal{G}_x} = P_x f \circ r|_{\mathcal{G}_x}.$$

If $\mathcal{G}$ integrates $\mathcal{V} = \Gamma(A)$, then $\pi(\Psi^m(\mathcal{G})) = \Psi^m_{\mathcal{V}}(\overline{M})$.

In nice situations, we can describe the norm closure of $\Psi^0(\mathcal{G})$, its image under π, and the quotient $\pi(\Psi^0(\mathcal{G}))/\mathcal{K}$.

We can say what it means for $(M, \overline{M}, A)$ to be "nice:"

1. We can find a groupoid $\mathcal{G}$ integrating A for which π is an isometric bijection.

2. $\mathcal{G}|_{\partial\overline{M}}$ is amenable.

Then $Q \in \mathfrak{A}$ is invertible, if and only if, its natural action Q_x on $L^2(\mathcal{G}_x)$ is invertible for all $x \in \partial\overline{M}$. But $Q_x = P_x(1 + P_x^* P_x)^{-1/2}$!

This gives $P_\alpha = P_x$ and $M_\alpha = \mathcal{G}_x$ for x in an equivalence class of units.

3. $\mathcal{G}_x^x := d^{-1}(x) \cap r^{-1}(x)$ acts on $\mathcal{G}_x$ and this action splits to a product (automatic if G_x^x is solvable, simply connected).

• **The integration problem will be discussed in the last lecture.**