Index theory, bordism and rho invariants

Paolo Piazza (Università di Roma "La Sapienza").

Spring school on
Index theory, Lie groupoids and boundary value problems
Regensburg, March 15 - 20, 2009

Dirac operators

- We are ultimately interested in certain analytic and geometric invariants of Dirac operators on riemannian manifolds (rho-invariants)
- Our final goal is to prove geometric theorems using these operators.
- We consider a riemannian manifold (M, g) and a complex hermitian vector bundle $E \to M$ endowed with a connection ∇^E;
- we assume the existence of a Clifford action $C^\infty(M, T^*M \otimes E) \xrightarrow{c} C^\infty(M, E)$
- an operator of Dirac type is obtained taking the composition $C^\infty(M, E) \xrightarrow{\nabla^E} C^\infty(M, T^*M \otimes E) \xrightarrow{c} C^\infty(M, E)$.Thus $D := c \circ \nabla^E$.
- If the Clifford action c is unitary and ∇^E is unitary and Clifford (i.e. compatible with Levi-Civita),then D is formally self-adjoint , i.e. $D = D^*$

Basic properties of Dirac operators.

- D is an elliptic differential operator
- hence if M is compact without boundary, then D is Fredholm
- if $\dim M = 2k$ then E is graded, $E = E^+ \oplus E^-$ and D is odd:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} \quad D^- = (D^+)^*$$

- if $\dim M = 2k$, $\mathrm{ind}(D) = 0$ (since $D = D^*$), but $\mathrm{ind}\, D^+ \neq 0$
- if $\dim M = 2k+1$ then $\mathrm{ind}(D) = 0$

Examples.

- The Gauss-Bonnet operator $d + d^*$ with $E = \Lambda^{\mathrm{evev}} M \oplus \Lambda^{\mathrm{odd}} M$;
- the spin-Dirac operator $D^{\mathrm{spin}} \equiv \not{D}$ on a spin manifold with $E = \not{S} = \not{S}^+ \oplus \not{S}^-$ the spinor bundle;
- the signature operator on an orientable manifold D^{sign} with $E = \Lambda^+ M \oplus \Lambda^- M$ defined in terms of Hodge-$\star$;
- the Dolbeault operator $\overline{\partial} + \overline{\partial}^*$. with $E = \Lambda^{0,\mathrm{evev}} M \oplus \Lambda^{0,\mathrm{odd}} M$
- (important) if D is any Dirac operator acting on sections of E and W is an auxiliary complex bundle endowed with a connection ∇^W, then we get a new Dirac operator D_W acting on sections of $E \otimes W$ by taking $c_E \otimes \mathrm{Id}_W$ as Clifford action and $\nabla^{E \otimes W}$ as a connection. We say that D_W is obtained from D by twisting with W.
- we shall be interested mainly in D^{spin} on a spin manifold and D^{sign} on an orientable manifold (plus their twisting).

Atiyah-Singer index theory.

- Atiyah-Singer index formula

$$\operatorname{ind} D^+ = \int_M AS(R^M, R^E) = < [AS(R^M, R^E), [M] >$$

- Right hand side is topological
- Geometric applications for Gauss-Bonnet, signature and Dolbeault: prove by Hodge-de Rham-Dolbeault that

$$\chi(M) = \operatorname{ind}(d + d^*)^+; \; \mathrm{sign}(M) = \operatorname{ind} D^{+,\mathrm{sign}}; \; \chi(M, \mathcal{O}) = \operatorname{ind}(\overline{\partial} + \overline{\partial}^*)^+$$

- apply Atiyah-Singer and get Chern-Gauss-Bonnet, Hirzebruch and Riemann-Roch:

$$\chi(M) = \int_M \mathrm{Pf}(M); \; \mathrm{sign}(M) = \int_M L(M); \; \chi(M, \mathcal{O}) = \int_M \mathrm{Td}(M)$$

More on the spin Dirac operator and on the signature operator

- Assume that M^{4k} is spin; then ind $D^{+,\mathrm{spin}} = \int_M \widehat{A}(M)$
- if g is of positive scalar curvature ($\equiv$ PSC) then D^{spin} is invertible because of Lichnerowicz formula
- $\Rightarrow$ ind $D^{+,\mathrm{spin}} = 0$
- $\Rightarrow$ obstructions to existence of PSC metrics ($\longrightarrow$ Thomas Schick)

Now we pass to the signature operataor on a $4k$-dimensional oriented manifold:

- Since ind $D^{+,\mathrm{sign}}$ is equal to the signature it follows that ind $D^{+,\mathrm{sign}}$ is a homotopic invariant
- Since ind $D^{+,\mathrm{sign}}$ is equal to the signature it follows that ind $D^{+,\mathrm{sign}}$ is a bordsim invariant.

More about the index on compact manifolds without boundary

- index depends only on 0-eigenvalue
- if "D is a boundary operator" then its index is zero (cobordism invariance of the index)
- It's a coarse invariant
- Example 1: the index of the signature operator cannot distinguish two homotopic but non-diffeomorphic manifolds.
- Example 2: the index of the spin Dirac operator cannot distinguish two metrics of PSC.
- Calderon (or Atiyah-Bott) formula:

$$\text{ind}\, D^+ = Tr(S^+) - Tr(S^-)$$

  where $S^\pm \in \Psi^{-\infty}$ are remainders in a parametrix construction
- parametrices can be localized near the diagonal
- $\Rightarrow$ index data are localized near the diagonal
- this is very special of the index; more sophisticated spectral invariant cannot be localized.

Dirac operators: eta invariants

- (M, g) is a now odd dimensional
- the eta invariant associated to a Dirac operator D is by definition

$$\eta(D) := \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}(D \exp(-(tD)^2) dt$$

- $\eta(D)$ is the value at $s = 0$ of the meromorphic continuation of $\sum_{\lambda \neq 0} \mathrm{sign}(\lambda) |\lambda|^{-s} \quad \mathrm{Re} s >> 0$.
- $\eta(D)$ measures the **spectral asymmetry** of the self-adjont op. D.
- $\eta(D)$ is a very sensitive invariant.
- Indeed, if $\{D_t\}$ is a one-parametr family of operators then (assuming for simplicity D_0 and D_1 invertible)

$$\eta(D_1) - \eta(D_0) = \int_M \mathrm{local} + \mathrm{SF}(\{D_t\})$$

- rho-invariants (defined next) are more stable objects.

- where does eta come from ?
- η is the boundary corection term in the index theorem on manifolds **with** boundary:
- **Atiyah-Patodi-Singer index theorem**: on an even dimensional manifold W with boundary equal to M and metric G of product type near the boundary:

$$\mathrm{ind}_{\mathrm{APS}}\,(D_W^+) = \int AS - \frac{\eta(D) + \dim(\mathrm{Ker}(D))}{2}$$

where AS is the Atiyah-Singer integrand.

- **Corollary:** $\mathrm{sign}(W) = \int L(W, \nabla^G) - \eta(D_M^{\mathrm{sign}})$, where $M = \partial W$.

Atiyah-Patodi-Singer rho invariant

- it is associated to the choice of a pair of finite dimensional unitary representations of $\pi_1(M) := \Gamma$ of the same dimension: $\lambda_1, \lambda_2 \colon \Gamma \to U(\mathbb{C}^N)$.
- we consider $L_j := \widetilde{M} \times_{\lambda_j} \mathbb{C}^N$ (a flat bundle endowed with a natural unitary connection).
- we can *twist* D by L_j obtaining two operators D_{L_1} and D_{L_2}.
- then the Atiyah-Patodi-Singer rho invariant is by definition

$$\rho(D)_{\lambda_1 - \lambda_2} := \eta(D_{L_1}) - \eta(D_{L_2})$$

- this is a **more stable** invariant (more on this later)
- particularly useful when $\pi_1(M)$ is **finite**
- for example: in distinguishing metrics of Positive Scalar Curvature (PSC)
- for example: in distinguishing the diffeomorphism type of lens spaces
- precise statements will be given later

A hierarchy of geometric structures

There is a hierarchy of geometric structures in index theory:

1. a compact manifold M, with $\partial M = \emptyset$
2. a fibration $X \to B$ with fiber M,
3. a Galois Γ-coverings $\widetilde{M} \to M$,
4. a Γ-equivariant fibration of manifolds without boundary $\widehat{M} \to T$, i.e.
   - Γ acts freely, properly discontinuously and cocompactly on $\widehat{M}$;
   - Γ acts on T by diffeomorphisms ,
   - the projection is a Γ-map.

   Notice that the quotient $X := \widehat{M}/\Gamma$ is foliated by the images of the fibers under the quotient map. Example: $\widetilde{M} \times T \to T$
5. general foliations.

- There is a corresponding hierarchy of numeric index theories
- There is a corresponding hierarchy in the case of manifolds **with** boundary, producing **for each index an eta invariant.**

A hierarchy of NUMERIC invariants

(We now assume that our Γ-equivariant fibration is such that T admits a Γ-invariant measure ν.This is a **non-trivial** assumption.)

	M	$X \to B$	$\widetilde{M}$	$X = (\widetilde{M} \times T)/\Gamma$
Dirac	D	$(D_b)_{b\in B}$	$\widetilde{D}$ (Γ-invariant)	$(\widetilde{D}_\theta)_{\theta\in T}$ (Γ-equiv.)
Index	Ind D	$\int_B$ Ind $D_b db$	$\mathrm{Ind}_{(2)}\ \widetilde{D}$	$\mathrm{Ind}_\nu(\widetilde{D}_\theta)$
theorems	AS	AS	Atiyah	Connes
eta	$\eta(D)$	$\int_B \eta(D_b)db$	$\eta_{(2)}(\widetilde{D})$	$\eta_\nu(\widetilde{D}_\theta)$
rho	APS	APS	Cheeger-Gromov	P.-Benameur
if $\partial(\ ) \neq \emptyset$	APS	APS	Ramachandran	Ramachandran

A glimpse of HIGHER index theory

Let us talk about higher invariants.
We do not assume anymore that T admits a Γ-invariant measure.

	$X \to B$	$\widetilde{M}$	$X = (\widetilde{M} \times T)/\Gamma$
Dirac	$(D_b)_{b\in B}$	$\widetilde{D}$ (Γ-invariant)	$(\widetilde{D}_\theta)_{\theta\in T}$ (Γ-equiv.)
Index class	$\in K^*(B)$	$\in K_*(C^*_r\Gamma)$	$\in K^*(C(T) \rtimes_r \Gamma)$
theorems	AS Bismut	Connes-Moscovici Lott	Connes Moriyoshi-Natsume Gorokowsky-Lott
higher eta	Bismut-Cheeger	Lott	Leichtnam-P. if Γ polyn. growth
higher rho	Azzali	only few examples	???
if $\partial(\ ) \neq \emptyset$	Bismut-Cheeger	Leichtnam-P.	Leichtnam-P. if Γ polyn. growth

Everything in red employs superconnections ...

The Cheeger-Gromov eta invariant

- $\widetilde{M} \to M$ a Γ-covering;
  we lift D to a Γ-invariant Dirac operator $\widetilde{D}$ on $\widetilde{M}$.
- The L^2-eta invariant is :

$$\eta_{(2)}(\widetilde{D}) = \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{(2)}(\widetilde{D} \exp(-(t\widetilde{D})^2) dt$$

  where $\mathrm{Tr}_{(2)}$ is the Von Neumann trace introduced by Atiyah,(i.e.

$$\mathrm{Tr}_{(2)}(\widetilde{D} \exp(-(t\widetilde{D})^2) = \int_{\mathcal{F}} \mathrm{tr}_x \widetilde{K}_t(x,x). \qquad (0)$$

  with $\mathcal{F}$ a fundamental domain for $\widetilde{M} \to M$.
- should introduce the Von Neumann algebra of Γ-invariant bounded operators on $L^2(\widetilde{M}, \widetilde{E})$; it comes with a trace; these operators are trace class and their trace is given by (0)

Index versus eta

- the index would be $\mathrm{Ind}_{(2)}(\widetilde{D}^+) = \mathrm{Tr}_{(2)}\, \Pi_+ - \mathrm{Tr}_{(2)}\, \Pi_-$ with $\Pi_\pm$ the orthogonal projections onto $\mathrm{Ker}\, \widetilde{D}^\pm$.
- Calderon's formula holds

$$\mathrm{Ind}_{(2)}(\widetilde{D}^+) = \mathrm{Tr}_{(2)}\, \widetilde{S}^+ - \mathrm{Tr}_{(2)}\, \widetilde{S}^-$$

  where $\widetilde{S}^\pm$ are Γ-invariant remainders localized near the diagonal.
- $\Rightarrow$ Atiyah's theorem $\mathrm{Ind}_{(2)}(\widetilde{D}^+) = \mathrm{Ind}(D^+)$
- $\mathrm{Ind}_{(2)}$ is stable and localizable
- $\eta_{(2)}(\widetilde{D})$ is very sensitive and not localizable,
- $\eta_{(2)}(\widetilde{D})$ is the boundary-correction term in the index theorem on a covering with boundary (Ramachandran).
- **Corollary (due to Vaillant and Lueck-Schick):** $\mathrm{sign}(\widetilde{W}) = \int L(W, \nabla^G) - \eta_{(2)}(\widetilde{D}^{\mathrm{sign}}_{\partial \widetilde{W}})$

Rho invariants

. The **Cheeger-Gromov** ρ-invariant is defined as the difference

$$\rho_{(2)}(\widetilde{D}) := \eta_{(2)}(\widetilde{D}) - \eta(D)$$

This is a **more stable** invariant:

Example

If $\{g_t\}$ is a path of metrics of **P**ositive **S**calar **C**urvature on a spin manifold, then $\rho(\widetilde{D}_t^{\mathrm{spin}})$ is constant;

Example

On an orientable manifold $\rho(\widetilde{D}^{\mathrm{sign}})$ is independent of the metric and an oriented-diffeomorphism invariant.

We can also consider **Lott's delocalized eta invariants:**
Fix a non-trivial conjugacy class $< g >$ of Γ (g is not the metric here). Define

$$\mathrm{Tr}_{<g>}(\widetilde{D}\exp(-(t\widetilde{D})^2)) := \sum_{h\in <g>} \int_{\mathcal{F}} \mathrm{tr}_x\, \widetilde{K}_t(x, hx).$$

This is called the **delocalized trace**. If the conjugacy class $< g >$ if finite then the **delocalized eta invariant** is defined as:

$$\eta_{<g>}(\widetilde{D}) := \frac{2}{\sqrt{\pi}} \int_0^\infty \mathrm{Tr}_{<g>}(\widetilde{D}\exp(-(t\widetilde{D})^2)\, dt.$$

Problems at $t = \infty$ if $| < g > | = \infty$, but OK if $< g >$ is of polynomial growth and $D = D^{\mathrm{spin}}$ for a metric of PSC.

Summary/Remarks.

- We have defined 3 invariants attached to a Dirac type operator on M and a Galois Γ-covering $\widetilde{M} \to M$:

$$\rho_{\lambda_1 - \lambda_2}(D)\,, \quad \rho_{(2)}(\widetilde{D}) \quad \text{and} \quad \eta_{<g>}(\widetilde{D})$$

- when M is oriented and $D = D^{\mathrm{sign}}$ these are **diffeomorphism invariants**
- when M is spin, $D = \not{D}$, these invariants are constant on the connected components of $\mathcal{R}^+(M)$

Remark. The analogous index-theoretic invariants (in even dimensions) are *always* zero because of "localizability" near the diagonal of the index:

- $\mathrm{ind}\, D_{L_1} = \mathrm{ind}\, D_{L_2}$ (from Atiyah-Singer, or directly)
- $\mathrm{ind}(D^+) = \mathrm{ind}_{(2)}(\widetilde{D}^+)$ (Atiyah's index theorem)
- $\mathrm{ind}_{<g>}(\widetilde{D}^+) = 0$.

Questions.

- **Question 1.** Are rho-invariants interesting ?
- **Question 2.** Are rho-invariants geometrically useful ?
- I will give answers to these questions
- Answers will involve operator algebras associated to Γ, K-theory for these operator algebras, higher index theory for Dirac operators and bordism theory

Γ is **torsion free**

Theorem

(Keswani '00. P.-Schick '05) Let M be an oriented odd dimensional manifold and let $\Gamma := \pi_1(M)$.
Assume that Γ is **torsion-free** *and that the max Baum-Connes assembly map is an isomorphism*

$$\mu_{max} \colon K_*(B\Gamma) \to K_*(C^*_{max}\Gamma).$$

Then $\rho_{(2)}(\widetilde{D}^{\mathrm{sign}})$ and $\eta_{<g>}(\widetilde{D}^{\mathrm{sign}})$ are **homotopy invariants**.
For $\eta_{<g>}(\widetilde{D}^{\mathrm{sign}})$ one can relax the hypothesis and ask only that the **reduced** *Baum-Connes map $\mu_{red} \colon K_*(B\Gamma) \to K_*(C^*_{red}\Gamma)$, is an isomorphism.*

Theorem

(P.-Schick '05) In the Positive Scalar Curvature spin case the invariants are all equal to zero under the same assumptions

What goes into the proof (for $D = D^{\rm sign}$)

- $\widetilde{D}^{\rm sign}$ defines an index class in $K_*(C^*\Gamma)$ which is a **homotopy invariant** (deep)
- so if M and M' are homotopy equivalent then

$$\mathrm{Ind}(\widetilde{D}^{\rm sign}_X) = 0 \text{ in } K_*(C^*\Gamma)$$

  where $X = M \sqcup (-M')$ with obvious $u_X : X \to B\Gamma$
- $K_*(B\Gamma) = \{(N, u : N \to B\Gamma, E \to N)\}/\mathcal{R}$
  (E is a vector bundle on N.)
  $\mathcal{R}$ is given by **bordism**, **direct sum** and **vector bundle modification**.
- μ send a class $[N, u : N \to B\Gamma, E]$ to the index class of $\widetilde{D}^{\rm sign}_E$
- By **injectivity** and $\mathrm{Ind}(\widetilde{D}^{\rm sign}_X) = 0$ in $K_*(C^*\Gamma)$ we know that $[X, u_X : X \to B\Gamma, \mathbf{1}] = \mathbf{0}$ in $K_*(B\Gamma)$ where $X = M \sqcup (-M')$.

- Now employ natural transformation of homology theories: $T(B\Gamma) : \Omega_*(B\Gamma) \to K_*(B\Gamma)$; $[M, u : M \to B\Gamma] \to [M, u : M \to B\Gamma, \mathbf{1}]$. We have proved that $[X, u_X : X \to B\Gamma] \in \text{Ker}\, T(B\Gamma)$
- use **algebraic topology** to describe Ker $T(B\Gamma)$, can then find a bordism between $X = M \sqcup (-M')$ and a special manifold N.
- develop a APS index theory in the C^*-algebraic framework
- define perturbed rho-invariants and show they are bordism invariant using this APS theory and the **surjectivity** of μ
- solve problem for perturbed rho-invariants using the bordism invariance and the structure of the special manifold (computation)
- connect perturbed rho invariants to unperturbed (hard).

Examples for the maximal BC map:

- (torsion free) discrete subgroups of $SU(n,1)$ and of $SO(n,1)$;
- torsion free amenable groups.

Examples for the reduced BC map:

- above examples;
- Gromov hyperbolic groups;
- cocompact discrete subgroups of $SL(3,\mathbb{C})$.

The Baum-Connes **conjecture** (still open) states that

$$\mu_{red} \colon K_*(B\Gamma) \to K_*(C^*_{red}\Gamma)$$

is **always** an isomoprhism if Γ is torsion free.

If Γ has property T then $\mu_{max} \colon K_*(B\Gamma) \to K_*(C^*_{max}\Gamma)$ is not surjective.

Comment: so, in the torsion-free case rho-invariants behave like indeces !! (under a Baum-Connes assumption).

Another comment (on the utility issue): with Schick we do give examples (with Γ satisfying BC, indeed even for $\Gamma = \mathbb{Z}$) of manifolds which are **homologically indistinguishable** but with different rho-invariants (thus **not-homotopically equivalent**)

Γ has torsion

- We start with $D = \not{D}$ and g of PSC.
- In this case the 3 rho-invariants are in general **non-zero**.
- Could be used in distinguishing metrics of PSC that are not path-connected or non-bordant (even modulo diffeomorphisms) on M.
- Results by Botvinnik-Gilkey when Γ is **finite** using APS eta.
- Now we look at the case Γ **infinite** and try to use $\rho_{(2)}$ and $\eta_{<g>}$

Theorem

(P.-Schick '06) M is spin of dimension $4k+3$, $k>0$, with g of PSC and $\Gamma = \pi_1(M)$ with torsion.

Then M admits **infinitely many** *different bordism classes of metric with PSC. They are distinguished by Cheeger-Gromov $\rho_{(2)}(\widetilde{D}^{\rm spin})$.*

The statement is true modulo the action of $\mathrm{Diffeo}(M)$ on the set $\mathcal{R}^+(M)$ of metrics with PSC. In particular:

$$|\pi_0(\mathcal{R}^+(M)/\mathrm{Diffeo}(M))| = \infty$$

What goes into the proof.

Fundamental to our analysis are the bordism groups $\mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma)$, $\Omega^{\mathrm{spin}}_n(B\Gamma)$ and the map $\mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \xrightarrow{\alpha} \Omega^{\mathrm{spin}}_n(B\Gamma)$This is part of Stolz' long exact sequence of bordism groups

$$\to \Omega^{\mathrm{spin}}_{n+1}(B\Gamma) \xrightarrow{t} R^{\mathrm{spin}}_{n+1}(B\Gamma) \xrightarrow{\delta} \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \to \Omega^{\mathrm{spin}}_n(B\Gamma) \to R^{\mathrm{spin}}_n(B\Gamma) \to$$

1. fix classifying map $u : M \to B\Gamma$ for universal covering
2. want to show that the set $\mathcal{A} := \{[M, u, h], h \in \mathcal{R}^+(M)\} \subset \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma)$ is infinite
3. using Gromow-Lawson/Shoen-Yau we prove that $\mathrm{Ker}(\alpha) \subset \mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma)$ acts freely on $\mathcal{A}$
4. $\rho_{(2)}$ and $\eta_{<g>}$ define homomorphisms $\mathrm{Pos}^{\mathrm{spin}}_n(B\Gamma) \to \mathbb{R}$ (APS L^2-index theory (Ramachandran/Leichtnam-Piazza))
5. $\rho_{(2)} : \mathrm{Pos}^{\mathrm{spin}}_n(B\mathbb{Z}_n) \to \mathbb{R}$ has infinite image (Botvinnik-Gilkey)
6. if $i : \mathbb{Z}_n \hookrightarrow \Gamma$, we get $Bi : B\mathbb{Z}_n \to B\Gamma$; if $x \in \mathrm{Pos}^{\mathrm{spin}}_n(B\mathbb{Z}_n)$ then $\rho_{(2)}((Bi)_* x) = \rho_{(2)}(x)$ (induction formula)

Using 1 $\to$ 6 one shows that $\rho_{(2)} : Ker(\alpha) \to \mathbb{R}$ has infinite image.

More refined results using $\eta_{<g>}$:

Theorem

(P.-Schick) Let $\dim M \equiv 4$ *mod 3; consider*
$\mathcal{C}_{fp} := \{< g > \mid g$ *finite order,* $< g >$ *has polynomial growth*$\}$
and the involution $\tau : \mathcal{C}_{fp} \to \mathcal{C}_{fp}$, $< g > \to < g^{-1} >$. **Then** *a group of rank* $|\mathcal{C}_{fp}/\tau|$ *acts freely on* $\mathcal{A} := \{[M, u, h], h \in \mathcal{R}^+(M)\}$

Conclusion: in this case one can prove that there are ∞^k distinct metrics of positive scalar curvature on M, with k depending on the cardinality of a set made of certain conjugacy classes.

Using $\eta_{<g>}$ we get similar results in dimension $4k + 1$.

- We now pass to the signature operator (Γ has always torsion)
- We know that for the signature operator the 3 rho-invariants are **diffeomorphism invariants**;
  they are **not** homotopy invariants; e.g. lens spaces.
- One can use these invariants in order to distinguish non-diffeomorphic manifolds within a fixed homotopy class.

Theorem

(Chang-Weinberger) If M is a compact oriented manifold of dimension $4k+3$, $k>0$, such that $\pi_1(M)=\Gamma$ has torsion, then there are infinitely many manifolds that are homotopic equivalent to M but not diffeomorphic to it.

Fundamental tool in the proof: L-theory groups $L_{k+1}(\mathbb{Z}\Gamma)$.

Conclusion.

- When Γ has torsion the spin rho-invariants are non-zero and useful in distinguishing PSC metrics;
- When Γ has torsion the signature rho-invariants are non-zero and useful in distinguishing non-diffeomorphic manifolds.
- When Γ is torsion free the rho-invariants behaves like indeces under a BC assumption: they are zero in the spin PSC case, they are homotopy invariants for the signature operator.

References for recent material: (both papers by P. Piazza and T. Schick)

Groups with torsions, bordism and rho-invariants.. *Pacific Journal of Mathematics* (2007)

Bordism, rho-invariants and the Baum-Connes conjecture *Journal of Noncommutative Geometry* Vol. 1, 2007.